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Inequalities

An Approach Through Problems

Second Edition

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Preface(First edition)

The International Mathematical Olympiad(IMO), which started as a simple contest among seven communist block countries in Europe in 1959, has now encompassed the whole world. With nearly 100 countries participating in this mega event, this has acquired a true international character. Mathematical olympiad has effused new enthusiasm in the last few generations of young students and really talented young minds have started getting attracted to the rare beauty of mathematics. Along with it, new ideas have emerged and many intricate problems woven around these ideas have naturally been discovered. This has enriched basic mathematics, strengthened the foundations of elementary mathematics and has posed challenging problems to the younger generation. In turn, high school mathematics has undergone a profound change.

Even though the concept of inequalities is old, the mathematical olympiad movement has generated new problems based on these inequalities and newer applications of these old inequalities. There are classical inequalities like the Arithmetic mean-Geometric mean inequality and the Cauchy-Schwarz inequality which are very old and which have innumerable applications. The main purpose of this book is to give a comprehensive presentation of inequalities and their use in mathematical olympiad problems. This book is also intended for those students who would like to participate in mathematical olympiad. Hopefully, this will fill a little vacuum that exists in the world of books. I have not touched on integral inequalities; they are not a part of olympiad mathematics.

The book is divided into six chapters. The first chapter describes all the classical inequalities which are useful for students who are interested in mathematical olympiad and similar contests. I have tried to avoid too much of theory; many results are taken for granted whenever a need for some advanced mathematics is required(especially results from calculus). Rather, I have put more emphasis on problems. The second chapter gives many useful techniques for deriving more inequalities. Here again the stress is on the application of different methods to problems. An important class of inequalities, called *geometric inequalities*, is based on geometric structures, like, triangles and quadrilaterals. Some important geometric inequalities are derived in the third chapter. In the fourth chapter, the problems(mainly taken from olympiad contests) whose solution(s) involve application of inequalities are discussed. The fifth chapter is simply a large collection of problems from various contests around the world and some problems are also taken from several problem journals. The reader is advised to try these problems on his own before looking into their possible solutions, which are discussed in the sixth chapter.

As I said earlier, the problems have been taken from various sources. I have

tried to give reference to them where ever possible and whenever I had one. I deeply regret and apologise for any inadvertent omission in mentioning the source.

This whole exercise of writing a book on inequalities is the outcome of my discussion with the bright students who have attended training camps and with my colleagues from the training camp. I am really grateful to all of them. Special thanks go to my colleagues, Prof. C.R.Pranesachar, from the Mathematical Olympiad cell and Prof. R.B.Bapat from the Indian Statistical Institute, New Delhi, for their encouragement. I also wish to acknowledge the support given by Prof. Arvindkumar, Director of the Homi Bhabha Centre for Science Education(TIFR), Mumbai. I would like to thank the referees for giving their valuable comments and invaluable tips to improve the quality of the material in this book.

Finally, the moral support I got from my family throughout this work is something I never forget.

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Preface(Second edition)

Eight years have elapsed after the first edition of this book has come out. More inequalities have been generated as a need for various national mathematical olympiads and International Mathematical Olympiads. New books have emerged and newer results have been added during these years. This has prompted me to revise the book adding more material to the existing one. New problems based on the old methods, new solutions to the old problems and new methods have been added to the book to make it richer.

A section on proving symmetric inequalities has been added to chapter 2. More problems have been solved based on these methods. More than 70 problems have been added to chapter 5 and their solutions have been given in chapter 6. It is my sincere wish that the new edition will help all those who aspire to study inequalities for mathematics competitions and for general interest.

I thank all my colleagues and students who have enriched me with their lively conversations and made this revision possible.

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Chapter 1

Some basic inequalities

1.1 Introduction

As we all know, one of the important properties of real numbers is *comparability*. We can compare two distinct real numbers and say that one is *smaller* or *larger* than the other. There is an inherent ordering $<$ on the real number system $\mathbb{R}$ which helps us to compare two real numbers. The basic properties of this ordering on $\mathbb{R}$ are:

(i) Given any two real numbers a and b, one and only one of the following three relations is true:

$$a < b \quad \text{or} \quad a = b \quad \text{or} \quad a > b;$$

(*law of trichotomy.*)

(ii) $a > 0$ and $b > 0$ imply $a + b > 0$;

(iii) $a > 0$ and $b > 0$ imply $ab > 0$.

Any new inequality we derive is totally dependent on these basic properties. These properties are used to derive the arithmetic mean-geometric inequality, the Cauchy-Schwarz inequality, Chebyshev's inequality, the rearrangement inequality, Hölder's and Minkowski's inequalities, and are also used in the study of convex and concave functions. Here are some easy consequences of these fundamental properties of ordering on $\mathbb{R}$:

(1) $a < b$, then $a + c < b + c$, for any real c;

(2) $a < b$ and $c > 0$ give $ac < bc$; if $c < 0$, we have $ac > bc$;

(3) $0 < a < b$ implies $0 < 1/b < 1/a$;

(4) $a < 0$ and $b < 0$, then $ab > 0$; $a < 0$ and $b > 0$ imply $ab < 0$;

(5) $a < b$ and $b < c$ together imply $a < c$ (*transitivity*);

(6) if $ac < bc$ and $c > 0$, we have $a < b$;

(7) $0 < a < 1$ implies $a^2 < a$; if $a > 1$, we have $a^2 > a$;

(8) for any real a, $a^2 \geq 0$;

(9) if a and b are positive and $a^2 < b^2$, we have $a < b$.

We emphasise here that the subtraction of inequalities is generally not allowed. If $a > b$ and $c > d$, we cannot guarantee either $a - c > b - d$ or $c - a > d - b$. The reason is obvious: $x > y$ implies $-x < -y$. Similarly, we cannot divide an inequality by another one. If $a > b$ and $c > d$, none of a, b, c, d equal 0, neither $\frac{a}{c} > \frac{b}{d}$ is true nor is $\frac{c}{a} > \frac{d}{b}$. Again the reason is simple: $x > y$ gives $\frac{1}{x} < \frac{1}{y}$ (x, y not equal to 0). On the other hand, we may add any two inequalities. If all the terms in two inequalities are positive, we may also multiply them : if a, b, c, d are positive and $a > b$, $c > d$ holds, the inequality $ac > bd$ also holds.

For any real number x, we define its *absolute value* by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Note that $|x| \geq 0$ and $|x| = 0$ if and only if $x = 0$. The absolute value function $x \to |x|$ has some nice properties:

(1) $|-x| = |x|$; $\quad |x|^2 = x^2$; $\quad |xy| = |x|\,|y|$; $\quad |x/y| = |x|/|y|$ for $y \neq 0$.

(2) $|x+y| \leq |x| + |y|$ and equality holds if and only if x and y have the same sign. This is known as the *triangle inequality.*

(3) $\Big||x| - |y|\Big| \leq |x - y|$.

On the other hand, there is no useful ordering on the complex number system $\mathbb{C}$; there is no natural ordering on $\mathbb{C}$ as enjoyed by $\mathbb{R}$. (This is the inherent universal principle that you have to pay some thing if you want to get some thing.) However, we can come down to the real number system using the absolute value of a complex number. For any complex number $z = a + ib$, we associate its *complex conjugate* by $\overline{z} = a - ib$ and absolute value by $|z| = \sqrt{a^2 + b^2}$. Note that $|z|^2 = z \cdot \overline{z}$. It is easy to check the following properties of $|z|$:

(1) $|z| \geq 0$; and $|z| = 0$ if and only if $z = 0$;

(2) $|z_1 z_2| = |z_1|\,|z_2|$;

(3) $|z_1 + z_2| \leq |z_1| + |z_2|$ and equality holds if and only if $z_1 = \lambda z_2$ for some positive real number λ, or one of z_1, z_2 is zero.

1.2 Arithmetic mean-Geometric mean inequality

The basic inequality in the real number system is $x^2 \geq 0$ for any real number x. This is so fundamental that any other inequality for real numbers is a consequence of this. Consider any two non-negative real numbers a and b.

Then we know that $\sqrt{a}$ and $\sqrt{b}$ are meaningful as real numbers. Since the square of a real number is always non-negative, $\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0$. This may be rewritten in the form

$$\frac{a+b}{2} \geq \sqrt{ab}. \tag{1.1}$$

The real number $\frac{a+b}{2}$ is called the *arithmetic mean* of a and b; similarly $\sqrt{ab}$ is known as the *geometric mean* of a and b. Thus the property that $x^2 \geq 0$ for a real number x implies that the arithmetic mean of two non-negative real numbers cannot be smaller than their geometric mean. Moreover, the derivation also shows that equality in (1.1) holds if and only if $a = b$.

This may be considered in a general setting. Starting with n non-negative real numbers $a_1, a_2, \ldots, a_n$, we define their arithmetic mean $A(a_1, a_2, \ldots, a_n)$ and geometric mean $G(a_1, a_2, \ldots, a_n)$ by

$$\begin{aligned} A\left(a_1, a_2, \ldots, a_n\right) &= \frac{a_1 + a_2 + \cdots + a_n}{n}, \\ G\left(a_1, a_2, \ldots, a_n\right) &= \left(a_1 a_2 \cdots a_n\right)^{1/n}. \end{aligned}$$

As in the case of two numbers, a comparison between these two *means* leads to the AM-GM inequality.

Theorem 1. Given any n non-negative real numbers $a_1, a_2, \ldots, a_n$, they satisfy the inequality

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \left(a_1 a_2 \cdots a_n\right)^{1/n}. \tag{1.2}$$

Equality holds in the inequality if and only if $a_1 = a_2 = \cdots = a_n$.

Proof: There are several proofs of this classical theorem. We give here a clever induction proof which was originated by Cauchy. It also illustrates how one can go ahead in an induction proof leaving out the validity of induction for some numbers and subsequently prove the validity for missing numbers using an interpolation argument.

We have already obtained the result for $n = 2$;

$$\frac{a_1 + a_2}{2} \geq \left(a_1 a_2\right)^{1/2}. \tag{1.3}$$

Here equality holds if and only if $a_1 = a_2$. Consider 4 non-negative real numbers a_1, a_2, a_3, a_4. Divide them in to two groups; $\{a_1, a_2\}$ and $\{a_3, a_4\}$. Applying the known inequality for each group, we obtain

$$\frac{a_1 + a_2}{2} \geq \left(a_1 a_2\right)^{1/2}, \quad \frac{a_3 + a_4}{2} \geq \left(a_3 a_4\right)^{1/2}.$$

This leads to

$$\begin{aligned}
\frac{a_1+a_2+a_3+a_4}{4} &= \frac{\left(\dfrac{a_1+a_2}{2}\right)+\left(\dfrac{a_3+a_4}{2}\right)}{2} \\
&\geq \frac{\left(a_1a_2\right)^{1/2}+\left(a_3a_4\right)^{1/2}}{2} \\
&\geq \left\{\left(a_1a_2\right)^{1/2}\left(a_3a_4\right)^{1/2}\right\}^{1/2} \\
&= \left(a_1a_2a_3a_4\right)^{1/4}.
\end{aligned}$$

Thus the inequality (1.2) is obtained for $n=4$. It may again be observed that equality holds here if and only if $a_1=a_2$, $a_3=a_4$ and $a_1a_2=a_3a_4$. Since all numbers are non-negative, it follows that $a_1=a_2=a_3=a_4$. Now this can be used to prove (1.2) for $n=8$. Now induction shows that (1.2) holds for $n=2^k$ for all $k\in\mathbb{N}$. Here again the condition for equality is that all the 2^k numbers be equal.

Now consider any n non-negative real numbers $a_1,a_2,\ldots,a_n$. We choose a natural number k such that $2^{k-1}\leq n<2^k$. Put

$$A=\frac{a_1+a_2+\cdots+a_n}{n},$$

and consider the set $\{a_1,a_2,\ldots,a_n,A,A,\ldots A\}$ of 2^k numbers; here A appears 2^k-n times. We apply the AM-GM inequality for these 2^k numbers; its validity has already been ascertained for such a collection. We thus obtain

$$a_1+a_2+\cdots+a_n+\underbrace{A+A+\cdots+A}_{2^k-n}\geq 2^k\left(a_1a_2\cdots a_n\underbrace{AA\cdots A}_{2^k-n}\right)^{1/2^k}.$$

This reduces to

$$A\geq\left(a_1a_2\cdots a_nA^{2^k-n}\right)^{1/2^k}.$$

A simple manipulation now yields $A^n\geq a_1a_2\cdots a_n$ which is equivalent to the inequality (1.2).

We also observe that equality holds if and only if all the numbers are equal.

■

Example 1.1. Let $a_1,a_2,\ldots,a_n$ be n positive real numbers whose product is 1. Prove that

$$\left(1+a_1\right)\left(1+a_2\right)\cdots\left(1+a_n\right)\geq 2^n.$$

Solution: Using the inequality $\left(1+a_j\right) \geq 2\sqrt{a_j}$ for $1 \leq j \leq n$, we obtain

$$\left(1+a_1\right)\left(1+a_2\right)\cdots\left(1+a_n\right) \geq 2^n\left(a_1 a_2 \cdots a_n\right)^{1/2} = 2^n.$$

■

Example 1.2. Show that for any natural number $n > 1$, the inequality

$$(2n)! < \left\{n(n+1)\right\}^n,$$

holds good.

Solution: We split $(2n)!$ as two products:

$$(2n)! = \left(1\cdot 3\cdot 5\cdots(2n-1)\right)\left(2\cdot 4\cdot 6\cdots(2n)\right).$$

Using the AM-GM inequality, we get

$$\begin{aligned} 2\cdot 4\cdot 6\cdots(2n) &= 2^n\left(1\cdot 2\cdot 3\cdots n\right) \\ &< 2^n\left(\frac{1+2+3+\cdots+n}{n}\right)^n = (n+1)^n, \end{aligned}$$

and

$$\begin{aligned} 1\cdot 3\cdot 5\cdots(2n-1) &< \left(\frac{1+3+5+\cdots+(2n-1)}{n}\right)^n \\ &= n^n. \end{aligned}$$

Combining these two inequalities, one can obtain

$$(2n)! < (n+1)^n\, n^n = \left\{n(n+1)\right\}^n,$$

as desired. ■

Example 1.3. If a, b, c are the sides of a triangle, prove that

$$\frac{3}{2} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2. \tag{1.4}$$

Solution: The first part of the above inequality is equivalent to

$$\begin{aligned} \frac{9}{2} &\leq \frac{a}{b+c} + 1 + \frac{b}{c+a} + 1 + \frac{c}{a+b} + 1 \\ &= \left(a+b+c\right)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right). \end{aligned}$$

If we introduce $a+b=x$, $b+c=y$ and $c+a=z$, this reduces to

$$9 \leq \left(x+y+z\right)\left(\frac{1}{x} + \frac{1}{z} + \frac{1}{z}\right),$$

which is a consequence of the AM-GM inequality. (We observe that, all we need here is the positivity of a, b, c; full force of the hypothesis that a, b, c are the sides of a triangle is not needed in this part.)

Suppose c is the largest among a, b, c. By the symmetry, we may assume $a \leq b \leq c$. In this case

$$\begin{aligned}\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} &\leq \frac{a}{a+c}+\frac{c}{c+a}+\frac{c}{a+b}\\ &= 1+\frac{c}{a+b}\\ &< 2,\end{aligned}$$

since $c < a + b$ by the triangle inequality. ■

If $a_1, a_2, \ldots, a_n$ are n positive real numbers, we define their *harmonic mean* by

$$H(a_1, a_2, \ldots, a_n) = \frac{n}{\dfrac{1}{a_1}+\dfrac{1}{a_2}+\cdots\dfrac{1}{a_n}}.$$

Thus the harmonic mean of n positive real numbers is equal to the reciprocal of the arithmetic mean of the reciprocals of the given numbers. Since

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n} \geq n\left(\frac{1}{a_1 a_2 \cdots a_n}\right)^{1/n},$$

it follows that

$$G(a_1, a_2, \ldots, a_n) \geq H(a_1, a_2, \ldots, a_n).$$

Thus we have for n positive real numbers $a_1, a_2, \ldots, a_n$, the inequality

$$A(a_1, a_2, \ldots, a_n) \geq G(a_1, a_2, \ldots, a_n) \geq H(a_1, a_2, \ldots, a_n),$$

or briefly AM $\geq$ GM $\geq$HM. This is often referred to as the AM-GM-HM inequality. Here again equality holds if and only if all the numbers are equal.

Example 1.4. For any four positive real numbers a_1, a_2, a_3, a_4, prove the inequality

$$\frac{a_1}{a_1+a_2}+\frac{a_2}{a_2+a_3}+\frac{a_3}{a_3+a_4}+\frac{a_4}{a_4+a_1} \leq \frac{a_1}{a_2+a_3}+\frac{a_2}{a_3+a_4}+\frac{a_3}{a_4+a_1}+\frac{a_4}{a_1+a_2}.$$

Solution: Adding

$$\frac{a_2}{a_1+a_2}+\frac{a_3}{a_2+a_3}+\frac{a_4}{a_3+a_4}+\frac{a_1}{a_4+a_1}$$

to both sides, the inequality can be written in the following equivalent form:

$$a_2\left(\frac{1}{a_3+a_4}+\frac{1}{a_1+a_2}\right)+a_3\left(\frac{1}{a_4+a_1}+\frac{1}{a_2+a_3}\right)$$
$$+a_4\left(\frac{1}{a_1+a_2}+\frac{1}{a_3+a_4}\right)+a_1\left(\frac{1}{a_4+a_1}+\frac{1}{a_2+a_3}\right)\geq 4.$$

However, using the AM-HM inequality, we know that

$$\begin{aligned}\frac{1}{a_3+a_4}+\frac{1}{a_1+a_2} &\geq \frac{4}{a_1+a_2+a_3+a_4},\\ \frac{1}{a_4+a_1}+\frac{1}{a_2+a_3} &\geq \frac{4}{a_1+a_2+a_3+a_4}.\end{aligned}$$

Hence, the required inequality follows. ■

Example 1.5. Let a, b, c be the sides of a triangle such that

$$\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b}=s,$$

where s is the semi-perimeter of the triangle. Prove that the triangle is equilateral.

Solution: Observe that

$$\begin{aligned}2s &= \frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2}\\ &\geq \frac{2}{\frac{1}{a}+\frac{1}{b}}+\frac{2}{\frac{1}{b}+\frac{1}{c}}+\frac{2}{\frac{1}{c}+\frac{1}{a}}\\ &= \frac{2ab}{a+b}+\frac{2bc}{b+c}+\frac{2ca}{c+a}\\ &= 2s.\end{aligned}$$

Here the AM-HM inequality has been used. Thus equality holds in the AM-HM inequality and hence $a=b=c$. ■

Example 1.6. Suppose a, b, c are positive real numbers. Prove the inequality

$$\left(\frac{a+b}{2}\right)\left(\frac{b+c}{2}\right)\left(\frac{c+a}{2}\right)\geq\left(\frac{a+b+c}{3}\right)\left(abc\right)^{2/3}. \tag{1.5}$$

Solution: Introduce new variables x, y, z by

$$x=\frac{a}{a+b+c},\quad y=\frac{b}{a+b+c},\quad z=\frac{c}{a+b+c},$$

so that $x+y+z=1$. Now the inequality (1.5) is equivalent to

$$\left(\frac{x+y}{2}\right)\left(\frac{y+z}{2}\right)\left(\frac{z+x}{2}\right) \geq \frac{1}{3}\left(xyz\right)^{2/3}. \tag{1.6}$$

This can be rewritten using $x+y+z=1$ in the following form:

$$\left(1-x\right)\left(1-y\right)\left(1-z\right) \geq \frac{8}{3}\left(xyz\right)^{2/3}. \tag{1.7}$$

Expanding the left hand side and using once again $x+y+z=1$, we may write (1.7) in the form

$$\left(xyz\right)^{1/3}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right) \geq \frac{8}{3}. \tag{1.8}$$

Now the AM-HM inequality gives

$$\left(xyz\right)^{1/3} \geq \frac{3}{\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}},$$

so that

$$\left(xyz\right)^{1/3}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right) \geq 3-\left(xyz\right)^{1/3}.$$

Hence it is sufficient to prove that

$$\left(xyz\right)^{1/3} \leq \frac{1}{3}.$$

Since $x+y+z=1$, this follows from the AM-GM inequality. ■

Now let us see how one may proceed to generalise the AM-GM inequality. Consider n non-negative real numbers $a_1, a_2, a_3, \ldots, a_n$ and n positive integers $k_1, k_2, k_3, \ldots, k_n$. Form a set in which each a_j appears exactly k_j times, $1 \leq j \leq n$. This set contains $k_1+k_2+k_3+\cdots+k_n$ non-negative real numbers. The arithmetic mean and the geometric mean of these numbers are

$$\frac{k_1a_1+k_2a_2+\cdots+k_na_n}{k_1+k_2+\cdots+k_n}, \quad \left(a_1^{k_1}a_2^{k_2}\cdots a_n^{k_n}\right)^{\frac{1}{k_1+k_2+\cdots+k_n}}$$

respectively. Now the AM-GM inequality gives

$$\frac{k_1a_1+k_2a_2+\cdots+k_na_n}{k_1+k_2+\cdots+k_n} \geq \left(a_1^{k_1}a_2^{k_2}\cdots a_n^{k_n}\right)^{\frac{1}{k_1+k_2+\cdots+k_n}}.$$

Suppose we have positive rational numbers $r_1, r_2, \ldots, r_n$. Then we can reduce all of them to a common denominator:

$$r_j = \frac{k_j}{m}, \quad 1 \leq j \leq n,$$

where m and k_j, $1 \leq j \leq n$ are natural numbers. It is not hard to see that

$$\frac{r_1 a_1 + r_2 a_2 + \cdots + r_n a_n}{r_1 + r_2 + \cdots + r_n} = \frac{k_1 a_1 + k_2 a_2 + \cdots + k_n a_n}{k_1 + k_2 + \cdots + k_n}.$$

Moreover, we have

$$\left(a_1^{r_1} a_2^{r_2} \cdots a_n^{r_n}\right)^{\frac{1}{r_1+r_2+\cdots+r_n}} = \left(a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}\right)^{\frac{1}{k_1+k_2+\cdots+k_n}}.$$

It follows that

$$\frac{r_1 a_1 + r_2 a_2 + \cdots + r_n a_n}{r_1 + r_2 + \cdots + r_n} \geq \left(a_1^{r_1} a_2^{r_2} \cdots a_n^{r_n}\right)^{\frac{1}{r_1+r_2+\cdots+r_n}}.$$

It is possible to further extend this inequality using a continuity argument. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are positive reals and $a_1, a_2, \ldots, a_n$ are non-negative reals, then

$$\frac{\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n}{\lambda_1 + \lambda_2 + \cdots + \lambda_n} \geq \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}\right)^{\frac{1}{\lambda_1+\lambda_2+\cdots+\lambda_n}}. \tag{1.9}$$

This is known as the *generalised* or the *weighted arithmetic mean- geometric mean* inequality. With obvious modifications, this can be further carried to the *weighted arithmetic mean- geometric mean-harmonic mean* inequality.

Example 1.7. Let a, b, c be positive real numbers. Prove that

$$a^a b^b c^c \geq (abc)^{(a+b+c)/3}.$$

Solution: Assigning the weights a, b, c to the numbers a, b, c respectively, the *weighted* GM-HM inequality may be invoked. This leads to

$$\sqrt[a+b+c]{a^a b^b c^c} \geq \frac{a+b+c}{\frac{a}{a} + \frac{b}{b} + \frac{c}{c}} = \frac{a+b+c}{3} \geq \sqrt[3]{abc},$$

which gives the required inequality. ∎

Example 1.8. Let $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1, b_2, \ldots, b_n$ be two sequences of non-negative real numbers such that for each k, $1 \leq k \leq n$, the inequality $b_1 b_2 \cdots b_k \geq a_1 a_2 \cdots a_k$ holds. Prove that

$$b_1 + b_2 + \cdots + b_n \geq a_1 + a_2 + \cdots + a_n. \tag{1.10}$$

Solution: We may assume that $a_j > 0$ for all j; otherwise delete those particular a_j's which are equal to 0. This does not affect the inequality. Consider the set

$$\left\{\frac{b_1}{a_1}, \frac{b_2}{a_2}, \ldots, \frac{b_n}{a_n}\right\},$$

where each b_j/a_j is attached with the weight a_j. The weighted arithmetic mean is

$$\frac{a_1\left(\frac{b_1}{a_1}\right)+a_2\left(\frac{b_2}{a_2}\right)+\cdots+a_n\left(\frac{b_n}{a_n}\right)}{a_1+a_2+\cdots+a_n}=\frac{b_1+b_2+\cdots+b_n}{a_1+a_2+\cdots+a_n}$$

and the weighted geometric mean is

$$\left(\left(\frac{b_1}{a_1}\right)^{a_1}\left(\frac{b_2}{a_2}\right)^{a_2}\cdots\left(\frac{b_n}{a_n}\right)^{a_n}\right)^{\frac{1}{a_1+a_2+\cdots+a_n}}.$$

Since $a_j \geq a_{j+1}$ and $b_1b_2\cdots b_j \geq a_1a_2\cdots a_j$ for $1 \leq j \leq n$, we have

$$\left(\frac{b_1b_2\cdots b_j}{a_1a_2\cdots a_j}\right)^{a_j-a_{j+1}} \geq 1,$$

for $1 \leq j \leq n$. Since

$$\begin{aligned}&\left(\frac{b_1}{a_1}\right)^{a_1}\left(\frac{b_2}{a_2}\right)^{a_2}\cdots\left(\frac{b_n}{a_n}\right)^{a_n}\\ &\qquad=\left(\frac{b_1b_2\cdots b_n}{a_1a_2\cdots a_n}\right)^{a_n}\left(\frac{b_1b_2\cdots b_{n-1}}{a_1a_2\cdots a_{n-1}}\right)^{a_n-a_{n-1}}\cdots\left(\frac{b_1}{a_1}\right)^{a_2-a_1},\end{aligned}$$

we conclude that

$$\left(\frac{b_1}{a_1}\right)^{a_1}\left(\frac{b_2}{a_2}\right)^{a_2}\cdots\left(\frac{b_n}{a_n}\right)^{a_n} \geq 1.$$

It follows from the weighted AM-GM inequality that

$$b_1+b_2+\cdots+b_n \geq a_1+a_2+\cdots+a_n.$$

Alternate Solution:

Put $\lambda_j = b_j/a_j$, $1 \leq j \leq n$. We have to show that

$$\sum_{j=1}^{n} a_j(\lambda_j - 1) \geq 0.$$

If $L_k = \sum_{j=1}^{k}(\lambda_j - 1)$ then

$$\begin{aligned}L_k &= \lambda_1+\lambda_2+\cdots+\lambda_k - k\\ &= \frac{b_1}{a_1}+\frac{b_2}{a_2}+\cdots+\frac{b_k}{a_k}-k\\ &\geq k\left(\frac{b_1b_2\cdots b_k}{a_1a_2\cdots a_k}\right)^{1/k}-k \geq 0,\end{aligned}$$

for all k. On the other hand, we observe that

$$\begin{aligned}\sum_{k=1}^{n} L_k\big(a_k - a_{k+1}\big) &= \sum_{k=1}^{n}\big(a_k - a_{k+1}\big)\Big\{\sum_{j=1}^{k}\big(\lambda_j - 1\big)\Big\} \\ &= \big(\lambda_1 - 1\big)a_1 + \big(\lambda_2 - 1\big)a_2 + \cdots + \big(\lambda_n - 1\big)a_n.\end{aligned}$$

(Here we have set $a_{n+1} = 0$.) Since $L_k \geq 0$ and $a_k \geq a_{k+1}$ for all k, it follows that

$$\big(\lambda_1 - 1\big)a_1 + \big(\lambda_2 - 1\big)a_2 + \cdots + \big(\lambda_n - 1\big)a_n \geq 0,$$

which is to be proved. ■

1.3 Cauchy-Schwarz inequality

Consider the real numbers a, b, c, d. We have

$$\begin{aligned}(ac + bd)^2 = a^2c^2 + 2acbd + b^2d^2 &\leq a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 \\ &= \big(a^2 + b^2\big)\big(c^2 + d^2\big),\end{aligned}$$

where we have used $2abcd \leq a^2d^2 + b^2c^2$. This is a consequence of the inequality $(ad - bc)^2 \geq 0$. Thus we obtain

$$\big|ac + bd\big| \leq \sqrt{a^2 + b^2}\,\sqrt{c^2 + d^2}.$$

This is the simplest form of the Cauchy-Schwarz inequality. Geometrically, if we consider two vectors (a, b) and (c, d) in the plane (here we represent vectors by coordinates), then $ac + bd$ gives the scalar product (or the *dot* product) of two vectors. If we represent (a, b) by $\vec{v_1}$ and (c, d) by $\vec{v_2}$, then

$$ac + bd = \vec{v_1} \cdot \vec{v_2} = \mid \vec{v_1} \mid \mid \vec{v_2} \mid \cos\theta,$$

where θ is the angle between $\vec{v_1}$ and $\vec{v_2}$. This shows that

$$\big|ac + bd\big| \leq \mid \vec{v_1} \mid \mid \vec{v_2} \mid = \sqrt{a^2 + b^2}\,\sqrt{c^2 + d^2}.$$

Thus the Cauchy-Schwarz inequality here represents the fact that the absolute value of the dot product of two vectors does not exceed the product of their lengths. This property extends to higher dimensions giving us the general form of the Cauchy-Schwarz inequality.

Theorem 2. Let $\{a_1, a_2, a_3, \ldots, a_n\}$ and $\{b_1, b_2, b_3, \ldots, b_n\}$ be any two sets of real numbers. Then

$$\left|\sum_{j=1}^{n} a_j b_j\right| \leq \left(\sum_{j=1}^{n} a_j^2\right)^{1/2} \left(\sum_{j=1}^{n} b_j^2\right)^{1/2}. \tag{1.11}$$

Equality holds if and only if there exists a constant λ such that $a_j = \lambda b_j$, for $1 \leq j \leq n$.

Proof: Let us write

$$A = \sum_{j=1}^{n} a_j^2, \quad B = \sum_{j=1}^{n} b_j^2, \quad C = \sum_{j=1}^{n} a_j b_j.$$

The inequality to be proved is $C^2 \leq AB$. If $B = 0$, then $b_j = 0$ for all j so that $C = 0$. Hence the inequality trivially holds. We may therefore assume that $B \neq 0$. Since B is the sum of squares of real numbers, B must be positive. Using the fact that the square of a real number is non-negative, we have

$$\begin{aligned} 0 \leq \sum_{j=1}^{n} \left(Ba_j - Cb_j\right)^2 &= B^2 \sum_{j=1}^{n} a_j^2 + C^2 \sum_{j=1}^{n} b_j^2 - 2BC \sum_{j=1}^{n} a_j b_j \\ &= B^2 A + BC^2 - 2BC^2 \\ &= B\left(BA - C^2\right). \end{aligned}$$

Since $B > 0$, it follows that $C^2 \leq AB$. We also infer that equality holds if and only if $Ba_j - Cb_j = 0$ for all j, $1 \leq j \leq n$. Taking $\lambda = C/B$, the condition for equality that $a_j = \lambda b_j$ for $1 \leq j \leq n$ is obtained. ∎

Remark 2.1: The Cauchy-Schwarz inequality extends to complex numbers as well with obvious modifications. Let us consider two sets $\{a_1, a_2, a_3, \ldots, a_n\}$ and $\{b_1, b_2, b_3, \ldots, b_n\}$ of complex numbers. Then the inequality

$$\left|\sum_{j=1}^{n} a_j \bar{b}_j\right| \leq \left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2} \left(\sum_{j=1}^{n} |b_j|^2\right)^{1/2}$$

holds good. Moreover equality holds here if and only if $a_j = \lambda \bar{b}_j$, $1 \leq j \leq n$, for some constant λ. The proof is similar to the one given above.

If we take n real numbers $a_1, a_2, \ldots, a_n$, then the Cauchy-Schwarz inequality gives

$$a_1 + a_2 + \cdots + a_n \leq \sqrt{n}\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}.$$

This may be written in the form

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \leq \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}}.$$

Here the left hand side is the arithmetic mean of $a_1, a_2, \ldots, a_n$ and the right side is the square root of the arithmetic mean of $a_1^2, a_2^2, \ldots, a_n^2$. This is often referred as the *Root-Mean-Square inequality* or some times called as the *RMS inequality* for short.

Example 1.9. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{a}{c} + \frac{c}{b} + \frac{b}{a}.$$

Solution: We write

$$\frac{a}{c}+\frac{c}{b}+\frac{b}{a}=\frac{a}{b}\cdot\frac{b}{c}+\frac{b}{c}\cdot\frac{c}{a}+\frac{c}{a}\cdot\frac{a}{b}.$$

Now applying the Cauchy-Schwarz inequality to the sets

$$\left\{\frac{a}{b},\frac{b}{c},\frac{c}{a}\right\} \text{ and } \left\{\frac{b}{c},\frac{c}{a},\frac{a}{b}\right\},$$

we get

$$\begin{aligned}\frac{a}{c}+\frac{c}{b}+\frac{b}{a} &\le \left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)^{1/2}\left(\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{a^2}{b^2}\right)^{1/2}\\ &= \frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}.\end{aligned}$$

■

Example 1.10. Let $a_1, a_2, \ldots, a_n$ and A be real numbers such that

$$A+\sum_{k=1}^{n} a_k^2 < \frac{1}{n-1}\left(\sum_{k=1}^{n} a_k\right)^2.$$

Prove that $A < 2a_l a_m$ for every pair $\{a_l, a_m\}$, $l \neq m$.

Solution: We may take $l = 1$ and $m = 2$; the proof for any other pair of indices is the same. Suppose $A \geq 2a_1a_2$. Then we see that

$$\begin{aligned}A+\sum_{k=1}^{n} a_k^2 &\geq 2a_1a_2+a_1^2+a_2^2+\sum_{k=3}^{n} a_k^2\\ &= (a_1+a_2)^2+\sum_{k=3}^{n} a_k^2\\ &\geq \frac{1}{n-1}\left(\sum_{k=1}^{n} a_k\right)^2.\end{aligned}$$

(Here we have applied the Cauchy-Schwarz inequality to the $(n-1)$ tuples $\{a_1+a_2, a_3, \ldots, a_n\}$ and $\{1, 1, \ldots, 1\}$.) But this contradicts the given condition. We conclude that $A < 2a_1a_2$. ■

Example 1.11. (Short-List, IMO-1993) Let a, b, c, d be positive real numbers. Prove that

$$\frac{a}{b+2c+3d}+\frac{b}{c+2d+3a}+\frac{c}{d+2a+3b}+\frac{d}{a+2b+3c} \geq \frac{2}{3}.$$

Solution: We write

$$\sum_{\text{cyclic}} a = \sum_{\text{cyclic}} \frac{\sqrt{a}}{\sqrt{b+2c+3d}} \left(\sqrt{a}\sqrt{b+2c+3d}\right),$$

where the sum is taken cyclically over a, b, c, d. Now we apply Cauchy-Schwarz inequality to the sets

$$\left\{ \frac{\sqrt{a}}{\sqrt{b+2c+3d}}, \frac{\sqrt{b}}{\sqrt{c+2d+3a}}, \frac{\sqrt{c}}{\sqrt{d+2a+3b}}, \frac{\sqrt{d}}{\sqrt{a+2b+3c}} \right\}$$

and

$$\left\{ \sqrt{a}\sqrt{b+2c+3d},\ \sqrt{b}\sqrt{c+2d+3a},\ \sqrt{c}\sqrt{d+2a+3b},\ \sqrt{d}\sqrt{a+2b+3c} \right\},$$

to get

$$\left(\sum_{\text{cyclic}} a\right)^2 \leq \left(\sum_{\text{cyclic}} \frac{a}{b+2c+3d}\right)\left(\sum_{\text{cyclic}} a(b+2c+3d)\right).$$

However, we have

$$\sum_{\text{cyclic}} a(b+2c+3d) = 4(ab+ac+ad+bc+bd+cd).$$

Thus it follows that

$$\sum_{\text{cyclic}} \frac{a}{b+2c+3d} \geq \frac{(a+b+c+d)^2}{4(ab+ac+ad+bc+bd+cd)}.$$

On the other hand,

$$\begin{aligned} &3(a+b+c+d)^2 - 8(ab+ac+ad+bc+bd+cd) \\ &\quad = \quad 3\left(a^2+b^2+c^2+d^2\right) - 2(ab+ac+ad+bc+bd+cd) \\ &\quad = \quad (a-b)^2+(a-c)^2+(a-d)^2+(b-c)^2+(b-d)^2+(c-d)^2 \\ &\quad \geq \quad 0. \end{aligned}$$

Combining these two, it follows that

$$\frac{(a+b+c+d)^2}{4(ab+ac+ad+bc+bd+cd)} \geq \frac{2}{3}$$

and hence

$$\sum_{\text{cyclic}} \frac{a}{b+2c+3d} \geq \frac{2}{3}.$$

■

Example 1.12. Let $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ and $b_1, b_2, \ldots, b_n$ be two sequences of real numbers such that

$$\sum_{j=1}^{k} a_j \leq \sum_{j=1}^{k} b_j,$$

for each k, $1 \leq k \leq n$. Prove that

$$\sum_{j=1}^{n} a_j^2 \leq \sum_{j=1}^{n} b_j^2.$$

Solution: Take $a_{n+1} = 0$. Since $a_k - a_{k+1} \geq 0$ for $1 \leq k \leq n$, we have

$$\left(a_k - a_{k+1}\right) \sum_{j=1}^{k} a_j \leq \left(a_k - a_{k+1}\right) \sum_{j=1}^{k} b_j,$$

for $1 \leq k \leq n$. Summing this over k, we obtain

$$a_1^2 + a_2^2 + \cdots + a_n^2 \leq a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Applying the Cauchy-Schwarz inequality to the right side, we get

$$a_1^2 + a_2^2 + \cdots + a_n^2 \leq \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)^{1/2} \left(b_1^2 + b_2^2 + \cdots + b_n^2\right)^{1/2}.$$

This simplifies to

$$a_1^2 + a_2^2 + \cdots + a_n^2 \leq b_1^2 + b_2^2 + \cdots + b_n^2.$$

■

Example 1.13. If a, b, c are positive real numbers, prove that

$$\sum \frac{a^2}{(a+b)(a+c)} \geq \frac{3}{4},$$

where the summation is taken cyclically over a, b, c.

Solution: We have

$$\begin{aligned}\left(\sum_{\text{cyclic}} a\right)^2 &= \left(\sum_{\text{cyclic}} \frac{a}{\sqrt{(a+b)(a+c)}} \sqrt{(a+b)(a+c)}\right)^2 \\ &\leq \left(\sum_{\text{cyclic}} \frac{a^2}{(a+b)(a+c)}\right) \left(\sum_{\text{cyclic}} (a+b)(a+c)\right),\end{aligned}$$

using the Cauchy-Schwarz inequality. Thus we obtain

$$\sum_{\text{cyclic}} \frac{a^2}{(a+b)(a+c)} \geq \frac{(a+b+c)^2}{(a+b)(a+c)+(b+c)(b+a)+(c+a)(c+b)}.$$

We need to show that

$$4(a+b+c)^2 \geq 3\Big((a+b)(a+c)+(b+c)(b+a)+(c+a)(c+b)\Big).$$

This simplifies to

$$a^2+b^2+c^2 \geq ab+bc+ca,$$

which again follows from the Cauchy-Schwarz inequality.

■

Example 1.14. (China, 1989) Let $a_1, a_2, a_3, \ldots, a_n$ be n positive real numbers which add up to 1. Prove that

$$\sum_{j=1}^{n} \frac{a_j}{\sqrt{1-a_j}} \geq \frac{1}{\sqrt{n-1}} \sum_{j=1}^{n} \sqrt{a_j}.$$

Solution: We may write

$$\sum_{j=1}^{n} \frac{a_j}{\sqrt{1-a_j}} = \sum_{j=1}^{n} \frac{1}{\sqrt{1-a_j}} - \sum_{j=1}^{n} \sqrt{1-a_j}.$$

Using the AM-GM inequality, we have

$$\begin{aligned} \sum_{j=1}^{n} \frac{1}{\sqrt{1-a_j}} &\geq n\left\{\frac{1}{\prod_{j=1}^{n}\sqrt{1-a_j}}\right\}^{1/n} \\ &= n\sqrt{\frac{1}{\Big(\prod_{j=1}^{n}(1-a_j)\Big)^{1/n}}}. \end{aligned}$$

However,

$$\Big(\prod_{j=1}^{n}(1-a_j)\Big)^{1/n} \leq \frac{1}{n}\sum_{j=1}^{n}(1-a_j) = \frac{n-1}{n}.$$

Thus we obtain,

$$\sum_{j=1}^{n} \frac{1}{\sqrt{1-a_j}} \geq n\sqrt{\frac{n}{n-1}}.$$

On the other hand, the Cauchy-Schwarz inequality gives

$$\sum_{j=1}^{n} \sqrt{1-a_j} \leq \sqrt{n}\sqrt{\sum_{j=1}^{n}(1-a_j)} = \sqrt{n(n-1)}.$$

We thus get

$$\sum_{j=1}^{n} \frac{1}{\sqrt{1-a_j}} - \sum_{j=1}^{n} \sqrt{1-a_j} \geq n\sqrt{\frac{n}{n-1}} - \sqrt{n(n-1)} = \frac{\sqrt{n}}{\sqrt{n-1}}.$$

We also have

$$\left(\sum_{j=1}^{n} \sqrt{a_j}\right)^2 \leq n \sum_{j=1}^{n} a_j = n.$$

Thus

$$\sqrt{n} \geq \sum_{j=1}^{n} \sqrt{a_j}$$

and hence

$$\sum_{j=1}^{n} \frac{a_j}{\sqrt{1-a_j}} \geq \frac{\sqrt{n}}{\sqrt{n-1}} \geq \frac{1}{\sqrt{n-1}} \sum_{j=1}^{n} \sqrt{a_j}.$$

■

1.4 Chebyshev's inequality

Let us consider the Cauchy-Schwarz inequality for two sets of real numbers $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$:

$$\sum_{j=1}^{n} a_j b_j \leq \left(\sum_{j=1}^{n} a_j^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} b_j^2\right)^{\frac{1}{2}}.$$

We may write this also in the form

$$\frac{1}{n}\left(\sum_{j=1}^{n} a_j b_j\right) \leq \left(\frac{1}{n}\sum_{j=1}^{n} a_j^2\right)^{\frac{1}{2}} \left(\frac{1}{n}\sum_{j=1}^{n} b_j^2\right)^{\frac{1}{2}}.$$

This shows that the Cauchy-Schwarz inequality relates the arithmetic means of $a_1^2, a_2^2, \ldots, a_n^2$ and $b_1^2, b_2^2, \ldots, b_n^2$ with that of the numbers $a_1b_1, a_2b_2, \ldots, a_nb_n$. How do we relate the arithmetic means of the numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ with that of $a_1b_1, a_2b_2, \ldots, a_nb_n$. Chebyshev's inequality answers this question.

Theorem 3. Let $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ be two sets of real numbers. Suppose either

$$a_1 \leq a_2 \leq \cdots \leq a_n \text{ and } b_1 \leq b_2 \leq \cdots \leq b_n;$$

or

$$a_1 \geq a_2 \geq \cdots \geq a_n \text{ and } b_1 \geq b_2 \geq \cdots \geq b_n;$$

i.e., both the sequences $\langle a_k \rangle$ and $\langle b_k \rangle$ are non-decreasing or both non-increasing. Then the inequality

$$\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)\left(\frac{b_1 + b_2 + \cdots + b_n}{n}\right) \leq \frac{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}{n}$$

holds. The inequality is strict unless at least one of the sequences is a constant sequence.

Proof: We have

$$\begin{aligned}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(a_j b_j - a_j b_k\right) &= \sum_{j=1}^{n}\left(n a_j b_j - a_j \sum_{k=1}^{n} b_k\right) \\ &= n\sum_{j=1}^{n} a_j b_j - \left(\sum_{j=1}^{n} a_j\right)\left(\sum_{k=1}^{n} b_k\right).\end{aligned}$$

Similarly, it is easy to obtain

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\left(a_k b_k - a_k b_j\right) = n\sum_{k=1}^{n} a_k b_k - \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{j=1}^{n} b_j\right).$$

Using these two we obtain

$$\begin{aligned}& n\sum_{k=1}^{n} a_k b_k - \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{j=1}^{n} b_j\right) \\ &\qquad = \frac{1}{2}\left\{\sum_{j=1}^{n}\sum_{k=1}^{n}\left(a_j b_j - a_j b_k + a_k b_k - a_k b_j\right)\right\} \\ &\qquad = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(a_j - a_k\right)\left(b_j - b_k\right).\end{aligned}$$

Since $\left(a_j - a_k\right)\left(b_j - b_k\right) \geq 0$ whether both the sequences are non-decreasing or both non-increasing, we get

$$n\sum_{k=1}^{n} a_k b_k - \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{j=1}^{n} b_j\right) \geq 0.$$

Here equality holds if and only if for each pair of indices j, k either $a_j = a_k$ or $b_j = b_k$. In particular taking $j = 1$, $k = n$ we see that $a_1 = a_n$ or $b_1 = b_n$. It follows that either $\langle a_j \rangle$ is a constant sequence or $\langle b_j \rangle$ is a constant sequence. ■

Remark 3.1: If the sequences $\langle a_j \rangle$ and $\langle b_j \rangle$ are oppositely ordered, i.e., either $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \ge b_2 \ge \cdots \ge b_n$, or $a_1 \ge a_2 \ge \cdots \ge a_n$ and $b_1 \le b_2 \le \cdots \le b_n$, then the inequality reverses;

$$\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)\left(\frac{b_1 + b_2 + \cdots + b_n}{n}\right) \ge \frac{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}{n}.$$

The proof is similar to the one given above, but with the crucial observation that $\bigl(a_j - a_k\bigr)\bigl(b_j - b_k\bigr) \le 0$ holds in this case for any pairs $\{a_j, a_k\}$ and $\{b_j, b_k\}$.

Remark 3.2: Chebyshev's inequality can be generalised to three or more sets of real numbers. Suppose $\langle a_{1,j} \rangle$, $\langle a_{2,j} \rangle$, $\langle a_{3,j} \rangle$ $\ldots$, $\langle a_{r,j} \rangle$, $1 \le j \le n$ be r sets of real numbers such that

$$0 \le a_{k,1} \le a_{k,2} \le \cdots \le a_{k,n},$$

for $1 \le k \le r$. Then the following inequality holds:

$$\left(\frac{1}{n}\sum_{j=1}^{n} a_{1,j}\right)\left(\frac{1}{n}\sum_{j=1}^{n} a_{2,j}\right)\left(\frac{1}{n}\sum_{j=1}^{n} a_{3,j}\right)\cdots\left(\frac{1}{n}\sum_{j=1}^{n} a_{r,j}\right) \le \frac{1}{n}\sum_{j=1}^{n} a_{1,j}a_{2,j}a_{3,j}\cdots a_{r,j}.$$

For $r \ge 3$, it is essential to consider only non-negative sequences.

Remark 3.3: Suppose $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are two real sequences such that either both are non-decreasing or both are non-increasing, and let $p_1, p_2, \ldots, p_n$ be a sequence of non-negative real numbers such that $\sum_{j=1}^{n} p_j$ is positive. Then the following inequality holds:

$$\left(\frac{\sum_{j=1}^{n} p_j a_j b_j}{\sum_{j=1}^{n} p_j}\right) \ge \left(\frac{\sum_{j=1}^{n} p_j a_j}{\sum_{j=1}^{n} p_j}\right)\left(\frac{\sum_{j=1}^{n} p_j b_j}{\sum_{j=1}^{n} p_j}\right).$$

The proof runs on similar lines. This gives Chebyshev's inequality with weights.

Example 1.15. Let a, b, c be positive real numbers and n a natural number. Prove that

$$\frac{a^n}{b+c} + \frac{b^n}{c+a} + \frac{c^n}{a+b} \ge \frac{a^{n-1} + b^{n-1} + c^{n-1}}{2}.$$

Solution: Since the expression is symmetric in a, b, c, we may assume that $a \ge b \ge c$. Then we have

$$\frac{a}{b+c} \ge \frac{b}{c+a} \ge \frac{c}{a+b}.$$

In fact

$$\begin{aligned}\frac{a}{b+c} \geq \frac{b}{c+a} \quad &\Longleftrightarrow \quad ac+a^2 \geq bc+b^2 \\ &\Longleftrightarrow \quad (a-b)(c+a+b) \geq 0 \\ &\Longleftrightarrow \quad a \geq b.\end{aligned}$$

Thus we have

$$a^{n-1} \geq b^{n-1} \geq c^{n-1} \text{ and } \frac{a}{b+c} \geq \frac{b}{c+a} \geq \frac{c}{a+b}.$$

Using Chebyshev's inequality for these numbers, we get

$$\left(a^{n-1}+b^{n-1}+c^{n-1}\right)\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right) \leq 3\left(\frac{a^n}{b+c}+\frac{b^n}{c+a}+\frac{c^n}{a+b}\right).$$

However, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2},$$

using the left part of the inequality (1.4). Hence

$$\frac{a^n}{b+c}+\frac{b^n}{c+a}+\frac{c^c}{a+b} \geq \frac{a^{n-1}+b^{n-1}+c^{n-1}}{2}.$$

∎

Example 1.16. Let $a_1, a_2, \ldots, a_n$ be n real numbers in the interval $[0,1]$ such that $a_1+a_2+\cdots+a_n=1$. Prove that

$$\sum_{j=1}^{n} \frac{n-a_j}{1+na_j} \geq \frac{n^2-1}{2}.$$

Solution: First we arrange a_j's such that $a_1 \leq a_2 \leq \cdots \leq a_n$. Then we have

$$n-a_1 \geq n-a_2 \geq \cdots \geq n-a_n.$$

The arrangement of a_j's shows that, $1+na_1 \leq 1+na_2 \leq \cdots \leq 1+na_n$, and hence

$$\frac{1}{1+na_1} \geq \frac{1}{1+na_2} \geq \cdots \geq \frac{1}{1+na_n}.$$

Using Chebyshev's inequality, we obtain

$$\begin{aligned}\sum_{j=1}^{n} \frac{n-a_j}{1+na_j} \quad &\geq \quad \frac{1}{n}\left(\sum_{j=1}^{n}(n-a_j)\right)\left(\sum_{j=1}^{n}\frac{1}{1+na_j}\right) \\ &\geq \quad \frac{n^2-1}{n}\left(\sum_{j=1}^{n}\frac{1}{1+na_j}\right).\end{aligned}$$

Using the AM-HM inequality, we also get

$$\sum_{j=1}^{n} \frac{1}{1+na_j} \geq \frac{n^2}{\sum_{j=1}^{n}(1+na_j)} = \frac{n^2}{2n} = \frac{n}{2}.$$

Combining these, we finally get

$$\sum_{j=1}^{n} \frac{n-a_j}{1+na_j} \geq \left(\frac{n^2-1}{n}\right)\left(\frac{n}{2}\right) = \frac{n^2-1}{2}.$$

■

1.5 Rearrangement inequality

Consider two sets of real numbers: $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$. We pose the following question: among all permutations $\langle a_j' \rangle$ of $\langle a_j \rangle$ and $\langle b_j' \rangle$ of $\langle b_j \rangle$, which permutation maximises the product $\sum a_j' b_j'$ and which minimizes it? The rearrangement inequality answers this question.

Theorem 4. Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ (or $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$) be two sequences of real numbers. If $a_1', a_2', \ldots, a_n'$ is any permutation of $a_1, a_2, \ldots, a_n$, the inequality

$$\sum_{j=1}^{n} a_j b_{n+1-j} \leq \sum_{j=1}^{n} a_j' b_j \leq \sum_{j=1}^{n} a_j b_j,$$

holds. Thus the sum $\sum_{j=1}^{n} a_j b_j$ is maximum when the two sequences $\langle a_j \rangle$ and $\langle b_j \rangle$ are ordered similarly (i.e., either both non-decreasing or both non-increasing). And the sum is minimum when $\langle a_j \rangle$ and $\langle b_j \rangle$ are ordered in opposite manner (i.e., one of them is increasing and the other decreasing).

Proof: We may assume that both a_j's and b_j's are non-decreasing; the proof is similar in the other case. Suppose $\langle a_j' \rangle \neq \langle a_j \rangle$. Let r be the largest index such that $a_r' \neq a_r$; i.e., $a_r' \neq a_r$ and $a_j' = a_j$ for $r < j \leq n$. This implies that a_r' is from the set $\{a_1, a_2, \ldots, a_{r-1}\}$ and $a_r' < a_r$. Further this also shows that $a_1', a_2' \ldots, a_r'$ is a permutation of $a_1, a_2, \ldots, a_r$. Thus we can find indices $k < r$ and $l < r$ such that $a_k' = a_r$ and $a_r' = a_l$. It follows that

$$a_k' - a_r' = a_r - a_l \geq 0, \quad b_r - b_k \geq 0.$$

We now interchange a'_r and a'_k to get a permutation $a''_1, a''_2, \dots, a''_n$ of $a'_1, a'_2, \dots, a'_n$; thus

$$\begin{cases} a''_j = a'_j, & \text{if } j \neq r, k \\ a''_r = a'_k = a_r, \\ a''_k = a'_r = a_l. \end{cases}$$

Let us consider the sums

$$S'' = a''_1 b_1 + a''_2 b_2 + \cdots + a''_n b_n, \quad S' = a'_1 b_1 + a'_2 b_2 + \cdots + a'_n b_n.$$

Consider the difference $S'' - S'$:

$$\begin{aligned} S'' - S' &= \sum_{j=1}^{n} \big(a''_j - a'_j\big) b_j \\ &= \big(a''_k - a'_k\big) b_k + \big(a''_r - a'_r\big) b_r \\ &= \big(a'_r - a'_k\big) b_k + \big(a'_k - a'_r\big) b_r \\ &= \big(a'_k - a'_r\big)\big(b_r - b_k\big). \end{aligned}$$

Since $a'_k - a'_r \geq 0$ and $b_r - b_k \geq 0$, we conclude that $S'' \geq S'$. Observe that the permutation $a''_1, a''_2, \dots, a''_n$ of $a'_1, a'_2, \dots, a'_n$ has the property that $a''_j = a'_j = a_j$ for $r < j \leq n$ and $a''_r = a'_k = a_r$. Hence the permutation $\langle a''_j \rangle$ in place of $\langle a'_j \rangle$ may be considered and the procedure can be continued as above. After at most $n-1$ such steps, we arrive at the original permutation $\langle a_j \rangle$ from $\langle a'_j \rangle$. At each step the corresponding sum is non-decreasing. Hence it follows that

$$a'_1 b_1 + a'_2 b_2 + \cdots + a'_n b_n \leq a_1 b_1 + a_2 b_2 + \cdots + a_n b_n. \tag{1.12}$$

To get the other inequality, let us put

$$c_j = a'_{n+1-j}, \quad d_j = -b_{n+1-j}.$$

Then $c_1, c_2, \dots, c_n$ is a permutation of $a_1, a_2, \dots, a_n$ and $d_1 \leq d_2 \leq \cdots \leq d_n$. Using the inequality (1.12) for the sequences $\langle c_j \rangle$ and $\langle d_j \rangle$, we get

$$c_1 d_1 + c_2 d_2 + \cdots + c_n d_n \leq a_1 d_1 + a_2 d_2 + \cdots + a_n d_n.$$

Substituting back c_j and d_j, we get

$$-\sum_{j=1}^{n} a'_{n+1-j}\, b_{n+1-j} \leq -\sum_{j=1}^{n} a_j b_{n+1-j}.$$

This reduces to

$$a'_1 b_1 + a'_2 b_2 + \cdots + a'_n b_n \geq a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1, \tag{1.13}$$

which gives the other part.

It is not difficult to find the condition under which equality holds in the inequality. If for each pair k, l with $1 \leq k < l \leq n$, either $a'_k = a'_l$ or $a'_k > a'_l$ and $b_k = b_l$, then equality holds in (1.12). A similar condition is true for equality in (1.13): for each k, l with $1 \leq k < l \leq n$, either $a'_{n+1-k} = a'_{n+1-l}$ or $a'_{n+1-k} > a'_{n+1-l}$ and $b_{n+1-k} = b_{n+1-l}$. ■

Many of the inequalities we have studied so far can be derived using the rearrangement inequality.

Corollary 4.1: Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be n given real numbers and $\beta_1, \beta_2, \ldots, \beta_n$ be a permutation of $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then

$$\sum_{j=1}^{n} \alpha_j \beta_j \leq \sum_{j=1}^{n} \alpha_j^2,$$

with equality if and only if $\langle \alpha_j \rangle = \langle \beta_j \rangle$.

Proof: Let $\alpha'_1, \alpha'_2, \ldots, \alpha'_n$ be a permutation of $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\alpha'_1 \leq \alpha'_2 \leq \cdots \leq \alpha'_n$. Then we can find a bijection σ of $\{1, 2, 3, \ldots, n\}$ onto itself such that $\alpha'_j = \alpha_{\sigma(j)}$, $1 \leq j \leq n$; i.e., σ is a permutation on the set $\{1, 2, 3, \ldots, n\}$. Let $\beta'_j = \beta_{\sigma(j)}$. Then $\beta'_1, \beta'_2, \ldots, \beta'_n$ is a permutation of $\alpha'_1, \alpha'_2, \ldots, \alpha'_n$. Applying the rearrangement inequality to $\alpha'_1, \alpha'_2, \ldots, \alpha'_n$ and $\beta'_1, \beta'_2, \ldots, \beta'_n$, we get

$$\sum_{j=1}^{n} \alpha'_j \beta'_j \leq \sum_{j=1}^{n} \left(\alpha'_j\right)^2 = \sum_{j=1}^{n} \alpha_j^2.$$

On the other hand, we observe that

$$\sum_{j=1}^{n} \alpha'_j \beta'_j = \sum_{j=1}^{n} \alpha_{\sigma(j)} \beta_{\sigma(j)} = \sum_{j=1}^{n} \alpha_j \beta_j,$$

since σ is a bijection on $\{1, 2, 3, \ldots, n\}$. It follows that

$$\sum_{j=1}^{n} \alpha_j \beta_j \leq \sum_{j=1}^{n} \alpha_j^2.$$

Suppose equality holds in the inequality and $\langle \alpha_j \rangle \neq \langle \beta_j \rangle$. Then $\langle \alpha'_j \rangle \neq \langle \beta'_j \rangle$. Let k be the largest index such that $\alpha'_k \neq \beta'_k$; i.e., $\alpha'_k \neq \beta'_k$ and $\alpha'_j = \beta'_j$ for $k < j \leq n$. Let m be the least integer such that $\alpha'_k = \beta'_m$. If $m > k$, then $\beta'_m = \alpha'_m$ and hence $\alpha'_k = \alpha'_m$. This implies that $\alpha'_k = \alpha'_{k+1} = \cdots = \alpha'_m$ and hence $\beta'_{k+1} = \cdots = \beta'_m$. But now we have a block of $m+1-k$ equal elements among α''s and $m-k$ elements among β''s. It follows that there is an $m_1 > m$ such that $\alpha'_k = \beta'_{m_1}$. Using m_1 as pivotal, we obtain $\alpha'_k = \alpha'_{k+1} = \cdots = \alpha'_m = \cdots = \alpha'_{m_1}$ and $\beta'_{k+1} = \cdots = \beta'_m = \cdots = \beta'_{m_1}$. This process cannot be continued indefinitely. We conclude that $\alpha'_k = \beta'_l$ for some $l < k$, thus forcing $m < k$.

Obviously $\beta'_m \neq \beta'_k$ by our choice of k. We know that equality holds if and only if for any two indices $r \neq s$, either $\alpha'_r = \alpha'_s$ or $\beta'_r = \beta'_s$. Since $\beta'_m \neq \beta'_k$, we must have $\alpha'_m = \alpha'_k$. But then we have $\alpha'_m = \alpha'_{m+1} = \cdots = \alpha'_k$. Using the minimality of m, we see that $k - m + 1$ equal elements $\alpha'_m, \alpha'_{m+1}, \cdots, \alpha'_k$ must be among $\beta'_m, \beta'_{m+1}, \cdots, \beta'_n$ and since β'_k is different from α'_k, we must have $\alpha'_k = \beta'_l$ for some $l > k$. But then using $\beta'_l = \alpha'_l$ we get

$$\alpha'_m = \alpha'_{m+1} = \cdots = \alpha'_k = \cdots = \alpha'_l.$$

Thus the number of equal elements gets enlarged to $l - m + 1 > k - m + 1$. Since this process cannot be continued indefinitely, we conclude that $\langle \alpha'_j \rangle = \langle \beta'_j \rangle$. It now follows that $\langle \alpha_j \rangle = \langle \beta_j \rangle$. ■

Corollary 4.2: Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive real numbers; let $\beta_1, \beta_2, \ldots, \beta_n$ be a permutation of $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then

$$\sum_{j=1}^{n} \frac{\beta_j}{\alpha_j} \geq n,$$

and equality holds if and only if $\langle \alpha_j \rangle = \langle \beta_j \rangle$.

Proof: Let $\alpha'_1, \alpha'_2, \ldots, \alpha'_n$ be a permutation of $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\alpha'_1 \leq \alpha'_2 \leq \cdots \leq \alpha'_n$. As in the proof of previous corollary, we can find a permutation σ of $\{1, 2, 3, \ldots, n\}$ such that $\alpha'_j = \alpha_{\sigma(j)}$ for $1 \leq j \leq n$. We define $\beta'_j = \beta_{\sigma(j)}$. Then $\langle \beta'_j \rangle$ is a permutation of $\langle \alpha'_j \rangle$. Using the rearrangement theorem, we get

$$\sum_{j=1}^{n} \beta'_j \left(-\frac{1}{\alpha'_j} \right) \leq \sum_{j=1}^{n} \alpha'_j \left(-\frac{1}{\alpha'_j} \right) = -n.$$

This gives the desired inequality. The condition for equality can be derived as earlier. ■

Example 1.17. Prove the AM-GM-HM inequality using the rearrangement inequality.

Solution: Let $a_1, a_2, \ldots, a_n$ be n positive real numbers. Put

$$G = \left(a_1 a_2 \cdots a_n\right)^{1/n}, \text{ and } \alpha_k = \frac{a_1 a_2 \cdots a_k}{G^k}, \text{ for } 1 \leq k \leq n.$$

Now we set

$$\beta_1 = \alpha_2, \beta_2 = \alpha_3, \ldots, \beta_{n-1} = \alpha_n, \beta_n = \alpha_1.$$

Now using corollary 4.2, we obtain

$$n \leq \sum_{j=1}^{n} \frac{\beta_j}{\alpha_j} = \sum_{j=1}^{n-1} \frac{\alpha_{j+1}}{\alpha_j} + \frac{\alpha_1}{\alpha_n}.$$

However,

$$\frac{\alpha_{j+1}}{\alpha_j} = \frac{\big(a_1 a_2 \cdots a_{j+1}\big)/G^{j+1}}{\big(a_1 a_2 \cdots a_j\big)/G^j} = \frac{a_{j+1}}{G}, \ \frac{\alpha_1}{\alpha_n} = \frac{a_1}{G}.$$

Thus we get

$$n \le \frac{a_1 + a_2 + \cdots + a_n}{G},$$

which is same as the AM-GM inequality. Here equality holds if and only if $\langle \alpha_j \rangle = \langle \beta_j \rangle$. This is equivalent to $\alpha_1 = \alpha_2 = \cdots = \alpha_n$ which in turn is equivalent to $a_1 = a_2 = \cdots = a_n$.

Using corollary 4.2, it may also be obtained that

$$n \le \sum_{j=1}^{n} \frac{\alpha_j}{\beta_j} = \frac{G}{a_2} + \frac{G}{a_3} + \cdots + \frac{G}{a_n} + \frac{G}{a_1}.$$

This implies that

$$\frac{n}{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \dfrac{1}{a_3} + \cdots + \dfrac{1}{a_n}} \le G = \big(a_1 a_2 \cdots a_n\big)^{1/n}.$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$. ■

Example 1.18. Prove the Cauchy-Schwarz inequality using the rearrangement inequality.

Solution: Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers. We have to show that

$$\Big(\sum_{j=1}^{n} a_j b_j\Big)^2 \le \Big(\sum_{j=1}^{n} a_j^2\Big)\Big(\sum_{j=1}^{n} b_j^2\Big).$$

If either $\sum_{j=1}^{n} a_j^2 = 0$ or $\sum_{j=1}^{n} b_j^2 = 0$, the inequality is immediate. Hence, it may be assumed that

$$A = \Big(\sum_{j=1}^{n} a_j^2\Big)^{1/2} \text{ and } B = \Big(\sum_{j=1}^{n} b_j^2\Big)^{1/2}$$

are both positive. Define

$$\begin{cases} \alpha_j = \frac{a_j}{A} & \text{for } 1 \le j \le n, \\ \alpha_{n+j} = \frac{b_j}{B} & \text{for } 1 \le j \le n. \end{cases}$$

We thus obtain $2n$ numbers, $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}, \ldots, \alpha_{2n}$. Now consider the permutation

$$\beta_j = \alpha_{n+j}, \quad \beta_{n+j} = \alpha_j, \quad 1 \le j \le n.$$

Using corollary 4.1, we obtain

$$\sum_{j=1}^{n} \alpha_j \alpha_{n+j} + \sum_{j=1}^{n} \alpha_{n+j} \alpha_j \leq \sum_{j=1}^{2n} \alpha_j^2 = 2.$$

Thus it follows that

$$2\left(\frac{\sum_{j=1}^{n} a_j b_j}{AB}\right) \leq 2.$$

Hence

$$\sum_{j=1}^{n} a_j b_j \leq AB = \left(\sum_{j=1}^{n} a_j^2\right)^{1/2} \left(\sum_{j=1}^{n} b_j^2\right)^{1/2}.$$

Here equality holds if and only if $\alpha_j = \alpha_{n+j}$, for $1 \leq j \leq n$. This is equivalent to

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n} = \lambda, \text{ a constant.}$$

■

Example 1.19. Let x, y, z be positive real numbers. Prove that

$$\sum (z+x)^2 (x+y)^2 (y+z-x)(z-x) \geq 0,$$

where the sum is taken cyclically over x, y, z.

Solution: Put $z + x = 2a$, $x + y = 2b$ and $y + z = 2c$. Then a, b, c are the sides of a triangle and the equivalent inequality is

$$\sum a^2 b^2 (3c - a - b)(c - b) \geq 0. \tag{1.14}$$

We may assume that $a \leq c$ and $b \leq c$. If $a \leq b \leq c$, then we have

$$\frac{s-a}{a} \geq \frac{s-b}{b} \geq \frac{s-c}{c},$$

where $2s = a + b + c$. If $b \leq a \leq c$, then

$$\frac{s-b}{b} \geq \frac{s-a}{a} \geq \frac{s-c}{c}.$$

In any case the rearrangement inequality gives

$$a\frac{(s-a)^2}{a^2} + b\frac{(s-b)^2}{b^2} + c\frac{(s-c)^2}{c^2} \leq a\frac{(s-b)^2}{b^2} + b\frac{(s-c)^2}{c^2} + c\frac{(s-a)^2}{a^2}.$$

Simplification gives the inequality (1.14). ■

Example 1.20. Let $a_1, a_2, \ldots, a_n$ be positive real numbers and put

$$\lambda = a_1 + a_2 + \cdots + a_n.$$

Prove that

$$\frac{a_1}{\lambda - a_1} + \frac{a_2}{\lambda - a_2} + \cdots + \frac{a_n}{\lambda - a_n} \geq \frac{n}{n-1}.$$

Solution: Note that the sum on the left side is symmetric in a_j's and hence we may assume that $a_1 \le a_2 \le \cdots \le a_n$. This implies that

$$\lambda - a_1 \ge \lambda - a_2 \ge \cdots \ge \lambda - a_n,$$

and hence

$$\frac{1}{\lambda - a_1} \le \frac{1}{\lambda - a_2} \le \cdots \le \frac{1}{\lambda - a_n}.$$

For any k, consider the permutation of $a_1, a_2, \dots, a_n$ defined by

$$\beta_j = \begin{cases} a_{k+j-1} & \text{for } 1 \le j \le n-k+1, \\ a_{k+j-1-n} & \text{for } n-k+2 \le j \le n. \end{cases}$$

Using the rearrangement inequality, we obtain

$$\begin{aligned} &\frac{a_1}{\lambda - a_1} + \frac{a_2}{\lambda - a_2} + \cdots + \frac{a_n}{\lambda - a_n} \\ &\qquad \ge \frac{a_k}{\lambda - a_1} + \frac{a_{k+1}}{\lambda - a_2} + \cdots + \frac{a_n}{\lambda - a_{n+k-1}} \\ &\qquad\qquad + \frac{a_1}{\lambda - a_{n+k}} + \cdots + \frac{a_{k-1}}{\lambda - a_n}. \end{aligned}$$

This is true for every k. Now summing over k, $2 \le k \le n$, we get

$$\begin{aligned} &\Big(n-1\Big)\left(\frac{a_1}{\lambda - a_1} + \frac{a_2}{\lambda - a_2} + \cdots + \frac{a_n}{\lambda - a_n}\right) \\ &\qquad \ge \sum_{j=1}^{n} \frac{1}{\lambda - a_j}\Big(\sum_{l \ne j} a_l\Big) = \sum_{j=1}^{n} \frac{\lambda - a_j}{\lambda - a_j} = n. \end{aligned}$$

This implies that

$$\sum_{j=1}^{n} \frac{a_j}{\lambda - a_j} \ge \frac{n}{n-1}.$$

(This can also be obtained using the Cauchy-Schwarz inequality.) ■

1.6 Hölder's and Minkowski's inequalities

There is a very useful generalisation of the Cauchy-Schwarz inequality which is known as Hölder's inequality. Given any two n-tuples $\big(a_1, a_2, \dots, a_n\big)$ and $\big(b_1, b_2, \dots, b_n\big)$ of real numbers, the Cauchy-Schwarz inequality states that

$$\Big|\sum_{j=1}^{n} a_j b_j\Big| \le \Big(\sum_{j=1}^{n} a_j^2\Big)^{1/2} \Big(\sum_{j=1}^{n} b_j^2\Big)^{1/2},$$

with equality if and only if $a_j = \lambda b_j$, $1 \le j \le n$, for some constant λ. Here squares and square-roots have special significance in the context of a more general inequality.

Theorem 5. Let $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$ be two n-tuples of real numbers and let p, q be two positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. (Such a pair of indices is called a pair of conjugate indices.) Then the inequality

$$\Big|\sum_{j=1}^{n} a_j b_j\Big| \le \Big(\sum_{j=1}^{n} |a_j|^p\Big)^{1/p} \Big(\sum_{j=1}^{n} |b_j|^q\Big)^{1/q}, \tag{1.15}$$

holds. Moreover this is an equality if and only if $|a_j|^p = \lambda |b_j|^q$, $1 \le j \le n$, for some real constant λ.

Proof: We first prove an auxiliary result which is useful for the proof of the above theorem.

If x, y are two positive real numbers, and p, q are positive reals such that $\{p, q\}$ is a pair of conjugate indices, then

$$\frac{x^p}{p} + \frac{y^q}{q} \ge xy.$$

There are many ways to prove this, but each one of them depends on some continuity argument. We can use the generalised AM-GM inequality to get

$$\frac{x^p}{p} + \frac{y^q}{q} \ge \Big((x^p)^{1/p} (y^q)^{1/q}\Big) = xy.$$

Or one may consider the function

$$f(t) = \frac{t^p}{p} + \frac{t^{-q}}{q},$$

on the interval $(0, \infty)$. Then f has a unique minimum at $t = 1$ and hence

$$\frac{t^p}{p} + \frac{t^{-q}}{q} \ge \frac{1}{p} + \frac{1}{q} = 1,$$

for all $t > 0$. Taking $t = x^{1/q} y^{-1/p}$, we obtain the desired inequality. It is also easy to see that equality holds at the minimum point $t = 1$ which corresponds to $x^p = y^q$.

Now taking

$$x = \frac{|a_k|}{\Big(\sum_{j=1}^{n} |a_j|^p\Big)^{1/p}}, \quad y = \frac{|b_k|}{\Big(\sum_{j=1}^{n} |b_j|^q\Big)^{1/q}},$$

we get

$$\frac{|a_k|^p}{p\Big(\sum_{j=1}^{n} |a_j|^p\Big)} + \frac{|b_k|^q}{q\Big(\sum_{j=1}^{n} |b_j|^q\Big)} \ge \frac{|a_k||b_k|}{\Big(\sum_{j=1}^{n} |a_j|^p\Big)^{1/p} \Big(\sum_{j=1}^{n} |b_j|^q\Big)^{1/q}}.$$

Now summing over k, we obtain

$$\frac{1}{p}+\frac{1}{q} \geq \frac{\sum\limits_{j=1}^{n} |a_k b_k|}{\Big(\sum\limits_{j=1}^{n} |a_j|^p\Big)^{1/p} \Big(\sum\limits_{j=1}^{n} |b_j|^q\Big)^{1/q}}.$$

Thus we have

$$\sum_{j=1}^{n} |a_k b_k| \leq \Big(\sum_{j=1}^{n} |a_j|^p\Big)^{1/p} \Big(\sum_{j=1}^{n} |b_j|^q\Big)^{1/q}.$$

This proves (1.15). Equality holds if and only if

$$\frac{|a_k|^p}{\sum\limits_{j=1}^{n} |a_j|^p} = \frac{|b_k|^q}{\sum\limits_{j=1}^{n} |b_j|^q},$$

for $1 \leq k \leq n$. Taking

$$\lambda = \sum_{j=1}^{n} |a_j|^p \Big/ \sum_{j=1}^{n} |b_j|^q,$$

it may seen that equality holds if and only if

$$|a_k|^p = \lambda |b_k|^q, \text{ for } 1 \leq k \leq n.$$

■

Remark 5.1: If we take $p = q = 2$, Hölder's inequality reduces to the Cauchy-Schwarz inequality.

Remark 5.2: If either of p and q is negative, the inequality (1.15) gets reversed. For example, if $p < 0$ and $q > 0$ (obviously p and q both cannot be negative simultaneously), we take

$$p_1 = -\frac{p}{q}, q_1 = \frac{1}{q}.$$

We observe that $p_1 > 0$, $p_2 > 0$ and

$$\frac{1}{p_1} + \frac{1}{q_1} = -\frac{q}{p} + q = q\Big(1 - \frac{1}{p}\Big) = 1.$$

Introducing $u_k = |a_k|^{-q}$, $v_k = |a_k|^q |b_k|^q$ and applying Hölder inequality to the collections $\langle u_k \rangle$, $\langle v_k \rangle$ with the conjugate indices p_1, q_1, we obtain

$$\begin{aligned}\sum_{k=1}^{n} |a_k|^{-q} |a_k|^q |b_k|^q &\leq \Big(\sum_{k=1}^{n} |a_k|^{-qp_1}\Big)^{\frac{1}{p_1}} \Big(\sum_{k=1}^{n} \big(|a_k||b_k|\big)^{qq_1}\Big)^{\frac{1}{q_1}} \\ &= \Big(\sum_{k=1}^{n} |a_k|^p\Big)^{-\frac{q}{p}} \Big(\sum_{k=1}^{n} |a_k b_k|\Big)^{q}.\end{aligned}$$

Rearranging this, we get

$$\sum_{k=1}^{n} |a_k b_k| \geq \Big(\sum_{k=1}^{n} |a_k|^p\Big)^{1/p} \Big(\sum_{k=1}^{n} |b_k|^q\Big)^{1/q},$$

which is the opposite of the inequality (1.15).

Remark 5.3: We can have Hölder's inequality with weights. Consider two n-tuples of real numbers $(a_1, a_2, \ldots, a_n)$, $(b_1, b_2, \ldots, b_n)$; let $(w_1, w_2, \ldots, w_n)$ be an n-tuple of positive real numbers. Then the following inequality holds:

$$\sum_{k=1}^{n} w_k |a_k b_k| \leq \Big(\sum_{k=1}^{n} w_k |a_k|^p\Big)^{1/p} \Big(\sum_{k=1}^{n} w_k |b_k|^q\Big)^{1/q},$$

for a pair p, q of positive conjugate indices. If either of p and q is negative, the inequality gets reversed.

An inequality related to Hölder's inequality is Minkowski's inequality. This is a generalisation of the well known triangle inequality: if a and b are any two arbitrary real numbers, then $|a+b| \leq |a| + |b|$. It can easily be generalised to n-tuples: if $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are two n-tuples of real numbers, then

$$\sum_{j=1}^{n} |a_j + b_j| \leq \sum_{j=1}^{n} |a_j| + \sum_{j=1}^{n} |b_j|.$$

For an n-tuple $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$, we define its *Euclidean norm* $||\boldsymbol{a}||_2$ by

$$||\boldsymbol{a}||_2 = \Big(\sum_{j=1}^{n} |a_j|^2\Big)^{1/2}.$$

This is precisely the Euclidean distance of $\boldsymbol{a}$ from the origin in $\mathbb{R}^n$. It is an easy consequence of the Cauchy-Schwarz inequality that

$$\Big(\sum_{j=1}^{n} |a_j + b_j|^2\Big)^{1/2} \leq \Big(\sum_{j=1}^{n} |a_j|^2\Big)^{1/2} + \Big(\sum_{j=1}^{n} |b_j|^2\Big)^{1/2},$$

for any two n-tuples $\big(a_1, a_2, \dots, a_n\big)$ and $\big(b_1, b_2, \dots, b_n\big)$. This can be easily seen by squaring on both sides. Thus we obtain

$$||\boldsymbol{a}+\boldsymbol{b}||_2 \leq ||\boldsymbol{a}||_2 + ||\mathbf{b}||_2\,,$$

for any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^n$. As in the case of Hölder's inequality, we can replace 2 by any positive real number p, with proper inequality sign.

Theorem 6. Let $p \geq 1$ be a real number; let $\big(a_1, a_2, \dots, a_n\big)$, $\big(b_1, b_2, \dots, b_n\big)$ be two n-tuples. Then

$$\Big(\sum_{j=1}^{n} \big|a_j + b_j\big|^p\Big)^{1/p} \leq \Big(\sum_{j=1}^{n} \big|a_j\big|^p\Big)^{1/p} + \Big(\sum_{j=1}^{n} \big|b_j\big|^p\Big)^{1/p}. \tag{1.16}$$

Here equality holds if and only if $a_j = \lambda b_j$ for some constant λ, $1 \leq j \leq n$.

Proof: We may assume $p > 1$, because the result for $p = 1$ is clear. Observe that

$$\begin{aligned}
\sum_{j=1}^{n} \big|a_j + b_j\big|^p &= \sum_{j=1}^{n} \big|a_j + b_j\big|^{p-1}\big|a_j + b_j\big| \\
&\leq \sum_{j=1}^{n} \big|a_j + b_j\big|^{p-1}\big|a_j\big| + \sum_{j=1}^{n} \big|a_j + b_j\big|^{p-1}\big|b_j\big|.
\end{aligned}$$

Let q be the conjugate index of p. We use Hölder's inequality to each sum on the right hand side. Thus we have

$$\sum_{j=1}^{n} \big|a_j + b_j\big|^{p-1}\big|a_j\big| \leq \Big(\sum_{j=1}^{n} \big|a_j\big|^p\Big)^{1/p}\Big(\sum_{j=1}^{n} \big|a_j + b_j\big|^{(p-1)q}\Big)^{1/q}.$$

Since p, q are conjugate indices, we get $(p-1)q = p$. It follows that

$$\sum_{j=1}^{n} \big|a_j + b_j\big|^{p-1}\big|a_j\big| \leq \Big(\sum_{j=1}^{n} \big|a_j\big|^p\Big)^{1/p}\Big(\sum_{j=1}^{n} \big|a_j + b_j\big|^{p}\Big)^{1/q}.$$

Similarly,

$$\sum_{j=1}^{n} \big|a_j + b_j\big|^{p-1}\big|b_j\big| \leq \Big(\sum_{j=1}^{n} \big|b_j\big|^p\Big)^{1/p}\Big(\sum_{j=1}^{n} \big|a_j + b_j\big|^{p}\Big)^{1/q}.$$

It now follows that

$$\sum_{j=1}^{n} \big|a_j + b_j\big|^p \leq \bigg\{\Big(\sum_{j=1}^{n} \big|a_j\big|^p\Big)^{1/p} + \Big(\sum_{j=1}^{n} \big|b_j\big|^p\Big)^{1/p}\bigg\}\Big(\sum_{j=1}^{n} \big|a_j + b_j\big|^{p}\Big)^{1/q}.$$

If we use $1-(1/q)=1/p$, we finally get

$$\left(\sum_{j=1}^{n}\left|a_j+b_j\right|^p\right)^{1/p}\leq\left(\sum_{j=1}^{n}\left|a_j\right|^p\right)^{1/p}+\left(\sum_{j=1}^{n}\left|b_j\right|^p\right)^{1/p}.$$

Using the conditions for equality in Hölder's inequality, we may obtain the conditions for equality in Minkowski's inequality: $a_j=\lambda b_j$, for all $1\leq j\leq n$, where λ is a real constant. ∎

Remark 6.1: For $0<p<1$, the inequality (1.16) gets reversed.

Remark 6.2: For any vector $\boldsymbol{a}=\left(a_1,a_2,\ldots,a_n\right)$ and $p>0$, we define

$$||\boldsymbol{a}||_p=\left(\sum_{j=1}^{n}|a_j|^p\right)^{1/p}.$$

Thus Minkowski's inequality can be put in the form

$$||\boldsymbol{a}+\boldsymbol{b}||_p\leq||\boldsymbol{a}||_p+||\boldsymbol{b}||_p\,,$$

for $p\geq 1$ and

$$||\boldsymbol{a}+\boldsymbol{b}||_p\geq||\boldsymbol{a}||_p+||\boldsymbol{b}||_p\,,$$

for $0<p<1$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are two vectors in $\mathbb{R}^n$.

Example 1.21. Let $a_1,a_2,\ldots,a_n$ be real numbers. Prove that

$$\left|\sum_{k=1}^{n}\frac{a_k}{k^{1/3}}\right|^3\leq\left(\sum_{k=1}^{n}|a_k|^{3/2}\right)^2\left(\sum_{k=1}^{n}\frac{1}{k}\right).$$

Solution: We take $p=3/2$ and $q=3$. Then p,q are conjugate indices and using Hölder's inequality we obtain,

$$\left|\sum_{k=1}^{n}\frac{a_k}{k^{1/3}}\right|\leq\left(\sum_{k=1}^{n}|a_k|^{3/2}\right)^{2/3}\left(\sum_{k=1}^{n}\frac{1}{k}\right)^{1/3}.$$

This gives the required inequality. ∎

Example 1.22. Let p,q be two positive real numbers such that $\dfrac{1}{p}+\dfrac{1}{q}=1$. Suppose $\left(a_1,a_2,\ldots,a_n\right)$ is an n-tuple of real numbers. Show that

$$\sum_{k=1}^{n}a_kb_k\leq C, \tag{1.17}$$

for every n-tuple $(b_1, b_2, \dots, b_n)$ of real numbers with $\sum_{k=1}^{n} |b_k|^q = 1$ if and only if

$$\Big(\sum_{k=1}^{n} |a_k|^p\Big)^{1/p} \leq C. \tag{1.18}$$

Solution: Suppose (1.17) holds for all n-tuples $(b_1, b_2, \dots, b_n)$ such that $\sum_{k=1}^{n} |b_k|^q = 1$. Consider the n-tuple defined by

$$b_k = \frac{|a_k|^{p-1}\ \mathrm{Sgn}(a_k)}{\Big(\sum_{k=1}^{n} |a_k|^p\Big)^{1/q}},$$

where

$$\mathrm{Sgn}(x) = \begin{cases} \frac{a_k}{|a_k|} & \text{if } a_k \neq 0, \\ 0 & \text{if } a_k = 0. \end{cases}$$

For this n-tuple, we have

$$\sum_{k=1}^{n} |b_k|^q = \frac{\sum_{k=1}^{n} |a_k|^{(p-1)q}}{\sum_{k=1}^{n} |a_k|^p} = \frac{\sum_{k=1}^{n} |a_k|^p}{\sum_{k=1}^{n} |a_k|^p} = 1.$$

Now the condition (1.17) gives

$$C \geq \sum_{k=1}^{n} a_k \left(\frac{|a_k|^{p-1}\ \mathrm{Sgn}(a_k)}{\Big(\sum_{k=1}^{n} |a_k|^p\Big)^{1/q}} \right) = \frac{\sum_{k=1}^{n} |a_k|^p}{\Big(\sum_{k=1}^{n} |a_k|^p\Big)^{1/q}} = \Big(\sum_{k=1}^{n} |a_k|^p\Big)^{1/p}.$$

This shows that (1.18) holds.

Conversely, suppose (1.18) holds and $(b_1, b_2, \dots, b_n)$ an n-tuple satisfying $\sum_{k=1}^{n} |b_k|^q = 1$. Then Hölder's inequality gives

$$\sum_{k=1}^{n} a_k b_k \leq \Big(\sum_{k=1}^{n} |a_k|^p\Big)^{1/p} \Big(\sum_{k=1}^{n} |b_k|^q\Big)^{1/q} \leq C.$$

Thus (1.17) holds. ■

Example 1.23. Let $\{a_1, a_2, \dots, a_n\}$, $\{b_1, b_2, \dots, b_n\}$, and $\{c_1, c_2, \dots, c_n\}$ be three sets, where a_j's, b_j's, c_j's are positive real numbers. If p, q, r are positive reals such that $(1/p) + (1/q) + (1/r) = 1$, then

$$\sum_{j=1}^{n} a_j b_j c_j \leq \left(\sum_{j=1}^{n} a_j^p\right)^{1/p} \left(\sum_{j=1}^{n} b_j^q\right)^{1/q} \left(\sum_{j=1}^{n} c_j^r\right)^{1/r}.$$

Solution: Using Hölder's inequality with exponents p and $(p-1)/p$, we get

$$\sum_{j=1}^{n} a_j b_j c_j \le \left(\sum_{j=1}^{n} a_j^p\right)^{1/p} \left(\sum_{j=1}^{n} (b_j c_j)^{p/(p-1)}\right)^{(p-1)/p}.$$

Now consider the exponents $(p-1)q/p$ and $(p-1)r/p$. We observe that

$$\frac{p}{(p-1)q} + \frac{p}{(p-1)r} = \frac{p}{p-1}\left(\frac{1}{q} + \frac{1}{r}\right) = 1,$$

because of $(1/p) + (1/q) + (1/r) = 1$. Thus Hölder's inequality with these exponents can be used to get

$$\left(\sum_{j=1}^{n} (b_j c_j)^{p/(p-1)}\right)^{(p-1)/p} \le \left(\sum_{j=1}^{n} b_j^q\right)^{1/q} \left(\sum_{j=1}^{n} c_j^r\right)^{1/r}.$$

Remark: This can be generalised to any number of sets with proper exponents. ■

1.7 Convex and Concave functions, Jensen's inequality

Most of the inequalities, we have studied so far, are in fact consequences of inequalities for a special class of functions, known as convex and concave functions. Consider the function $f(x) = x^2$ defined on $\mathbb{R}$. If we take any two points on the graph of $f(x) = x^2$, then the chord joining these points always lies above this graph. In fact taking $a < b$, and the point $\lambda a + (1-\lambda)b$ between a and b, we see that

$$\big(\lambda a + (1-\lambda)b\big)^2 - \lambda a^2 - (1-\lambda)b^2 = -\lambda(1-\lambda)(a-b)^2 \le 0.$$

Thus we see that

$$f\big(\lambda a + (1-\lambda)b\big) \le \lambda f(a) + (1-\lambda)f(b),$$

for $f(x) = x^2$. This is taken as defining property of a convex function. We shall see that the class of convex functions obey a general inequality known as Jensen's inequality.

Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is said to be convex if for all x, y in I and λ in the interval $[0,1]$, the inequality

$$f\big(\lambda x + (1-\lambda)y\big) \le \lambda f(x) + (1-\lambda)f(y), \tag{1.19}$$

holds. If the inequality is strict for all $x \ne y$, we say that f is strictly convex on I. If the inequality in (1.19) is reversed, i.e.,

$$f\big(\lambda x + (1-\lambda)y\big) \ge \lambda f(x) + (1-\lambda)f(y),$$

for all $x, y \in I$ and $\lambda \in [0,1]$, then we say f is concave. If the inequality is strict for all $x \neq y$, we say that f is strictly concave.

There are other equivalent characterisations of a convex function. Suppose $f : I \to \mathbb{R}$ is convex and x_1, x_2, x_3 are in I such that $x_1 < x_2 < x_3$. Take

$$\lambda = \frac{x_3 - x_2}{x_3 - x_1},$$

so that

$$1 - \lambda = \frac{x_2 - x_1}{x_3 - x_1}, \text{ and } x_2 = \lambda x_1 + (1-\lambda)x_3.$$

We have

$$\begin{aligned} f\big(x_2\big) &= f\big(\lambda x_1 + (1-\lambda)x_3\big) \\ &\leq \lambda f\big(x_1\big) + (1-\lambda) f\big(x_3\big) \\ &= \frac{x_3 - x_2}{x_3 - x_1} f\big(x_1\big) + \frac{x_2 - x_1}{x_3 - x_1} f\big(x_3\big). \end{aligned}$$

We may write this in the form

$$\frac{f\big(x_1\big) - f\big(x_2\big)}{x_1 - x_2} \leq \frac{f\big(x_2\big) - f\big(x_3\big)}{x_2 - x_3},$$

for all $x_1 < x_2 < x_3$ in I. This may also be written in a more symmetric form:

$$\frac{f\big(x_1\big)}{\big(x_1 - x_2\big)\big(x_1 - x_3\big)} + \frac{f\big(x_2\big)}{\big(x_2 - x_1\big)\big(x_2 - x_3\big)} + \frac{f\big(x_3\big)}{\big(x_3 - x_1\big)\big(x_3 - x_2\big)} \geq 0.$$

The convexity of a function has an intrinsic geometric interpretation. If we look at the graph of a convex function, this is a subset of the plane and the line joining any two points on the graph always lies above the graph. Suppose $\boldsymbol{z}_1 = (a, f(a))$ and $\boldsymbol{z}_2 = (b, f(b))$ are two points on the graph of f. Then the equation of the line joining these two points may be written in the form

$$L(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

Now any point between a and b is of the form $x = \lambda a + (1-\lambda)b$ for some $\lambda \in [0,1]$. Hence

$$\begin{aligned} L(x) &= L\big(\lambda a + (1-\lambda)b\big) \\ &= f(a) + \frac{f(b) - f(a)}{b - a}\big(\lambda a + (1-\lambda)b - a\big) \\ &= f(a) + \big(1-\lambda\big)\big(f(b) - f(a)\big) \\ &= \lambda f(a) + (1-\lambda) f(b) \\ &\geq f\big(\lambda a + (1-\lambda)b\big) = f(x). \end{aligned}$$

Thus $\big(x, L(x)\big)$ lies above $(x, f(x))$, a point on the graph of f.

There is still another way of looking at these things. A subset E of the plane $\mathbb{R}^2$ is said to be convex if for every pair of points $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ in E, the line joining $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ lies entirely in E. With every function $f : I \to \mathbb{R}$, we associate a subset of $\mathbb{R}^2$ by

$$E(f) = \big\{(x, y) \,:\, a \le x \le b, f(x) \le y\big\}.$$

Theorem 7. The function $: I \to \mathbb{R}$ is convex if and only if $E(f)$ is a convex subset of $\mathbb{R}^2$.

Proof: Suppose f is convex. Let $\boldsymbol{z}_1 = (x_1, y_1)$ and $\boldsymbol{z}_2 = (x_2, y_2)$ be two points of $E(f)$. Consider any point on the line joining $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$. It is of the form

$$\begin{aligned} \boldsymbol{z} &= \lambda \boldsymbol{z}_1 + (1-\lambda)\boldsymbol{z}_2 \\ &= \big(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2\big), \end{aligned}$$

for some $\lambda \in [0, 1]$. Observe that $a \le \lambda x_1 + (1-\lambda)x_2 \le b$. Moreover,

$$\begin{aligned} f\big(\lambda x_1 + (1-\lambda)x_2\big) &\le \lambda f\big(x_1\big) + \big(1-\lambda\big) f\big(x_2\big) \\ &\le \lambda y_1 + (1-\lambda)y_2. \end{aligned}$$

Hence it follows that $\boldsymbol{z} \in E(f)$, proving that $E(f)$ is convex.

Conversely, suppose $E(f)$ is convex. Let x_1, x_2 be two points in I and let $\boldsymbol{z}_1 = \big(x_1, f(x_1)\big)$, $\boldsymbol{z}_2 = \big(x_2, f(x_2)\big)$. Then $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ are in $E(f)$. By convexity of $E(f)$, the point $\lambda \boldsymbol{z}_1 + (1-\lambda)\boldsymbol{z}_2$ also lies in $E(f)$ for each $\lambda \in [0, 1]$. Thus

$$\big(\lambda x_1 + (1-\lambda)x_2, \lambda f(x_1) + (1-\lambda) f(x_2)\big) \in E(f),$$

for all $\lambda \in [0, 1]$. The definition of $E(f)$ shows that

$$f\big(\lambda x_1 + (1-\lambda)x_2\big) \le \lambda f(x_1) + (1-\lambda) f(x_2),$$

for all $\lambda \in [0, 1]$. This shows that f is convex on the interval I. ■

The convexity of a function implies something about the slope of its graph. The following theorem gives a complete description.

Theorem 8. Let $f : I \to \mathbb{R}$ be a convex function and $a \in I$ be a fixed point. Define a function $P : I \setminus \{a\} \to \mathbb{R}$ by

$$P(x) = \frac{f(x) - f(a)}{x - a}.$$

Then P is a non-decreasing function on $I \setminus \{a\}$.

Proof: Suppose f is convex on I and let x, y be two points in I, $x \neq a$, $x \neq b$ such that $x < y$. Then exactly one of following three possibilities hold:

$$a < x < y; \quad x < a < y; \quad \text{or } x < y < a.$$

Consider the case $a < x < y$; other cases may be taken care of similarly. We write

$$x = \frac{x-a}{y-a}y + \frac{y-x}{y-a}a.$$

The convexity of f shows that

$$f\left(\frac{x-a}{y-a}y + \frac{y-x}{y-a}a\right) \leq \frac{x-a}{y-a}f(y) + \frac{y-x}{y-a}f(a).$$

This is equivalent to

$$\frac{f(x)-f(a)}{x-a} \leq \frac{f(y)-f(a)}{y-a}.$$

Thus $P(x) \leq P(y)$. This shows that $P(x)$ is a non-decreasing function for $x \neq a$. ■

Interestingly, the converse is also true; if $P(x)$ is a non-decreasing function on $I \setminus \{a\}$ for every $a \in I$, then $f(x)$ is convex. The proof is not difficult. Fix $x < y$ in I and let $a = \lambda x + (1-\lambda)y$ where $\lambda \in (0,1)$. (The cases $\lambda = 0$ or 1 are obvious.) In this case

$$\begin{aligned} P(x) &= \frac{f(x)-f(a)}{x-a} &= \frac{f(x)-f(a)}{(1-\lambda)(x-y)} \\ P(y) &= \frac{f(y)-f(a)}{y-a} &= \frac{f(y)-f(a)}{\lambda(y-x)}. \end{aligned}$$

The condition $P(x) \leq P(y)$ implies that $f(a) \leq \lambda f(x) + (1-\lambda)f(y)$. Hence the convexity of f follows.

There is a useful, easy way of deciding whether a function is convex or concave for twice differentiable functions. If f is convex on an interval I and if its second derivative exists on I, then f is convex(strictly convex) on I if $f''(x) \geq 0 (> 0)$ for all $x \in I$. (Here $f''(x)$ denotes the second derivative of f at x.) Similarly f is concave(strictly concave) on I if $f''(x) \leq 0 (< 0)$ for all $x \in I$.

Here are some examples of convex and concave functions.

1. The function $f(x) = x^\alpha$ is concave for $0 < \alpha \leq 1$ and convex for $1 \leq \alpha < \infty$ on the interval $(0, \infty)$. We observe that

$$f''(x) = \alpha(\alpha-1)x^{\alpha-2}.$$

Hence $f''(x) \geq 0$ on $(0, \infty)$ if $\alpha \geq 1$ and $f''(x) \leq 0$ for $0 < \alpha \leq 1$.

2. Consider $f(x) = e^x$. This is convex on $\mathbb{R} = (-\infty, \infty)$. Here $f''(x) = e^x > 0$ for all x in $(-\infty, \infty)$ and hence e^x is strictly convex.

3. The inverse of exponential function, namely the logarithmic function $f(x) = \ln x$ is concave on $(0, \infty)$. In this case $f''(x) = -1/x^2$ which is negative on $(0, \infty)$. Thus $f(x) = \ln x$ is strictly concave on $(0, \infty)$.

4. The function $f(x) = \sin x$ is concave on $(0, \pi)$. We have $f''(x) = -\sin x$ and hence $f''(x) < 0$ on $(0, \pi)$.

5. If a function f is convex on I, then $-f$ is concave on I.

The definition of a convex function involves inequality for two points x, y; refer to (1.19). But this can be extended to any finite number of points.

Theorem 9. (Jensen's inequality) Let $f : I \to \mathbb{R}$ be a convex function. Suppose $x_1, x_2, \ldots, x_n$ are points in I and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real numbers in the interval $[0, 1]$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$. Then

$$f\Big(\sum_{j=1}^{n} \lambda_j x_j\Big) \le \sum_{j=1}^{n} \lambda_j f(x_j). \tag{1.20}$$

Proof: We use the induction on n. For $n = 2$, this is precisely the definition of a convex function. Suppose the inequality (1.20) is true for all $k < n$; i.e., for $k < n$, if $x_1, x_2, \ldots, x_k$ are k points in I and $\lambda_1, \lambda_2, \ldots, \lambda_k$ are real numbers in $[0, 1]$ such that $\sum_{j=1}^{k} \lambda_j = 1$, then

$$f\Big(\sum_{j=1}^{k} \lambda_j x_j\Big) \le \sum_{j=1}^{k} \lambda_j f(x_j).$$

Now consider any n points $x_1, x_2, \ldots, x_n$ in the interval I and real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ in the interval $[0, 1]$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$. Let us put

$$y_1 = \frac{\sum_{j=1}^{n-1} \lambda_j x_j}{\sum_{j=1}^{n-1} \lambda_j}, \quad y_2 = x_n, \quad \alpha_1 = \sum_{j=1}^{n-1} \lambda_j, \quad \alpha_2 = \lambda_n.$$

Observe that $\alpha_2 = 1 - \alpha_1$, and y_1, y_2 are in I. Using the convexity of f, we get

$$\begin{aligned} f(\alpha_1 y_1 + \alpha_2 y_2) &= f(\alpha_1 y_1 + (1 - \alpha_1) y_2) \\ &\le \alpha_1 f(y_1) + (1 - \alpha_1) f(y_2) \\ &= \alpha_1 f(y_1) + \alpha_2 f(y_2). \end{aligned}$$

However, we have

$$\alpha_1 y_1 + \alpha_2 y_2 = \sum_{j=1}^{n} \lambda_j x_j.$$

Now consider $f(y_1)$. If

$$\mu_l = \frac{\lambda_l}{\sum_{j=1}^{n-1}\lambda_j}, \quad 1 \le l \le n-1$$

then it can be easily verified that $\sum_{l=1}^{n-1}\mu_l = 1$. Using the induction hypothesis, we get

$$f\Big(\sum_{l=1}^{n-1}\mu_l x_l\Big) \le \sum_{l=1}^{n-1}\mu_l f(x_l).$$

Since

$$\sum_{l=1}^{n-1}\mu_l x_l = y_1,$$

we get

$$f(y_1) \le \frac{\sum_{l=1}^{n-1}\lambda_l f(x_l)}{\sum_{j=1}^{n-1}\lambda_j} = \frac{\sum_{j=1}^{n-1}\lambda_j f(x_j)}{\alpha_1}.$$

Thus we obtain

$$\begin{aligned} f\Big(\sum_{j=1}^{n}\lambda_j f(x_j)\Big) &\le \alpha_1\left(\frac{\sum_{j=1}^{n-1}\lambda_j f(x_j)}{\sum_{j=1}^{n-1}\lambda_j}\right) + \lambda_n f(x_n) \\ &= \sum_{j=1}^{n}\lambda_j f(x_j). \end{aligned}$$

This completes the proof by the induction. ■

Remark 9.1: If $f : I \to \mathbb{R}$ is concave, then the inequality (1.20) gets reversed. If $x_1, x_2, \ldots, x_n$ are points in I and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real numbers in the interval $[0,1]$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$, then we have the inequality

$$f\Big(\sum_{j=1}^{n}\lambda_j x_j\Big) \ge \sum_{j=1}^{n}\lambda_j f(x_j). \tag{1.21}$$

Remark 9.2: Using the concavity of $f(x) = \ln x$ on $(0,\infty)$, the AM-GM inequality may be proved. If $x_1, x_2, \ldots, x_n$ are points in $(0,\infty)$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real numbers in the interval $[0,1]$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$, then we have

$$\ln\Big(\sum_{j=1}^{n}\lambda_j x_j\Big) \ge \sum_{j=1}^{n}\lambda_j \ln(x_j).$$

Taking $\lambda_j = 1/n$ for all j,

$$\ln\Big(\sum_{j=1}^{n}\frac{x_j}{n}\Big) \ge \frac{1}{n}\sum_{j=1}^{n}\ln x_j = \sum_{j=1}^{n}\ln\big(x_j^{1/n}\big).$$

Using the fact that $g(x) = \exp(x)(= e^x)$ is strictly increasing on the interval $(-\infty, \infty)$, this leads to

$$\begin{aligned} \frac{1}{n}\sum_{j=1}^{n} x_j &\geq \exp\Big(\sum_{j=1}^{n} \ln\left(x_j^{1/n}\right)\Big) \\ &= \prod_{j=1}^{n} \exp\Big(\ln\left(x_j^{1/n}\right)\Big) \\ &= \Big(x_1 x_2 \cdots x_n\Big)^{1/n}. \end{aligned}$$

In fact, the generalised AM-GM inequality may also be proved by this method . We have

$$\ln\Big(\sum_{j=1}^{n} \lambda_j x_j\Big) \geq \sum_{j=1}^{n} \lambda_j \ln\left(x_j\right) = \sum_{j=1}^{n} \ln x_j^{\lambda_j},$$

for all points $x_1, x_2, \dots, x_n$ in $(0, \infty)$ and $\lambda_1, \lambda_2, \dots, \lambda_n$ in the interval $[0, 1]$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$. Using the property of exponential function, we obtain

$$\sum_{j=1}^{n} \lambda_j x_j \geq \prod_{j=1}^{n} x_j^{\lambda_j}.$$

Now for any n positive real numbers $\alpha_1, \alpha_2, \dots, \alpha_n$, consider

$$\lambda_j = \frac{\alpha_j}{\sum_{k=1}^{n} \alpha_k}.$$

Observe that λ_j are in $[0, 1]$ and $\sum_{j=1}^{n} \lambda_j = 1$. These choices of λ_j give

$$\frac{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n}{\alpha_1 + \alpha_2 + \cdots + \alpha_n} \geq \Big(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}\Big)^{1/(\alpha_1+\alpha_2+\cdots+\alpha_n)},$$

which is the generalised AM-GM inequality.

Remark 9.3: We use the function $f(x) = x^p$ to derive Hölder's inequality. We know that $f(x) = x^p$ is convex for $p \geq 1$ and concave for $0 < p < 1$ on the interval $(0, \infty)$. Hence for any n-tuple $\big(x_1, x_2, \dots, x_n\big)$ of real numbers and $\lambda_1, \lambda_2, \dots, \lambda_n$ in $[0, 1]$, we have

$$\Big(\sum_{j=1}^{n} \lambda_j x_j\Big)^p \leq \sum_{j=1}^{n} \lambda_j x_j^p \text{ for } p \geq 1,$$

and

$$\Big(\sum_{j=1}^{n} \lambda_j x_j\Big)^p \geq \sum_{j=1}^{n} \lambda_j x_j^p \text{ for } 0 < p < 1.$$

Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be two n-tuples of real numbers and $p > 1$ be a given real number. Let q be the conjugate of p; $\dfrac{1}{p} + \dfrac{1}{q} = 1$. We may assume $b_j \neq 0$ for all j; otherwise we may delete all those b_j which are *zero* without affecting the inequality. We now set

$$t = \sum_{j=1}^{n} |b_j|^q, \quad \lambda_k = \frac{|b_k|^q}{t}, \quad x_k = \frac{|a_k|}{|b_k|^{q-1}}.$$

We observe that λ_k are in $[0,1]$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$. Using the convexity of x^p, we get

$$\left(\sum_{j=1}^{n} \lambda_j x_j\right)^p \leq \sum_{j=1}^{n} \lambda_j x_j^p.$$

This implies that

$$\left(\sum_{k=1}^{n} \frac{|b_k|^q}{t} \frac{|a_k|}{|b_k|^{q-1}}\right)^p \leq \sum_{k=1}^{n} \frac{|b_k|^q}{t} \frac{|a_k|^p}{|b_k|^{(q-1)p}} = \frac{1}{t} \sum_{k=1}^{n} |a_k|^p,$$

since $(q-1)p = q$. Simplification gives

$$\sum_{k=1}^{n} |b_k a_k| \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} t^{1-(1/p)} = \left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |b_k|^q\right)^{1/q}.$$

If $0 < p < 1$, then $f(x) = x^p$ is concave and the inequality is reversed:

$$\sum_{k=1}^{n} |b_k a_k| \geq \left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |b_k|^q\right)^{1/q}.$$

We observe that for $0 < p < 1$, its conjugate index q is negative.

There is an interesting generalisation of the rearrangement inequality to convex functions.

Theorem 10. Suppose $f : I \to \mathbb{R}$ is a convex function; $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ are two sequences of real numbers in I such that $a_1 + b_1 \in I$ and $a_n + b_n \in I$. Let $a_1', a_2' \ldots, a_n'$ be a permutation of $a_1, a_2, \ldots, a_n$. Then the inequality

$$\sum_{j=1}^{n} f\big(a_j + b_{n+1-j}\big) \leq \sum_{j=1}^{n} f\big(a_j' + b_j\big) \leq \sum_{j=1}^{n} f\big(a_j + b_j\big).$$

holds.

Proof: We follow the proof of theorem 4. Suppose $\langle a_j' \rangle \neq \langle a_j \rangle$ and r be the largest index such that $a_r' \neq a_r$; i.e., $a_r' \neq a_r$ but $a_j' = a_j$ for $r < j \leq n$. Hence a_r' is in the set $\{a_1, a_2, \ldots, a_{r-1}\}$ and $a_r' < a_r$. Since $a_j = a_j'$ for $r < j \leq n$, we see that $(a_1', a_2' \ldots, a_r')$ is a permutation of $(a_1, a_2, \ldots, a_r)$. Hence we can find $k < r$, $l < r$ such that $a_k' = a_r$ and $a_r' = a_l$. We deduce that $a_k' - a_r' = a_r - a_l \geq 0$ and $b_r - b_k \geq 0$. We now interchange a_r' and a_k' to get a permutation $(a_1'', a_2'', \ldots, a_n'')$ of $(a_1', a_2' \ldots, a_n')$; thus

$$a_j'' = a_j', \text{ for } j \neq r, k, \quad a_r'' = a_k' = a_r, \quad a_k'' = a_r' = a_l.$$

Let us write

$$S'' = \sum_{j=1}^{n} f(a_j'' + b_j), \quad S' = \sum_{j=1}^{n} f(a_j' + b_j).$$

We have

$$\begin{aligned} S'' - S' &= f(a_r'' + b_r) + f(a_k'' + b_k) - f(a_r' + b_r) - f(a_k' + b_k) \\ &= f(a_r + b_r) + f(a_l + b_k) - f(a_l + b_r) - f(a_r + b_k). \end{aligned}$$

We observe that

$$a_l + b_k < a_r + b_k \text{ and } a_l + b_r < a_r + b_r.$$

These give

$$\begin{aligned} a_l + b_k < a_r + b_k \leq a_r + b_r, \\ a_l + b_k \leq a_l + b_r < a_r + b_r. \end{aligned}$$

If x_1, x_2, x_3 are in I, then the convexity of f implies that

$$(x_3 - x_1) f(x_2) \leq (x_3 - x_2) f(x_1) + (x_2 - x_1) f(x_3).$$

Taking $x_1 = a_l + b_k$, $x_2 = a_r + b_k$ and $x_3 = a_r + b_r$, we get

$$(a_r + b_r - a_l - b_k) f(a_r + b_k) \leq (b_r - b_k) f(a_l + b_k) + (a_r - a_l) f(a_r + b_r).$$

Similarly, taking $x_1 = a_l + b_k$, $x_2 = a_l + b_r$ and $x_3 = a_r + b_r$, we get

$$(a_r + b_r - a_l - b_k) f(a_l + b_r) \leq (a_r - a_l) f(a_l + b_k) + (b_r - b_k) f(a_r + b_r).$$

Adding these two, we obtain

$$\begin{aligned} &(a_r + b_r - a_l - b_k)\{f(a_r + b_k) + f(a_l + b_r)\} \\ &\qquad \leq (a_r + b_r - a_l - b_k)\{f(a_l + b_k) + f(a_r + b_r)\}. \end{aligned}$$

Since $a_l + b_k < a_r + b_r$, we deduce that

$$f(a_r + b_k) + f(a_l + b_r) \leq f(a_l + b_k) + f(a_r + b_r).$$

This proves that $S'' - S' \geq 0$.

Now we observe that the permutation $(a_1'', a_2'', \dots, a_n'')$ has the property $a_r'' = a_r$ and $a_j'' = a_j$, for $r < j \leq n$. We may consider the $(a_1'', a_2'', \dots, a_n'')$ in place $(a_1', a_2', \dots, a_n')$ and proceed as above. After at most $n-1$ steps we arrive at the original numbers $\langle a_j \rangle$ from $\langle a_j' \rangle$, and at each stage the corresponding sum is non-decreasing. We finally get

$$\sum_{j=1}^{n} f\big(a_j' + b_j\big) \leq \sum_{j=1}^{n} f\big(a_j + b_j\big).$$

For proving the other inequality, we proceed along the same lines. We define $c_j = a_{n+1-j}$ so that $c_1 \geq c_2 \geq \cdots \geq c_n$. We have to show that

$$\sum_{j=1}^{n} f\big(a_{n+1-j} + b_j\big) \leq \sum_{j=1}^{n} f\big(a_j' + b_j\big).$$

We also set $c_j' = a_j'$. Thus the inequality becomes

$$\sum_{j=1}^{n} f\big(c_j + b_j\big) \leq \sum_{j=1}^{n} f\big(c_j' + b_j\big),$$

where $(c_1', c_2', \dots, c_n')$ is a permutation of $(c_1, c_2, \dots, c_n)$. We assume that $\langle c_j' \rangle \neq \langle c_j \rangle$ and let r be the smallest index such that $c_r' \neq c_r$; i.e., $c_r' \neq c_r$ but $c_j' = c_j$ for $1 \leq j < r$. This forces that $c_r' \in \big\{c_{r+1}, c_{r+2}, \dots, c_n\big\}$ and $c_r' < c_r$. Observe that $(c_r', c_{r+1}', \dots, c_n')$ is a permutation of $(c_r, c_{r+1}, \dots, c_n)$. Hence we can find $k > r$, $l > r$ such that $c_k' = c_r$ and $c_r' = c_l$. This implies that $c_k' - c_r' = c_r - c_l \geq 0$ and $b_k - b_r \geq 0$. Now we interchange c_r' and c_k' to get a permutation $(c_1'', c_2'', \dots, c_n'')$ of $(c_1', c_2', \dots, c_n')$; thus

$$c_j'' = c_j', \text{ for } j \neq r, k, \quad c_r'' = c_k' = c_r, \quad c_k'' = c_r' = c_l.$$

Now we compute the difference between

$$S'' = \sum_{j=1}^{n} f\big(c_j'' + b_j\big), \quad S' = \sum_{j=1}^{n} f\big(c_j' + b_j\big),$$

and obtain

$$\begin{aligned} S'' - S' &= f\big(c_r'' + b_r\big) + f\big(c_k'' + b_k\big) - f\big(c_r' + b_r\big) - f\big(c_k' + b_k\big) \\ &= f\big(c_r + b_r\big) + f\big(c_l + b_k\big) - f\big(c_l + b_r\big) - f\big(c_r + b_k\big). \end{aligned}$$

We observe that

$$\begin{aligned} c_l + b_r &\leq c_l + b_k < c_r + b_k, \\ c_l + b_r &\leq c_r + b_r < c_r + b_k. \end{aligned}$$

The convexity of f gives

$$\big(c_r + b_k - c_l - b_r\big) f\big(c_l + b_k\big) \le \big(c_r - c_l\big) f\big(c_l + b_r\big) + \big(b_k - b_r\big) f\big(c_r + b_k\big),$$

and

$$\big(c_r + b_k - c_l - b_r\big) f\big(c_r + b_r\big) \le \big(b_k - b_r\big) f\big(c_l + b_r\big) + \big(c_r - c_l\big) f\big(c_r + b_k\big).$$

Addition of these two gives

$$\begin{aligned}\big(c_r + b_k - c_l - b_r\big)&\big\{f\big(c_l + b_k\big) + f\big(c_r + b_r\big)\big\}\\ &\le \big(c_r + b_k - c_l - b_r\big)\big\{f\big(c_l + b_r\big) + f\big(c_r + b_k\big)\big\}.\end{aligned}$$

Since $c_r + b_k - c_l - b_r \neq 0$, we get

$$f\big(c_l + b_k\big) + f\big(c_r + b_r\big) \le f\big(c_l + b_r\big) + f\big(c_r + b_k\big).$$

This shows that $S'' \le S'$. We also observe that the new sequence $\langle c''_j \rangle$ has the property: $c''_r = c_r$ and $c''_j = c_j$ for $1 \le j < r$. We may consider now the sequence $\langle c''_j \rangle$ in place of $\langle c'_j \rangle$ and continue the above argument. At each stage the sum never increases. After at most $n-1$ steps we arrive at the sequence $\langle c_j \rangle$. Hence the corresponding sum cannot exceed that of $S' = \sum_{j=1}^{n} f\big(c'_j + b_j\big)$. We thus get

$$\sum_{j=1}^{n} f\big(c_j + b_j\big) \le \sum_{j=1}^{n} f\big(c'_j + b_j\big),$$

which is to be proved. ■

Example 1.24. Show that for all $n \ge 1$, the inequality

$$\prod_{j=1}^{n} j^j \ge \Big(\frac{n+1}{2}\Big)^{n(n+1)/2}$$

holds.

Solution: Let us put $a_j = b_j = j$, $a'_j = n + 1 - j$ for $1 \le j \le n$. Then we see that $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$. Moreover $a'_1 \ge a'_2 \ge \cdots \ge a'_n$. Consider the function $f(x) = x \ln x$. This is a convex function on $(0, \infty)$. In fact we see that

$$f'(x) = \ln x + 1, \text{ and } f''(x) = \frac{1}{x} > 0 \text{ for } x > 0.$$

Now the rearrangement inequality applied to $f(x)$ gives

$$\sum_{j=1}^{n} (n + 1 - j + j) \ln(n + 1 - j + j) \le \sum_{j=1}^{n} (j + j) \ln(j + j).$$

This simplifies to

$$n(n+1)\ln(n+1) \le 2\sum_{j=1}^{n} j\ln j + n(n+1)\ln 2.$$

This further reduces to

$$\frac{n(n+1)}{2}\ln\left(\frac{n+1}{2}\right) \le \sum_{j=1}^{n} j\ln j = \ln\Big(\prod_{j=1}^{n} j^j\Big).$$

Now taking exponentiation both sides we get the desired result. ■

The concavity of the log function on $(0,\infty)$ also helps us to derive a very interesting inequality known as Bernoulli's inequality.

Example 1.25. **(Bernoulli's inequality)** Prove that for $x > -1$

$$\big(1+x\big)^a \le 1 + ax, \text{ if } 0 < a < 1,$$

and

$$\big(1+x\big)^a \ge 1 + ax \text{ if } 1 \le a < \infty.$$

Solution: We use the concavity of $f(x) = \ln(x)$ on $(0,\infty)$. Since $x > -1$, we see that $1 + x > 0$. If $0 < a < 1$, we have

$$\begin{aligned} \ln\big(1+ax\big) &= \ln\big(a(1+x)+1-a\big) \\ &\ge a\ln\big(1+x\big) + \big(1-a\big)\ln(1) \\ &= a\ln\big(1+x\big). \end{aligned}$$

The exponentiation gives $\big(1+x\big)^a \le 1+ax$. Suppose $1 \le a < \infty$. Then again

$$\begin{aligned} \ln\big(1+x\big) &= \ln\left(\frac{a-1}{a} + \frac{1}{a}(1+ax)\right) \\ &\ge \frac{a-1}{a}\ln(1) + \frac{1}{a}\ln\big(1+ax\big) \\ &= \frac{1}{a}\ln\big(1+ax\big). \end{aligned}$$

This gives $\big(1+x\big)^a \ge 1+ax$. ■

Example 1.26. Let $\boldsymbol{a} = \big(a_1,a_2,a_3,\ldots,a_n\big)$ be an n-tuple of positive real numbers and $\big(w_1,w_2,w_3,\ldots,w_n\big)$ be another n-tuple of positive real numbers (called

weights). Define the weighted mean of order r by

$$M_r(\boldsymbol{a}) = \begin{cases} \left(\dfrac{\sum_{j=1}^n w_j a_j^r}{\sum_{j=1}^n w_j}\right)^{1/r} & \text{if } r \neq 0, \text{ and } |r| < \infty, \\ \left(\prod_{j=1}^n a_j^{w_j}\right)^{\left(1/\sum_{j=1}^n w_j\right)}, & \text{if } r = 0, \\ \min\{a_1, a_2, a_3, \ldots, a_n\}, & \text{if } r = -\infty, \\ \max\{a_1, a_2, a_3, \ldots, a_n\}, & \text{if } r = \infty \end{cases}$$

Suppose r, s are two real numbers such that $r < s$. Prove that

$$M_{-\infty}(\boldsymbol{a}) \leq M_r(\boldsymbol{a}) \leq M_s(\boldsymbol{a}) \leq M_\infty(\boldsymbol{a}).$$

Solution: We prove this by considering several cases.

(a) Suppose $r = 0$. Then we have $s > 0$ and hence

$$\begin{aligned} \left(M_0(\boldsymbol{a})\right)^s &= \left(\prod_{j=1}^n a_j^{w_j}\right)^{\left(s/\sum_{j=1}^n w_j\right)} \\ &= \left(\prod_{j=1}^n \left(a_j^s\right)^{w_j}\right)^{\left(1/\sum_{j=1}^n w_j\right)} \\ &\leq \frac{\sum_{j=1}^n w_j a_j^s}{\sum_{j=1}^n w_j} \quad \text{(by the generalised AM-GM)} \\ &= \left(M_s(\boldsymbol{a})\right)^s. \end{aligned}$$

(b) Suppose $0 < r < s$. Put $\alpha = s/r$ and observe that $\alpha > 1$. Hence the function $f(x) = x^\alpha$ is convex on $(0, \infty)$. Hence Jensen's inequality gives

$$\left(\frac{\sum_{j=1}^n w_j a_j^r}{\sum w_k}\right)^\alpha \leq \sum_{j=1}^n \frac{w_j}{\sum w_k}\left(a_j^{r\alpha}\right) = \frac{\sum_{j=1}^n w_j a_j^s}{\sum w_k}.$$

This shows that $\left(M_r(\boldsymbol{a})\right)^{r\alpha} \leq \left(M_s(\boldsymbol{a})\right)^s$. Taking s-th root, we get $M_r(\boldsymbol{a}) \leq M_s(\boldsymbol{a})$.

(c) If $s = 0$, then $r < 0$ and we take $t = -r$. Using the generalised AM-GM inequality, we have

$$\begin{aligned} \frac{\sum \frac{w_j}{a_j^t}}{\sum w_k} &\geq \left(\prod_{j=1}^n \frac{1}{a_j^{tw_j}}\right)^{\left(1/\sum w_k\right)} \\ &= \left(\prod_{j=1}^n \frac{1}{a_j^{w_j}}\right)^{\left(t/\sum w_k\right)} \end{aligned}$$

Simplification gives

$$\left(\frac{\sum \frac{w_j}{a_j^t}}{\sum w_k}\right)^{(1/t)} \geq \left(\prod_{j=1}^{n} \frac{1}{a_j^{w_j}}\right)^{\left(1/\sum w_k\right)}.$$

This is equivalent to

$$\left(\frac{\sum w_j a_j^r}{\sum w_k}\right)^{1/r} \leq \left(\prod_{j=1}^{n} a_j^{w_j}\right)^{\left(1/\sum w_k\right)}.$$

Thus we obtain $M_r(\boldsymbol{a}) \leq M_0(\boldsymbol{a})$.

(d) If $r < 0 < s$, then we have

$$M_r(\boldsymbol{a}) \leq M_0(\boldsymbol{a}) \leq M_s(\boldsymbol{a})$$

from (c) and (d).

(e) Suppose $r < s < 0$. Then we have $0 < -s < -r$ and hence

$$M_{-s}(\boldsymbol{a}^{-1}) \leq M_{-r}(\boldsymbol{a}^{-1}),$$

where

$$\boldsymbol{a}^{-1} = \left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}\right)$$

But then

$$M_{-s}(\boldsymbol{a}^{-1}) = \frac{1}{M_s(\boldsymbol{a})}, \quad M_{-r}(\boldsymbol{a}^{-1}) = \frac{1}{M_r(\boldsymbol{a})}.$$

Hence we get $M_r(\boldsymbol{a}) \leq M_s(\boldsymbol{a})$.

(f) Obviously

$$\min\left\{a_1, a_2, a_3, \ldots, a_n\right\} \leq M_r(\boldsymbol{a}) \leq \max\left\{a_1, a_2, a_3, \ldots, a_n\right\}.$$

Combining all these, we get the desired inequality. ■

1.8 Inequalities for symmetric functions

Given n complex numbers $\boldsymbol{a} = \langle a_1, a_2, a_3, \ldots, a_n \rangle$, we consider the polynomial whose roots are $a_1, a_2, a_3, \ldots, a_n$;

$$P(x) = \prod_{j=1}^{n} (x - a_j).$$

Expanding this we obtain

$$P(x) = x^n + \sum_{k=1}^{n} (-1)^k \binom{n}{k} u_k x^{n-k},$$

where

$$\begin{aligned}
u_1 = u_1(\boldsymbol{a}) &= \frac{\sum_{j=1}^{n} a_j}{\binom{n}{1}}, \\
u_2 = u_2(\boldsymbol{a}) &= \frac{\sum_{j<k} a_j a_k}{\binom{n}{2}}, \\
\vdots \quad & \quad \vdots \\
u_n = u_n(\boldsymbol{a}) &= \frac{\prod_{j=1}^{n} a_j}{\binom{n}{n}}.
\end{aligned}$$

These functions $u_1, u_2, u_3, \ldots, u_n$ are called the elementary symmetric functions in the variables $a_1, a_2, a_3, \ldots, a_n$. We may also write them in the form $u_r(\langle a_k \rangle)$ to show the dependence on the sequence $a_1, a_2, a_3, \ldots, a_n$. We observe that the permutation of $a_1, a_2, a_3, \ldots, a_n$ does not alter any of the u_j's, $1 \le j \le n$. We take $u_0(\boldsymbol{a}) = 1$.

We further introduce $v_k = u_k^{1/k}$. We observe that v_1 is simply the arithmetic mean of $a_1, a_2, a_3, \ldots, a_n$ and v_n is their geometric mean.

Theorem 11. (Newton's inequalities) Let $a_1, a_2, a_3, \ldots, a_n$ be a sequence of non-negative real numbers. We have the inequalities

$$v_n \le v_{n-1} \le \cdots \le v_1.$$

Proof: Let $a = \min\{a_j : 1 \le j \le n\}$ and $b = \max\{a_j : 1 \le j \le n\}$. Then $P(x) = 0$ has n roots in $[a, b]$. Hence the derivative polynomial $P'(x)$ has $n-1$ zeros in $[a, b]$, counted according to multiplicity. (If $P(x) = 0$ has a root α of multiplicity l, then α is also a root of $P'(x) = 0$ of multiplicity $l-1$.) It follows that for any positive integer k, $P^{(k)}(x) = 0$ has $n-k$ roots in $[a, b]$. (Here $P^{(k)}(x)$ denotes the k-th derivative of $P(x)$.)

We observe that

$$P^{(n-2)}(x) = \frac{n!}{2}(x^2 - 2u_1 x + u_2).$$

This has real roots by previous observation. Hence we get

$$u_2 \le u_1^2. \tag{1.22}$$

We also observe that

$$\binom{n}{2} \frac{u_{n-2}}{u_n} = \sum_{j<k} \frac{1}{a_j a_k}, \quad \binom{n}{1} \frac{u_{n-1}}{u_n} = \sum_{j=1}^{n} \frac{1}{a_j}.$$

Applying (1.22) for the numbers $1/a_1, 1/a_2, \ldots, 1/a_n$, we get

$$\frac{1}{\binom{n}{2}} \sum_{j<k} \frac{1}{a_j a_k} \le \frac{1}{\binom{n}{1}^2} \left(\sum_{j=1}^{n} \frac{1}{a_j} \right)^2 .$$

This reduces to

$$u_n u_{n-2} \le u_{n-1}^2. \tag{1.23}$$

We show that $u_{r-1}u_{r+1} \le u_r^2$ for $r = 2, 3, \ldots, n-1$. We use the induction on the number of a_j's. For $n = 3$, we have $r = 2$ and hence we have to prove that $u_1 u_3 \le u_2^2$. This follows from (1.23) with $n = 3$. Suppose the result holds for a set of $n-1$ numbers. We have

$$P'(x) = n \left\{ x^{n-1} + \sum_{k=1}^{n-1} (-1)^k \binom{n-1}{k} u_k x^{n-1-k} \right\}.$$

Let the roots of $P'(x) = 0$ be $b_1, b_2, \ldots, b_{n-1}$. Introduce

$$\binom{n-1}{r} z_r = \sum_{j_1<j_2<\cdots<j_r} b_{j_1} b_{j_2} \cdots b_{j_r}.$$

Then we see that

$$\begin{aligned} P'(x) &= n(x-b_1)(x-b_2)\cdots(x-b_{n-1}) \\ &= n \left\{ x^{n-1} + \sum_{k=1}^{n-1} (-1)^k \binom{n-1}{k} z_k x^{n-1-k} \right\}. \end{aligned}$$

It follows that $z_r = u_r$ for $1 \le r \le n-1$. Now, the induction hypothesis holds for the set $\{b_1, b_2, \ldots, b_{n-1}\}$. This gives $z_{r-1}z_{r+1} \le z_r^2$ for $2 \le r \le n-2$. Using $z_r = u_r$ for $1 \le r \le n-1$, we infer that $u_{r-1}u_{r+1} \le u_r^2$ for $2 \le r \le n-2$. For $r = n-1$, this follows from (1.23). This completes the induction. We observe that $u_2 \le u_1^2$, $(u_1u_3)^2 \le u_2^4$, $(u_2u_4)^3 \le u_3^6$ and so on. Multiplying all these, we get

$$u_1^2 u_2^4 u_3^6 \cdots u_{r-1}^{2r-2} u_r^{r-1} u_{r+1}^r \le u_1^2 u_2^4 \cdots u_r^{2r}.$$

This reduces to $u_{r+1}^r \le u_r^{r+1}$ and hence it follows that $v_{r+1} \le v_r$ for $r = 1, 2, \ldots, n-1$.

■

Chapter 2

Techniques for proving inequalities

2.1 Introduction

There are several standard ways of proving a given inequality. We have already seen how to obtain the AM-GM inequality using forward and backward induction. One can also use the known standard inequalities or use ideas from calculus. In some cases trigonometric substitutions simplify the result. We take each of these separately and illustrate them using examples.

2.2 Use of induction

In many problems on inequalities, the *Principle of Mathematical Induction* is a powerful tool that can be deployed to prove the result. The technique is to use induction on the number of variables or some times on the degree (or power).

Example 2.1. Let $x_1, x_2, \ldots, x_n$ be n real numbers in the interval $[0,1]$ and suppose $\sum_{j=1}^{n} x_j = m + r$, where m is an integer and $0 \leq r < 1$. Prove that

$$\sum_{j=1}^{n} x_j^2 \leq m + r^2.$$

Solution: We show by induction on n that

$$\sum_{j=1}^{n} x_j^2 \leq \left[\sum_{j=1}^{n} x_j\right] + \left(\sum_{j=1}^{n} x_j - \left[\sum_{j=1}^{n} x_j\right]\right)^2, \tag{2.1}$$

where $[x]$ denotes the integral part of x. If $n = 1$, then x_1 lies in the interval $[0,1]$ and (2.1) reduces to

$$x_1^2 \leq [x_1] + \Big(x_1 - [x_1]\Big)^2,$$

which is true since $[x_1] = 0$ or 1. Suppose (2.1) holds for $1 \leq n \leq N-1$. Put

$$\begin{aligned} S_{N-1} &= \sum_{j=1}^{N-1} x_j = k + t, \quad 0 \leq t < 1, \\ S_N &= \sum_{j=1}^{N} x_j = m + r, \quad 0 \leq r < 1. \end{aligned}$$

We show that

$$x_N^2 \le [S_N] + \big(S_N - [S_N]\big)^2 - [S_{N-1}] - \big(S_{N-1} - [S_{N-1}]\big)^2. \tag{2.2}$$

This is equivalent to

$$(m-k+r-t)^2 \le m + r^2 - k - t^2.$$

We can further reduce it to another equivalent statement:

$$2(r-t)(m-k-t) \le (m-k) - (m-k)^2. \tag{2.3}$$

If $m = k$, then $t \le r$ and hence (2.3) reduces to $-2(r-t)t \le 0$, which is true. If $m = k+1$, then $r \le t$ since $x_n \le 1$ and (2.3) is equivalent to $2(r-t)(1-t) \le 0$. This again is true. Thus (2.2) holds for N. Hence

$$\begin{aligned}\sum_{j=1}^{N} x_j^2 &= \sum_{j=1}^{N-1} x_j^2 + x_N^2 \\ &\le [S_{N-1}] + \big(S_{N-1} - [S_{N-1}]\big)^2 + x_n^2 \\ &\le [S_N] + \big(S_N - [S_N]\big)^2.\end{aligned}$$

This completes induction.

Here is a generalisation of the previous result(CRUX-1992).

Let $\alpha > 0$ and let $f : [0, \alpha] \to \mathbb{R}$ be a convex function. Let $x_1, x_2, \ldots, x_n$ be $n(\ge 2)$ points in the interval $[0, \alpha]$ such that $\sum_{j=1}^{n} x_j = m\alpha + r$, where m is an integer and $0 \le r < \alpha$. Then

$$\sum_{j=1}^{n} f(x_j) \le (n-m-1)f(0) + mf(\alpha) + f(r).$$

A proof of this may be based on the properties of convex functions and the principle of induction as used above. ■

Example 2.2. Let $x_1 \ge x_2 \ge \cdots \ge x_n > 0$ be n real numbers. Prove that

$$\prod_{j=1}^{n} \frac{(1+x_j)}{2} \le \frac{1 + x_1 + x_1x_2 + \cdots + x_1x_2\cdots x_n}{n+1}.$$

Solution: We use induction on n. If $n = 1$, the result is immediate. Suppose it holds for all $m \le n-1$. Take any n real numbers such that $x_1 \ge x_2 \ge \cdots \ge x_n > 0$. We apply the induction hypothesis to the sequence $x_2 \ge x_3 \ge \cdots \ge x_n > 0$ and obtain

$$\prod_{j=2}^{n} \frac{(1+x_j)}{2} \le \frac{1 + x_2 + x_2x_3 + \cdots + x_2x_3\cdots x_n}{n}.$$

Multiplying both the sides by $(1+x_1)/2$, this takes the form

$$\prod_{j=1}^{n} \frac{(1+x_j)}{2} \le \Big(\frac{1+x_1}{2}\Big)\Big(\frac{1+R}{n}\Big),$$

where

$$R = x_2 + x_2x_3 + \cdots + x_2x_3\cdots x_n.$$

Thus the inequality to be proved is

$$\frac{(1+x_1)(1+R)}{2n} \le \frac{1+x_1+x_1R}{n+1}.$$

This is equivalent to

$$R\Big((n+1)-(n-1)x_1\Big) \le (n-1)(1+x_1).$$

If $n+1 \le (n-1)x_1$, then the left hand side is non-positive and the right hand side is non-negative. Hence the result is true in this case. Thus we may assume that $(n+1)-(n-1)x_1 > 0$. Thus it is sufficient to deduce that

$$R \le \frac{(n-1)(1+x_1)}{(n+1)-(n-1)x_1}.$$

But, observe that

$$R = x_2 + x_2x_3 + \cdots + x_2x_3\cdots x_n \le x_1 + x_1^2 + \cdots + x_1^{n-1}.$$

Therefore it is enough to show that

$$\big(x_1 + x_1^2 + \cdots + x_1^{n-1}\big)\big((n+1)-(n-1)x_1\big) \le (n-1)(1+x_1).$$

Equivalently, we need to prove that

$$(n-1)x_1^n - 2\big(x_1 + x_1^2 + \cdots + x_1^{n-1}\big) + (n-1) \ge 0.$$

Consider the polynomial

$$P(x) = (n-1)x^n - 2\big(x + x^2 + \cdots + x^{n-1}\big) + (n-1).$$

Obviously $P(1) = 0$. We also have

$$P'(x) = n(n-1)x^{n-2} - 2\big(1 + 2x + 3x^2 + \cdots + (n-1)x^{n-2}\big).$$

This shows that $P'(1) = 0$. Thus $P(x) = 0$ has a double root at $x = 1$. However, the number of sign changes in $P(x)$ is 2 and hence the number of positive roots of $P(x) = 0$ cannot exceed 2. We conclude that $P(x) > 0$ for $x > 1$. But, $P(x)$ is a reciprocal polynomial and hence $x^nP\big(1/x\big) = P(x)$. It follows that $P(x) > 0$ for $0 < x < 1$. Thus $P(x) \ge 0$ for $x > 0$. This completes the proof. ■

2.3 Application of known inequalities

We have already built up a library of some basic inequalities. These may be used in proving a new inequality. Only some insight into the problem leads to the correct inequality to be chosen. Sometimes many inequalities may be used in conjunction to prove the result.

Example 2.3. If a, b, c are positive real numbers such that $a+b+c=1$, prove that

$$8abc \leq (1-a)(1-b)(1-c) \leq \frac{8}{27}.$$

Solution: Since a, b, c are positive and $a+b+c=1$, none of these can exceed 1. Thus each of $1-a, 1-b, 1-c$ is positive. Using the AM-GM inequality, we obtain

$$(1-a)(1-b)(1-c) \leq \left(\frac{3-a-b-c}{3}\right)^3 = \frac{8}{27},$$

which is the second part of the required inequality. On the other hand, we also see that

$$\begin{aligned}(1-a)(1-b)(1-c) &= 1-(a+b+c)+(ab+bc+ca)-abc \\ &= ab+bc+ca-abc.\end{aligned}$$

Thus the first part of the inequality is equivalent to

$$9abc \leq ab+bc+ca.$$

We may write this last inequality in the form

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 9.$$

This is an immediate consequence of the AM-HM inequality, since $a+b+c=1$. More generally, it is not hard to see that for any n positive real numbers $a_1, a_2, \ldots, a_n$, the inequality

$$\left(a_1+a_2+\cdots+a_n\right)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right) \geq n^2,$$

holds. ■

Example 2.4. Let a, b, c be positive real numbers, and let p, q, r be three numbers in the interval $[0, 1/2]$. Suppose $a+b+c = p+q+r = 1$. Prove that

$$abc \leq \frac{pa+qb+rc}{8}.$$

Solution: We make use of the generalised AM-GM inequality: if α, β, γ are non-negative real numbers such that $\alpha+\beta+\gamma=1$, then for positive u, v, w, we have

$$\alpha u+\beta v+\gamma w\geq u^{\alpha}v^{\beta}w^{\gamma}.$$

Thus we obtain

$$p\cdot\frac{1}{bc}+q\cdot\frac{1}{ca}+r\cdot\frac{1}{ab}\geq\left(\frac{1}{bc}\right)^{p}\left(\frac{1}{ca}\right)^{q}\left(\frac{1}{ab}\right)^{r}$$

and

$$\left(bc\right)^{p}\left(ca\right)^{q}\left(ab\right)^{r}\leq p\cdot bc+q\cdot ca+r\cdot ab.$$

These two lead to

$$\begin{aligned}\frac{abc}{pa+qb+rc}&=\frac{1}{p\cdot\frac{1}{bc}+q\cdot\frac{1}{ca}+r\cdot\frac{1}{ab}}\\&\leq\frac{1}{\left(\frac{1}{bc}\right)^{p}\left(\frac{1}{ca}\right)^{q}\left(\frac{1}{ab}\right)^{r}}\\&=\left(bc\right)^{p}\left(ca\right)^{q}\left(ab\right)^{r}\\&\leq p\cdot bc+q\cdot ca+r\cdot ab.\end{aligned}$$

Assume $a\leq b\leq c$. Then

$$\begin{aligned}p\cdot bc+q\cdot ca+r\cdot ab&\leq pbc+(q+r)ac\\&=pbc+(1-p)ac.\end{aligned}$$

Since $0\leq p\leq 1/2$, we get

$$\left(\frac{1}{2}-p\right)ac\leq\left(\frac{1}{2}-p\right)bc.$$

This implies that

$$\begin{aligned}pbc+(1-p)ac&=\frac{1}{2}bc+\frac{1}{2}ac-\left(\frac{1}{2}-p\right)bc+\left(\frac{1}{2}-p\right)ac\\&\leq\frac{1}{2}bc+\frac{1}{2}ac\\&=\frac{1}{2}c(a+b).\end{aligned}$$

But we know that

$$(a+b)c\leq\left(\frac{a+b+c}{2}\right)^{2}=\frac{1}{4}.$$

Combining all these, we end up with

$$\frac{abc}{pa+qb+rc}\leq\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{8}.$$

We also observe that $a=b=1/4$, $c=1/2$, $p=q=1/2$, $r=0$ gives equality.

■

Example 2.5. For $n \geq 2$, and real numbers $0 < x_1 < x_2 < \cdots < x_n \leq 1$, prove that

$$\frac{n\sum_{k=1}^n x_k}{\sum_{k=1}^n x_k + nx_1x_2\cdots x_n} \geq \sum_{k=1}^n \frac{1}{1+x_k}.$$

Solution: If $a_k > 0$ for $1 \leq k \leq n$, then we know that

$$\Big(\sum_{k=1}^n a_k\Big)\Big(\sum_{k=1}^n a_k^{-1}\Big) \geq n^2,$$

as an application of the AM-HM inequality. This implies that

$$\begin{aligned}\Big(\sum_{k=1}^n a_k\Big)\Big(\sum_{k=1}^n a_k^{-1}(1-a_k)\Big) &= \Big(\sum_{k=1}^n a_k\Big)\Big(\sum_{k=1}^n a_k^{-1}\Big) - n\sum_{k=1}^n a_k \\ &\geq n\Big(n - \sum_{k=1}^n a_k\Big) \\ &= n\Big(\sum_{k=1}^n (1-a_k)\Big).\end{aligned}$$

Taking

$$a_k = \frac{x_k}{1+x_k}, \text{ for } 1 \leq k \leq n,$$

we obtain

$$\sum_{k=1}^n x_k^{-1}\Big(n - \sum_{k=1}^n \frac{1}{1+x_k}\Big) \geq n\sum_{k=1}^n \frac{1}{1+x_k}.$$

Multiplying by $x_1x_2\cdots x_n$ and using the inequality

$$\sum_{k=1}^n x_k \geq x_1x_2\cdots x_n \sum_{k=1}^n x_k^{-1}, >$$

we get

$$\Big(\sum_{k=1}^n x_k\Big)\Big(n - \sum_{k=1}^n \frac{1}{1+x_k}\Big) \geq nx_1x_2\cdots x_n \sum_{k=1}^n \frac{1}{1+x_k}.$$

This implies that

$$n\sum_{k=1}^n x_k \geq \Big(\sum_{k=1}^n x_k + nx_1x_2\cdots x_n\Big)\sum_{k=1}^n \frac{1}{1+x_k}.$$

This is equivalent to the given inequality. ∎

Example 2.6. Let $a_1, a_2, \cdots, a_n$ be positive real numbers. Prove that

$$\frac{a_1^2}{a_1+a_2} + \frac{a_2^2}{a_2+a_3} + \cdots + \frac{a_n^2}{a_n+a_1} \geq \frac{1}{2}\big(a_1 + a_2 + \cdots + a_n\big).$$

Solution: We put

$$\alpha_j = \frac{a_j}{a_1 + a_2 + \cdots + a_n} \text{ and } b_j = \frac{a_j}{a_j + a_{j+1}},$$

where $a_{n+1} = a_1$. Then $\sum_{j=1}^n \alpha_j = 1$. Now the weighted AM-HM inequality gives

$$\frac{\displaystyle\sum_{j=1}^n \alpha_j b_j}{\displaystyle\sum_{j=1}^n \alpha_j} \geq \frac{\displaystyle\sum_{j=1}^n \alpha_j}{\displaystyle\sum_{j=1}^n \frac{\alpha_j}{b_j}}.$$

We observe that

$$\left(\sum_{j=1}^n a_j\right)\left(\sum_{j=1}^n \frac{\alpha_j}{b_j}\right) = \sum_{j=1}^n (a_j + a_{j+1}) = 2\left(\sum_{j=1}^n a_j\right),$$

so that $\displaystyle\sum_{j=1}^n \frac{\alpha_j}{b_j} = 2$. Since $\sum_{j=1}^n \alpha_j = 1$, we get $\sum_{j=1}^n \alpha_j b_j \geq 1/2$. Thus we obtain

$$\sum_{j=1}^n \frac{a_j^2}{a_j + a_{j+1}} = \left(\sum_{j=1}^n a_j\right)\left(\sum_{j=1}^n \alpha_j b_j\right) \geq \frac{1}{2}\left(\sum_{j=1}^n a_j\right),$$

which is the desired inequality. ∎

Example 2.7. If a, b, c are positive real numbers, show that

$$\frac{abc\left(a+b+c+\sqrt{a^2+b^2+c^2}\right)}{\left(a^2+b^2+c^2\right)\left(ab+bc+ca\right)} \leq \frac{3+\sqrt{3}}{9}.$$

When does equality hold?

Solution: The Cauchy-Schwarz inequality applied to (a, b, c) and $(1, 1, 1)$ gives

$$a+b+c \leq \sqrt{3}\sqrt{a^2+b^2+c^2}.$$

On the other hand, the AM-GM inequality gives

$$a^2+b^2+c^2 \geq 3(abc)^{2/3}, \quad ab+bc+ca \geq 3(abc)^{2/3}.$$

Combining these two, we have

$$\begin{aligned}
\frac{abc\Big(a+b+c+\sqrt{a^2+b^2+c^2}\Big)}{\big(a^2+b^2+c^2\big)\big(ab+bc+ca\big)} &\leq \frac{abc\big(1+\sqrt{3}\,\big)\big(a^2+b^2+c^2\big)^{\frac{1}{2}}}{\big(a^2+b^2+c^2\big)\big(ab+bc+ca\big)} \\
&= \frac{abc\big(1+\sqrt{3}\,\big)}{\big(ab+bc+ca\big)\big(a^2+b^2+c^2\big)^{\frac{1}{2}}} \\
&\leq \frac{abc\big(1+\sqrt{3}\,\big)}{3\sqrt{3}\big(abc\big)^{\frac{1}{3}}\big(abc\big)^{\frac{2}{3}}} \\
&= \frac{3+\sqrt{3}}{9}.
\end{aligned}$$

Equality holds if and only if $a=b=c$. ∎

Example 2.8. (IMO, 1995) Let a,b,c be positive reals such that $abc=1$. Prove that

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}.$$

Solution: Taking $x=1/a$, $y=1/b$, $z=1/c$, we have $xyz=1$ and the inequality to be proved is

$$\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\geq\frac{3}{2}.$$

Using the Cauchy-Schwarz inequality, we have

$$(x+y+z)^2\leq 2\left(\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\right)\Big(x+y+z\Big).$$

Thus

$$\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\geq\frac{x+y+z}{2}.$$

But the AM-GM inequality gives

$$x+y+z\geq 3(xyz)^{1/3}=3.$$

It follows that

$$\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\geq\frac{3}{2}.$$

We prove, the following more general assertion:
Let a,b,c be positive reals such that $abc=1$. Then

$$\frac{1}{a^x(b+c)}+\frac{1}{b^x(c+a)}+\frac{1}{c^x(a+b)}\geq\frac{3}{2}, \tag{2.4}$$

holds if $x \geq 2$ or $x \leq -1$. For each x, $-1 < x < 2$, we can find positive real numbers a, b, c such that $abc = 1$ and (2.4) is not true.

For any real x, define

$$f(x; a, b, c) = \frac{1}{a^x(b+c)} + \frac{1}{b^x(c+a)} + \frac{1}{c^x(a+b)}.$$

We show that $f(x; a, b, c)$ is a non-decreasing function for $x \geq 1$. Let $x > y \geq 1$ be two reals. Suppose $a \leq b \leq c$. Then $a^{x-y} \leq b^{x-y} \leq c^{x-y}$ is true. We also observe that $a^y(b+c) \leq b^y(c+a) \leq c^y(a+b)$. In fact, $a^y(b+c) \leq b^y(c+a)$ is equivalent to $ab(a^{y-1} - b^{y-1}) \leq c(b^y - a^y)$ which is true because $y - 1 \geq 0$ and $a \leq b$. Similarly, $b^y(c+a) \leq c^y(a+b)$ holds. Using Chebyshev's inequality, we obtain

$$\begin{aligned}
&\frac{1}{a^x(b+c)} + \frac{1}{b^x(c+a)} + \frac{1}{c^x(a+b)} \\
&= \frac{1}{a^{x-y}a^y(b+c)} + \frac{1}{b^{x-y}b^y(c+a)} + \frac{1}{c^{x-y}c^y(a+b)} \\
&\geq \frac{1}{3}\left(\frac{1}{a^{x-y}} + \frac{1}{b^{x-y}} + \frac{1}{c^{x-y}}\right)\left(\frac{1}{a^y(b+c)} + \frac{1}{b^y(c+a)} + \frac{1}{c^y(a+b)}\right) \\
&\geq \frac{1}{a^y(b+c)} + \frac{1}{b^y(c+a)} + \frac{1}{c^y(a+b)}.
\end{aligned}$$

We have used the AM-GM inequality in the last leg. We get $f(x; a, b, c) \geq f(y; a, b, c)$ if $x > y \geq 1$. Thus $f(x; a, b, c)$ is a non-decreasing function on the interval $[1, \infty)$.

Now consider $f(2; a, b, c)$. We write

$$\begin{aligned}
f(2; a, b, c) &= \frac{1}{a^2(b+c)} + \frac{1}{b^2(c+a)} + \frac{1}{c^2(a+b)} \\
&= \frac{bc}{ab+ac} + \frac{ca}{bc+ab} + \frac{ab}{bc+ca} \\
&= (ab+bc+ca)\left(\frac{1}{ab+bc} + \frac{1}{bc+ca} + \frac{1}{ca+ab}\right) - 3 \\
&\geq \frac{9}{2} - 3 \geq \frac{3}{2}.
\end{aligned}$$

It follows that $f(x; a, b, c) \geq 3/2$ for $x \geq 2$.

We now show that the result is not true for $1/2 \leq x < 2$. Take $x = 2 - e$ where $e > 0$. Put $a = d^2, b = 1/d, c = 1/d$ so that $abc = 1$. Then

$$f(2-e; a, b, c) = \frac{1}{2d^{3-2e}} + \frac{2d^{3-e}}{d^3+1}$$

and $f(2-e; a, b, c) \to 0$ as $d \to \infty$, if $e < 3/2$. If $e = 3/2$ then $f(2-e; a, b, c) \to 1/2 < 3/2$. In any case (2.4) is not true. Thus (2.4) fails for $1/2 \leq x < 2$.

For x in the interval $(-\infty, 1/2)$, we replace a, b, c by $1/p$, $1/q$, $1/r$ respectively. Then $f(x; a, b, c) = f(1-x; p, q, r)$. But $1-x$ lies in the interval $(1/2, \infty)$. It follows that (2.4) holds for $x \leq -1$ and for x in the interval $(-1, 1/2)$ there are a, b, c such that $abc = 1$ and (2.4) fails. ■

Example 2.9. Let $n \geq 3$ be an integer and let $x_1, x_2, \ldots, x_n$ be positive numbers such that

$$\sum_{j=1}^{n} \frac{1}{1+x_j} = 1.$$

Prove that

$$\sum_{j=1}^{n} \sqrt{x_j} \geq (n-1) \sum_{j=1}^{n} \frac{1}{\sqrt{x_j}}.$$

Solution: We may take $x_1 \leq x_2 \leq \cdots \leq x_n$. We claim that under this condition we have the inequality:

$$\frac{\sqrt{x_j}}{1+x_j} \geq \frac{\sqrt{x_k}}{1+x_k},$$

whenever $1 \leq j \leq k \leq n$. In fact, we have

$$1 > \frac{1}{1+x_j} + \frac{1}{1+x_k} = \frac{2+x_j+x_k}{(1+x_j)(1+x_k)},$$

from the given condition. This gives

$$1 + x_j + x_k + x_j x_k > 2 + x_j + x_k.$$

Hence $x_j x_k > 1$. But we also have

$$\frac{\sqrt{x_j}}{1+x_j} - \frac{\sqrt{x_k}}{1+x_k} = \frac{\left(\sqrt{x_j} - \sqrt{x_k}\right)\left(1 - \sqrt{x_j x_k}\right)}{(1+x_j)(1+x_k)} \geq 0,$$

since $\sqrt{x_j} \leq \sqrt{x_k}$ and $1 \leq \sqrt{x_j x_k}$. Thus we obtain

$$\frac{\sqrt{x_1}}{1+x_1} \geq \frac{\sqrt{x_2}}{1+x_2} \geq \cdots \geq \frac{\sqrt{x_n}}{1+x_n}.$$

Since we also know that

$$\frac{1}{\sqrt{x_1}} \geq \frac{1}{\sqrt{x_2}} \geq \cdots \geq \frac{1}{\sqrt{x_n}},$$

by the assumed ordering of x_j's, we may apply Chebyshev's inequality. We get

$$\frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{\sqrt{x_j}}\right)\left(\sum_{j=1}^{n} \frac{\sqrt{x_j}}{1+x_j}\right) \leq \sum_{j=1}^{n} \frac{1}{1+x_j} = 1 \tag{2.5}$$

However using the AM-HM inequality, we obtain

$$\left(\sum_{j=1}^{n} \frac{1+x_j}{\sqrt{x_j}}\right)\left(\sum_{j=1}^{n} \frac{\sqrt{x_j}}{1+x_j}\right) \geq n^2.$$

We may write this in the form

$$\left(\sum_{j=1}^{n} \frac{\sqrt{x_j}}{1+x_j}\right)\left(\sum_{j=1}^{n} \sqrt{x_j}\right) + \left(\sum_{j=1}^{n} \frac{\sqrt{x_j}}{1+x_j}\right)\left(\sum_{j=1}^{n} \frac{1}{\sqrt{x_j}}\right) \geq n^2.$$

Now using (2.5), we obtain

$$n^2 \leq n + \left(\sum_{j=1}^{n} \frac{\sqrt{x_j}}{1+x_j}\right)\left(\sum_{j=1}^{n} \sqrt{x_j}\right).$$

This reduces to

$$\left(\sum_{j=1}^{n} \frac{\sqrt{x_j}}{1+x_j}\right)\left(\sum_{j=1}^{n} \sqrt{x_j}\right) \geq n^2 - n = n(n-1). \tag{2.6}$$

However (2.5) may also written in the form

$$\sum_{j=1}^{n} \frac{\sqrt{x_j}}{1+x_j} \leq \frac{n}{\displaystyle\sum_{j=1}^{n} \frac{1}{\sqrt{x_j}}}.$$

Using this in (2.6), we get

$$n(n-1) \leq \left(\sum_{j=1}^{n} \sqrt{x_j}\right)\left(n \Big/ \sum_{j=1}^{n} \frac{1}{\sqrt{x_j}}\right).$$

This gives

$$\sum_{j=1}^{n} \sqrt{x_j} \geq (n-1) \sum_{j=1}^{n} \frac{1}{\sqrt{x_j}}.$$

Alternate Solution:

Using

$$\sum_{j=1}^{n} \frac{1}{1+x_j} = 1,$$

we easily obtain

$$\sum_{j=1}^{n} \frac{x_j}{1+x_j} = n-1.$$

Hence

$$\begin{aligned}
&\sum_{j=1}^{n}\sqrt{x_j}-(n-1)\sum_{j=1}^{n}\frac{1}{\sqrt{x_j}}\\
&\quad=\quad\Big(\sum_{j=1}^{n}\frac{1}{1+x_j}\Big)\Big(\sum_{j=1}^{n}\sqrt{x_j}\Big)-\Big(\sum_{j=1}^{n}\frac{x_j}{1+x_j}\Big)\Big(\sum_{j=1}^{n}\frac{1}{\sqrt{x_j}}\Big)\\
&\quad=\quad\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{x_j-x_k}{(1+x_k)\sqrt{x_j}}\\
&\quad=\quad\sum_{j>k}\frac{\big(\sqrt{x_j}\sqrt{x_k}-1\big)\big(\sqrt{x_j}-\sqrt{x_k}\big)^2\big(\sqrt{x_j}+\sqrt{x_k}\big)}{\sqrt{x_j}\sqrt{x_k}(1+x_j)(1+x_k)}.
\end{aligned}$$

Here we have used

$$\frac{1}{(1+x_k)\sqrt{x_j}}-\frac{1}{(1+x_j)\sqrt{x_k}}=\frac{\big(\sqrt{x_j}\sqrt{x_k}-1\big)\big(\sqrt{x_j}-\sqrt{x_k}\big)}{\sqrt{x_j}\sqrt{x_k}(1+x_j)(1+x_k)}.$$

Thus it is sufficient to prove that $x_jx_k\geq 1$ for any pair j,k such that $j\neq k$. But this follows from the given condition:

$$1\geq\frac{1}{1+x_j}+\frac{1}{1+x_k}=\frac{2+x_j+x_k}{1+x_j+x_k+x_jx_k},$$

which implies that $x_jx_k\geq 1$. ■

Example 2.10. (Hardy's inequality) Let $a_1,a_2,\dots,a_N$ be N positive real numbers and let $p>1$. Put

$$\alpha_k=\frac{a_1+a_2+\cdots+a_k}{k},\ \text{for } 1\leq k\leq N.$$

Prove that

$$\sum_{k=1}^{N}\alpha_k^p\leq\Big(\frac{p}{p-1}\Big)^p\sum_{k=1}^{N}a_k^p.$$

Solution: Let $A_k=a_1+a_2+\cdots+a_k$, We have

$$\begin{aligned}
\alpha_n^p-\frac{p}{p-1}\alpha_n^{p-1}a_n &= \alpha_n^p-\frac{p}{p-1}\alpha_n^{p-1}\big(A_n-A_{n-1}\big)\\
&= \alpha_n^p-\frac{p}{p-1}\alpha_n^{p-1}\big(n\alpha_n-(n-1)\alpha_{n-1}\big)\\
&= \alpha_n^p\Big(1-\frac{np}{p-1}\Big)+\frac{(n-1)p}{p-1}\alpha_n^{p-1}\alpha_{n-1}.
\end{aligned}$$

Using the generalised AM-GM inequality, we have

$$\frac{(n-1)p}{p-1}\alpha_n^{p-1}\alpha_{n-1}\leq\frac{n-1}{p-1}\Big\{(p-1)\alpha_n^p+\alpha_{n-1}^p\Big\}.$$

Thus we get

$$\begin{aligned} \alpha_n^p - \frac{p}{p-1}\alpha_n^{p-1}a_n &\leq \alpha_n^p\Big\{1 - \frac{np}{p-1} + (n-1)\Big\} + \frac{n-1}{p-1}\alpha_{n-1}^p \\ &= \frac{1}{p-1}\Big\{(n-1)\alpha_{n-1}^p - n\alpha_n^p\Big\}. \end{aligned}$$

Taking $\alpha_0 = 0$ and summing over n from 1 to N, we get

$$\sum_{n=1}^N \alpha_n^p - \frac{p}{p-1}\sum_{n=1}^N \alpha_n^{p-1}a_n \leq -\frac{N\alpha_N^p}{p-1} \leq 0.$$

Thus we obtain,

$$\begin{aligned} \sum_{n=1}^N \alpha_n^p &\leq \frac{p}{p-1}\sum_{n=1}^N \alpha_n^{p-1}a_n \\ &\leq \frac{p}{p-1}\Big(\sum_{n=1}^N a_n^p\Big)^{1/p}\Big(\sum_{n=1}^N \alpha_n^{(p-1)q}\Big)^{1/q}, \end{aligned}$$

where q is the conjugate index of p; we have used Hölder's inequality. Hence $(p-1)q = p$ and we obtain

$$\sum_{n=1}^N \alpha_n^p \leq \frac{p}{p-1}\Big(\sum_{n=1}^N a_n^p\Big)^{1/p}\Big(\sum_{n=1}^N \alpha_n^p\Big)^{1/q}.$$

This simplifies to

$$\sum_{k=1}^N \alpha_k^p \leq \Big(\frac{p}{p-1}\Big)^p \sum_{k=1}^N a_k^p.$$

■

Example 2.11. Let $a_1, a_2, \ldots, a_n$ be n positive real numbers. Prove that

$$\prod_{j=1}^n a_j^{a_j} \geq \Big(\prod_{j=1}^n a_j\Big)^{\left(\sum_{j=1}^n a_j\right)/n}.$$

Solution: Consider the function $f(x) = x\ln x$. This is convex on $(0,\infty)$. In fact $f''(x) = 1/x$ which is positive. Using Jensen's inequality, we obtain

$$\sum_{j=1}^n \frac{1}{n}a_j \ln a_j \geq \Big(\sum_{j=1}^n \frac{1}{n}a_j\Big)\ln\Big(\sum_{j=1}^n \frac{1}{n}a_j\Big).$$

However, the AM-GM inequality gives

$$\frac{1}{n}\sum_{j=1}^n a_j \geq \Big(\prod_{j=1}^n a_j\Big)^{1/n}.$$

Using the monotonicity of $\ln x$, we obtain

$$\frac{1}{n}\sum_{j=1}^{n} a_j \ln a_j \geq \left(\sum_{j=1}^{n}\frac{1}{n}a_j\right)\ln\left(\prod_{j=1}^{n} a_j\right)^{1/n}.$$

This implies that

$$\sum_{j=1}^{n} \ln a_j^{a_j} \geq \left(\frac{1}{n}\sum_{j=1}^{n} a_j\right)\ln\left(\prod_{j=1}^{n} a_j\right).$$

Using the fact that the logarithm takes multiplication to addition, and using the monotonicity of exponential function, we obtain the desired inequality. ■

Example 2.12. Let $a_1, a_2, \ldots, a_n$ be n positive real numbers. Show that for any permutation $a_1', a_2', \ldots, a_n'$ of the sequence $a_1, a_2, \ldots, a_n$, the inequality

$$\prod_{j=1}^{n} a_j^{a_j} \geq \prod_{j=1}^{n} a_j^{a_j'},$$

holds.

Solution: We may assume that $a_1 \geq a_2 \geq \cdots \geq a_n$. Then we have

$$\ln a_1 \geq \ln a_2 \geq \cdots \geq \ln a_n.$$

We apply the rearrangement inequality to the sequences $a_1, a_2, \ldots, a_n$ and $\ln a_1, \ln a_2, \ldots, \ln a_n$. We obtain

$$\sum_{j=1}^{n} a_j \ln a_j \geq \sum_{j=1}^{n} a_j' \ln a_j.$$

This implies

$$\sum_{j=1}^{n} \ln a_j^{a_j} \geq \sum_{j=1}^{n} \ln a_j^{a_j'}.$$

The exponentiation gives the result;

$$\prod_{j=1}^{n} a_j^{a_j} \geq \prod_{j=1}^{n} a_j^{a_j'}.$$

■

2.4 Use of calculus in inequalities

Many inequalities follow directly from the monotonicity of the function involved or by calculating its global maxima and minima. For a differentiable function, the monotonicity can be established using the nature of its derivatives. A point of local minima or maxima for a differentiable function is precisely that point at which its derivative vanishes. We state these results here without giving proofs; the proofs need the techniques of differential calculus.

Theorem 12. Let $f : (a, b) \to \mathbb{R}$ be a differentiable function and suppose $f'(x) \geq 0$ for all $x \in (a, b)$. Then $f(x)$ is a non-decreasing function on (a, b). If $f'(x) > 0$ on (a, b), then $f(x)$ is strictly increasing on (a, b).

Similarly, $f'(x) \leq 0$ for all $x \in (a, b)$ implies that $f(x)$ is non-increasing on (a, b) and $f'(x) < 0$ for all $x \in (a, b)$ implies that $f(x)$ is strictly decreasing on (a, b).

Theorem 13. Suppose $f : (a, b) \to \mathbb{R}$ be a differentiable function and λ is a point where a local extremum occurs for f; i.e., there exists a neighborhood $(\lambda - \delta, \lambda + \delta)$ of λ such that either $f(x) \leq f(\lambda)$ for all $x \in (\lambda - \delta, \lambda + \delta)$ or $f(x) \geq f(\lambda)$ for all $x \in (\lambda - \delta, \lambda + \delta)$. Then $f'(\lambda) = 0$. Suppose further f is twice differentiable and $f''(\lambda) \neq 0$. If $f''(\lambda) < 0$, $f(\lambda)$ is a local maximum; and if $f''(\lambda) > 0$, then $f(\lambda)$ is a local minimum. If $f''(\lambda) = 0$, then $f(\lambda)$ is neither a maximum nor a minimum(a point of *inflexion*).

For proofs of these statements, refer to any book on Calculus. We consider examples which make use of these theorems in their solutions.

Example 2.13. Show that for any $x > 0$, the inequality

$$x^\alpha - \alpha x + \alpha - 1 \quad \begin{cases} \geq 0, & \text{if } \alpha > 1 \text{ or } \alpha < 0, \\ \leq 0, & \text{if } 0 < \alpha < 1. \end{cases} \tag{2.7}$$

Solution: We consider the function $f(x) = x^\alpha - \alpha x + \alpha - 1$ on the interval $(0, \infty)$. This is a differentiable function and

$$f'(x) = \alpha x^{\alpha-1} - \alpha = \alpha\big(x^{\alpha-1} - 1\big).$$

This shows that $f'(x) = 0$ only if $x = 1$ (Note that $\alpha \neq 1$). Moreover $f''(x) = \alpha\big(\alpha - 1\big)x^{\alpha-2}$ which is positive at $x = 1$ if either $\alpha < 0$ or $\alpha > 1$, and negative at $x = 1$ if $0 < \alpha < 1$. Hence $x = 1$ is a unique point of minimum if either $\alpha < 0$ or $\alpha > 1$, and a unique point of maximum if $0 < \alpha < 1$. Since $f(1) = 0$, we conclude that $f(x) \geq f(1) = 0$ if either $\alpha < 0$ or $\alpha > 1$, and $f(x) \leq f(1) = 0$ if $0 < \alpha < 1$. This proves the result. ■

Interestingly, the inequalities (2.7) may be used to derive many of the known inequalities. We show here how we can derive the generalised AM-GM inequality and Young's inequality. Suppose x_1 and x_2 are two positive real numbers

and let $x = x_1/x_2$. Take any α with $0 < \alpha < 1$. Then we have

$$\left(\frac{x_1}{x_2}\right)^{\alpha} - \alpha\left(\frac{x_1}{x_2}\right) + \alpha - 1 \leq 0.$$

It may be rewritten as

$$x_1^{\alpha} x_2^{1-\alpha} \leq \alpha x_1 + \big(1 - \alpha\big) x_2.$$

This proves the generalised AM-GM inequality for non-negative real numbers x_1, x_2 and weights α, $1 - \alpha$. Now we use the principle of induction to prove the general case. Assume that for any $n - 1$ non-negative real numbers y_1,y_2,y_3,...,y_{n-1} and positive weights λ_1,λ_2,λ_3,...,λ_{n-1} with $\sum_{j=1}^{n-1} \lambda_j = 1$, we have

$$\prod_{j=1}^{n-1} y_j^{\lambda_j} \leq \sum_{j=1}^{n-1} \lambda_j y_j.$$

Take any n non-negative numbers x_1,x_2,x_3,...,x_n, and positive real numbers α_1,α_2,α_3,...,α_n such that $\sum_{j=1}^{n} \alpha_j = 1$. Put

$$\begin{aligned} &y_j = x_j, \quad \lambda_j = \alpha_j, \text{ for } 1 \leq j \leq n-2, \\ &y_{n-1} = x_{n-1}^{\alpha_{n-1}/\lambda_{n-1}} x_n^{\alpha_n/\lambda_{n-1}}, \quad \lambda_{n-1} = \alpha_{n-1} + \alpha_n. \end{aligned}$$

We observe that $\lambda_j > 0$ for $1 \leq j \leq n - 1$ and

$$\sum_{j=1}^{n-1} \lambda_j = \sum_{j=1}^{n} \alpha_j = 1.$$

Using the induction hypothesis, we obtain

$$\begin{aligned} \prod_{j=1}^{n} x_j^{\alpha_j} &= \prod_{j=1}^{n-1} y_j^{\lambda_j} \\ &\leq \sum_{j=1}^{n-1} \lambda_j y_j \\ &= \sum_{j=1}^{n-2} \alpha_j x_j + \big(\alpha_{n-1} + \alpha_n\big)\Big(x_{n-1}^{\alpha_{n-1}/\lambda_{n-1}} x_n^{\alpha_n/\lambda_{n-1}}\Big) \\ &\leq \sum_{j=1}^{n-2} \alpha_j x_j + \big(\alpha_{n-1} + \alpha_n\big)\left(\frac{\alpha_{n-1}}{\lambda_{n-1}} x_{n-1} + \frac{\alpha_n}{\lambda_{n-1}} x_n\right) \\ &= \sum_{j=1}^{n} \alpha_j x_j, \end{aligned}$$

since $\lambda_{n-1} = \alpha_{n-1} + \alpha_n$. Equality holds if and only if all x_j's are equal.

We also establish Young's inequality which is key to Hölder's inequality. Suppose a and b are positive reals and p, q be a nonzero reals such that $\frac{1}{p}+\frac{1}{q}=1$. We take $\alpha = 1/p$ and $x = a/b$. We get $\left(\frac{a}{b}\right)^{1/p} - \frac{1}{p}\left(\frac{a}{b}\right) + \frac{1}{p} - 1 \geq 0$ if $p < 1$ and ≤ 0 if $p > 1$. This reduces to

$$\begin{aligned} a^{1/p}b^{1/q} &\geq \frac{a}{p}+\frac{b}{q} \quad \text{if } p<1, \\ a^{1/p}b^{1/q} &\leq \frac{a}{p}+\frac{b}{q} \quad \text{if } p>1. \end{aligned}$$

Now replacing a by a^p and b by b^q, we get

$$\begin{aligned} ab &\geq \frac{a^p}{p}+\frac{b^q}{q} \quad \text{if } p<1, \\ ab &\leq \frac{a^p}{p}+\frac{b^q}{q} \quad \text{if } p>1, \end{aligned}$$

which is Young's inequality.

Example 2.14. Show that $e^x > 1+x+(x^2/2)$ for $x > 0$, and $e^x < 1+x+(x^2/2)$ for $x < 0$.

Solution: Let $f(x) = e^x - 1 - x - (x^2/2)$. We have

$$f'(x) = e^x - 1 - x, \quad f''(x) = e^x - 1.$$

Hence $f''(x) > 0$ for $x > 0$ and $f''(x) < 0$ for $x < 0$. This implies that $f'(x)$ is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$. We thus infer that $f'(x) > f'(0) = 0$ for all real $x \neq 0$. It follows that $f(x)$ is increasing on $(-\infty, 0)$ and $(0, \infty)$. This implies that $f(x) < f(0)$ for $x < 0$, and $f(x) > f(0)$ for $x > 0$. Since $f(0) = 0$, it follows that $f(x) > 0$ for $x > 0$, and $f(x) < 0$ for $x < 0$, which is the desired inequality. ■

Example 2.15. Prove *Jordan's* inequality that

$$\frac{2}{\pi} \leq \frac{\sin x}{x} \leq 1,$$

for all $x \in [0, \pi/2]$.

Solution: Consider $f(x) = \sin x$. Since $f'(x) = \cos x$ and $f''(x) = -\sin x$, we see that $f''(x) \leq 0$ on $[0, \pi/2]$. Hence $f(x)$ is a concave function on $[0, \pi/2]$. Take any $x \in [0, \pi/2]$ and set $\lambda = 2x/\pi$. Using the concavity of $f(x) = \sin x$, we have

$$\begin{aligned} \sin(x) &= \sin\big((1-\lambda)0 + \lambda\pi/2\big) \\ &\geq (1-\lambda)\sin\big(0\big) + \lambda \sin\big(\pi/2\big) \\ &= \lambda = \frac{2x}{\pi}. \end{aligned}$$

Thus we obtain
$$\frac{\sin x}{x} \geq \frac{2}{\pi}.$$
On the other hand, consider $g(x) = \sin x - x$ on $[0, \pi/2]$. We again note that $g'(x) = \cos x - 1 \leq 0$ for all $x \in [0, \pi/2]$. Hence g is a non-increasing function on $[0, \pi/2]$. This implies that $g(x) \leq g(0) = 0$. Thus we get, $\sin x \leq x$ for all $x \in [0, \pi/2]$. Using the known result that $\lim_{x \to 0} \sin x/x = 1$, we conclude that $\sin x/x \leq 1$ for all $x \in [0, \pi/2]$. Thus we have Jordan's inequality,
$$\frac{2}{\pi} \leq \frac{\sin x}{x} \leq 1,$$
valid for all $x \in [0, \pi/2]$. Here we have to take the limiting value for $x = 0$. ■

Example 2.16. Prove *Bernoulli's* inequality that for $x > -1$,
$$\begin{aligned} (1+x)^\alpha &> 1 + \alpha x, \quad \text{if } \alpha > 1 \text{ or } \alpha < 0, \\ (1+x)^\alpha &< 1 + \alpha x, \quad \text{if } 0 < \alpha < 1. \end{aligned}$$

Solution: Consider the function $f(x) = (1+x)^\alpha - 1 - \alpha x$ for $x > -1$ and $\alpha \neq 1$. Then we see that
$$f'(x) = \alpha\{(1+x)^{\alpha-1} - 1\}.$$
Suppose $\alpha > 1$. Since $x > -1$, we see that $(1+x)^{\alpha-1} > 1$ if $x > 0$, and $(1+x)^{\alpha-1} < 1$ for $-1 < x < 0$. Thus $f'(x) > 0$ if $x > 0$, and $f'(x) < 0$ if $-1 < x < 0$. This shows that $f(x)$ is decreasing on $(-1, 0)$, and increasing on $(0, \infty)$. Hence $f(x) > f(0)$ for all $x > -1$, $x \neq 0$. We obtain
$$(1+x)^\alpha > 1 + \alpha x \quad \text{if } x > -1,\ x \neq 0,\ \alpha > 1.$$
If $\alpha < 0$, then $1 - \alpha > 1$, and hence $(1+x)^{\alpha-1} - 1 < 0$ for $1 + x > 1$; and $(1+x)^{\alpha-1} - 1 > 0$ for $0 < 1 + x < 1$. Thus $f'(x) > 0$ for $1 + x > 1$, and $f'(x) < 0$ for $0 < 1+x < 1$. Hence $f(x)$ is decreasing in $(-1, 0)$, and increasing on $(0, \infty)$. We see that $f(x) > 0$ for $x > -1$, and $x \neq 0$ in this case. This gives
$$(1+x)^\alpha > 1 + \alpha x, \quad \text{if } x > -1,\ x \neq 0,\ \alpha < 0.$$
Consider the case in which $0 < \alpha < 1$. Then $\alpha - 1 < 0$. Hence $(1+x)^{\alpha-1} - 1 < 0$ for $x > 0$, and $(1+x)^{\alpha-1} - 1 > 0$ for $-1 < x < 1$. Thus, it follows that $f'(x) > 0$ on $(-1, 0)$, and $f'(x) < 0$ on $(0, \infty)$. We deduce that $f(x)$ is increasing on $(-1, 0)$, and decreasing on $(0, \infty)$. This gives $f(x) < f(0)$, and hence
$$(1+x)^\alpha < 1 + \alpha x, \quad \text{if } x > -1,\ x \neq 0,\ 0 < \alpha < 1.$$
■

Example 2.17. Let $a_1,a_2,a_3,\ldots,a_n$ be $n>1$ distinct real numbers. Prove that

$$\min_{1\le j<k\le n}\left(a_j-a_k\right)^2 \le \frac{12}{n(n^2-1)}\left(\sum_{j=1}^n a_j^2-\frac{1}{n}\left(\sum_{j=1}^n a_j\right)^2\right).$$

Solution: We first prove that

$$\min_{1\le j<k\le n}\left(a_j-a_k\right)^2 \le \frac{12}{n(n^2-1)}\left(\sum_{j=1}^n a_j^2\right), \tag{2.8}$$

for any n real numbers $a_1,a_2,a_3,\ldots,a_n$. We may assume $\sum_{j=1}^n a_j^2=1$. We may also enumerate a_j's such that

$$a=a_1\le a_2\le a_3\le\cdots\le a_n.$$

Put

$$\mu^2=\frac{12}{n(n^2-1)}.$$

Suppose the inequality is not true. Then $a_{j+1}-a_j>\mu$ for $1\le j\le n-1$. Thus

$$a_j>a+(j-1)\mu,$$

for $2\le j\le n$. Hence

$$\begin{aligned}
1=\sum_{j=1}^n a_j^2 &> \sum_{j=1}^n\left(a+(j-1)\mu\right)^2\\
&= \sum_{j=0}^{n-1}\left(a+j\mu\right)^2\\
&= na^2+2a\mu\sum_{j=1}^{n-1}j+\mu^2\sum_{j=1}^{n-1}j^2\\
&= na^2+a\mu n(n-1)+\mu^2\frac{n(n-1)(2n-1)}{6}\\
&= n\left(a+\frac{\mu(n-1)}{2}\right)^2+1\\
&> 1.
\end{aligned}$$

This contradiction proves the inequality (2.8).

We take $b_j=a_j+t$ where t is a real variable and apply the inequality (2.8) to the sequence $\langle b_j\rangle$ to get

$$\min_{1\le j<k\le n}\left(a_j-a_k\right)^2 \le \frac{12}{n(n^2-1)}\left(\sum_{j=1}^n\left(a_j+t\right)^2\right),$$

valid for all t. Consider the function $f(t) = \sum_{j=1}^{n} \left(a_j + t\right)^2$. Expanding this, we obtain

$$f(t) = \sum_{j=1}^{n} a_j^2 + 2\left(\sum_{j=1}^{n} a_j\right)t + nt^2,$$

which is a quadratic polynomial in t. This quadratic polynomial attains its extremum at t_0, where $f'(t_0) = 0$. Solving this, we see that

$$t_0 = -\frac{1}{n}\left(\sum_{j=1}^{n} a_j\right).$$

Since $f''(t_0) = 2n > 0$, we infer that f has a unique minimum at t_0. Hence $f(t) \geq f(t_0)$, for all t. But we see that

$$\begin{aligned} f(t_0) &= \sum_{j=1}^{n} a_j^2 - \frac{2}{n}\left(\sum_{j=1}^{n} a_j\right)^2 + \frac{1}{n}\left(\sum_{j=1}^{n} a_j\right)^2 \\ &= \sum_{j=1}^{n} a_j^2 - \frac{1}{n}\left(\sum_{j=1}^{n} a_j\right)^2. \end{aligned}$$

This implies that

$$\begin{aligned} \min_{1 \leq j < k \leq n} \left(a_j - a_k\right)^2 &\leq \frac{12}{n(n^2-1)} f(t_0) \\ &= \frac{12}{n(n^2-1)}\left(\sum_{j=1}^{n} a_j^2 - \frac{1}{n}\left(\sum_{j=1}^{n} a_j\right)^2\right). \end{aligned}$$

■

Example 2.18. Let a, b, x, y be real numbers such that $ay - bx = 1$. Prove that

$$a^2 + b^2 + x^2 + y^2 + ax + by \geq \sqrt{3}.$$

Solution: We can find u, v, θ, ϕ such that $a = u\cos\theta$, $b = u\sin\theta$, $x = v\cos\phi$ and $y = v\sin\phi$. Using $ay - bx = 1$, we infer that $uv\sin(\theta - \phi) = 1$. Hence $uv \geq 1$ (remember u and v are positive). We obtain

$$\cos^2(\theta - \phi) = 1 - \sin^2(\theta - \phi) = 1 - \frac{1}{u^2v^2} = \left(u^2v^2 - 1\right)/u^2v^2.$$

We observe that $u^2 = a^2 + b^2$, $v^2 = x^2 + y^2$ and $ax + by = uv\cos(\theta - \phi)$. Using $u^2 + v^2 \geq 2uv$, we obtain

$$a^2 + b^2 + x^2 + y^2 + ax + by \geq 2uv - \sqrt{u^2v^2 - 1}.$$

Thus, it is sufficient to prove that

$$2uv - \sqrt{u^2v^2 - 1} \geq \sqrt{3}.$$

Consider the function $f(t) = 2t - \sqrt{t^2 - 1}$ for $t \geq 1$. We have

$$f'(t) = 2 - \frac{t}{\sqrt{t^2 - 1}},$$

which gives $f'(t_0) = 0$ on $[1, \infty)$ if and only if $t_0 = 2/\sqrt{3}$. We see that

$$f''(t) = \frac{1}{\left(\sqrt{t^2 - 1}\right)^3}.$$

Hence $f''(t_0) = 3\sqrt{3} > 0$. This shows that f has a unique minimum at $t_0 = 2/\sqrt{3}$ on $[1, \infty)$. Thus we have

$$f(t) \geq f(t_0) = \sqrt{3}.$$

It follows that

$$2uv - \sqrt{u^2v^2 - 1} \geq \sqrt{3},$$

which is what we have to prove. ■

Another important technique for proving inequalities is to use the fact that between any two real zeros of a differentiable function, there is always a zero of its derivative. This follows from Rolle's theorem in calculus. We have already used this idea while proving Newton's inequality in an earlier chapter. Existence of real zeros for the derivative puts certain restrictions on the function and this would lead to inequalities.

Example 2.19. Prove that in a triangle with sides a, b, c, circum-radius R and in-radius r, the inequality

$$9r\left(4R + r\right) \leq 3s^2 \leq \left(4R + r\right)^2,$$

holds, where $s = (a + b + c)/2$ is the semi-perimeter.

Solution: Consider the monic polynomial whose roots are a, b, c. We have

$$\begin{aligned}
s^2 + r\left(4R + r\right) &= s^2 + 4Rr + r^2 \\
&= s^2 + \frac{abc}{\Delta} \cdot \frac{\Delta}{s} + \frac{\Delta^2}{s^2} \\
&= s^2 + \frac{abc}{s} + \frac{\Delta^2}{s^2} \\
&= \frac{s^3 + abc + (s - a)(s - b)(s - c)}{s} \\
&= ab + bc + ca,
\end{aligned}$$

and

$$4sRr = \frac{abc}{\Delta} \cdot \Delta = abc.$$

Here Δ denotes the area of the given triangle. Observe that we have used the well known Heron's formula: $\Delta^2 = s(s-a)(s-b)(s-c)$. Thus a, b, c are the roots of the equation

$$p(x) = x^3 - 2sx^2 + \Big(s^2 + r\big(4R + r\big)\Big)x - 4sRr = 0.$$

Now $p(x) = 0$ has three real roots. Hence Rolle's theorem shows that $p'(x) = 0$ has two real roots. But

$$p'(x) = 3x^2 - 4sx + s^2 + r\big(4R + r\big).$$

This is a quadratic equation and it has two real roots if and only if its discriminant is non-negative. Computing the discriminant, we obtain

$$16s^2 - 12\Big(s^2 + r\big(4R + r\big)\Big) \geq 0.$$

This leads to

$$9r\big(4R + r\big) \leq 3s^2.$$

We also know that

$$r_a = \frac{\Delta}{s - a}, \quad r = \frac{\Delta}{s}.$$

Hence

$$a = \frac{s(r_a - r)}{r_a}.$$

Similarly, we can get

$$b = \frac{s(r_b - r)}{r_b}, \quad c = \frac{s(r_c - r)}{r_c}.$$

Here r_a, r_b, r_c are the ex-radii of the triangle ABC.
Putting $x = s(y - r)/y$ in $p(x) = 0$, we obtain an equation in y;

$$q(y) = y^3 - \big(4R + r\big)y^2 + s^2y - s^2r = 0.$$

Observe that r_a, r_b, r_c are the roots of $q(y) = 0$. Hence $q'(y) = 0$ also has real roots. But

$$q'(y) = 3y^2 - 2\big(4R + r\big) + s^2,$$

and its discriminant is $4\big(4R + r\big)^2 - 12s^2$. We thus obtain the inequality

$$3s^2 \leq \big(4R + r\big)^2.$$

■

2.5 Trigonometric substitutions

Some of the inequalities may be proved using suitable trigonometric substitutions. We have already encountered one such use in Example 2.18. We shall study a few more such examples here.

Example 2.20. Let a, b, c be real numbers such that $0 \le a, b, c \le 1$. Prove that

$$\sqrt{a(1-b)(1-c)} + \sqrt{b(1-c)(1-a)} + \sqrt{c(1-a)(1-b)} \le 1 + \sqrt{abc}.$$

Solution: Since a, b, c are in $[0,1]$, we can find x, y, z in the interval $[0, \pi/2]$ such that

$$a = \sin^2 x, b = \sin^2 y, c = \sin^2 z.$$

Thus the inequality to be proved gets transformed to

$$\sin x \cos y \cos z + \sin y \cos z \cos x + \sin z \cos x \cos y \le 1 + \sin x \sin y \sin z.$$

This is equivalent to

$$\sin x\big(\cos y \cos z - \sin y \sin z\big) + \cos x\big(\sin y \cos z + \cos y \sin z\big) \le 1.$$

On simplification, this reduces to $\sin(x+y+z) \le 1$, which follows from the property of the *sine* function. ■

Example 2.21. Let x, y, z be positive real numbers such that $xy + yz + zx = 1$. Prove that

$$\frac{2x\big(1-x^2\big)}{\big(1+x^2\big)^2} + \frac{2y\big(1-y^2\big)}{\big(1+y^2\big)^2} + \frac{2z\big(1-z^2\big)}{\big(1+z^2\big)^2} \le \frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2}.$$

Solution: Consider the first term on the left side:

$$\frac{2x\big(1-x^2\big)}{\big(1+x^2\big)^2} = \frac{2x}{1+x^2} \cdot \frac{1-x^2}{1+x^2}.$$

We know that

$$\frac{2\tan(\alpha/2)}{1+\tan^2(\alpha/2)} = \sin\alpha, \qquad \frac{1-\tan^2(\alpha/2)}{1+\tan^2(\alpha/2)} = \cos\alpha.$$

Hence, this suggests the substitutions

$$x = \tan(\alpha/2), \quad y = \tan(\beta/2), \quad z = \tan(\gamma/2).$$

We recall that the function $x \mapsto \tan x$ is a one-one function of $(0, \pi/2)$ on to $(0, \infty)$. The inequality now transforms to

$$\sin\alpha\cos\alpha + \sin\beta\cos\beta + \sin\gamma\cos\gamma \le \frac{1}{2}\big(\sin\alpha + \sin\beta + \sin\gamma\big).$$

This is equivalent to

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma \leq \sin\alpha + \sin\beta + \sin\gamma.$$

We observe that

$$\begin{aligned} z = \frac{1-xy}{x+y} &= \frac{1-\tan(\alpha/2)\tan(\beta/2)}{\tan(\alpha/2)+\tan(\beta/2)} \\ &= \tan\left(\frac{\pi}{2}-\frac{\alpha}{2}-\frac{\beta}{2}\right). \end{aligned}$$

This shows that

$$\tan(\gamma/2) = \tan\left(\frac{\pi}{2}-\frac{\alpha}{2}-\frac{\beta}{2}\right).$$

Since α and β are in $(0,\pi/2)$, so is $\frac{\pi}{2}-\frac{\alpha}{2}-\frac{\beta}{2}$. Since $x \mapsto \tan x$ is one-one on $(0,\pi/2)$, we conclude that

$$\frac{\gamma}{2} = \frac{\pi}{2}-\frac{\alpha}{2}-\frac{\beta}{2}.$$

This gives $\alpha+\beta+\gamma=\pi$. Now $f(x) = -\sin x$ is convex on $(0,\pi)$. Suppose $\alpha \leq \beta \leq \gamma$ and let α', β', γ' be a permutation of α, β, γ. Using Theorem 10 on page 41, we obtain

$$-\sin\left(\alpha'+\alpha\right) - \sin\left(\beta'+\beta\right) - \sin\left(\gamma'+\gamma\right) \leq -\sin\left(2\alpha\right) - \sin\left(2\beta\right) - \sin\left(2\gamma\right).$$

Taking $\alpha' = \beta$, $\beta' = \gamma$ and $\gamma' = \alpha$, we obtain

$$\sin\left(2\alpha\right) + \sin\left(2\beta\right) + \sin\left(2\gamma\right) \leq \sin\left(\alpha+\beta\right) + \sin\left(\beta+\gamma\right) + \sin\left(\gamma+\alpha\right).$$

This reduces to

$$\sin\left(2\alpha\right) + \sin\left(2\beta\right) + \sin\left(2\gamma\right) \leq \sin\alpha + \sin\beta + \sin\gamma,$$

proving the inequality needed. ■

Example 2.22. Show that for any three real numbers a, b, c, the inequality

$$\left(ab+bc+ca-1\right)^2 \leq \left(a^2+1\right)\left(b^2+1\right)\left(c^2+1\right),$$

holds.

Solution: Put $a = \tan\alpha$, $b = \tan\beta$, $c = \tan\gamma$, where α, β, γ are in the interval $(-\pi/2, \pi/2)$. Then

$$a^2+1 = \sec^2\alpha, \quad b^2+1 = \sec^2\beta, \quad c^2+1 = \sec^2\gamma,$$

and the inequality to be proved is

$$\Big((ab+bc+ca-1)\cos\alpha\cos\beta\cos\gamma\Big)^2 \leq 1.$$

However, we observe that

$$\begin{aligned}(ab+bc)\cos\alpha\cos\beta\cos\gamma &= \sin\alpha\sin\beta\cos\gamma+\cos\alpha\sin\beta\sin\gamma\\ &= \sin\beta\sin(\alpha+\gamma),\\ (ca-1)\cos\alpha\cos\beta\cos\gamma &= \sin\alpha\cos\beta\sin\gamma-\cos\alpha\cos\beta\cos\gamma\\ &= -\cos\beta\cos(\alpha+\gamma).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\Big((ab+bc+ca-1)\cos\alpha\cos\beta\cos\gamma\Big)^2\\ &\qquad= \Big(\sin\beta\sin(\alpha+\gamma)-\cos\beta\cos(\alpha+\gamma)\Big)^2\\ &\qquad= \cos^2\big(\alpha+\beta+\gamma\big)\leq 1.\end{aligned}$$

■

Example 2.23. For any three positive real numbers x, y, z such that $x+y+z = xyz$, prove the inequality

$$\frac{x}{\sqrt{1+x^2}}+\frac{y}{\sqrt{1+y^2}}+\frac{z}{\sqrt{1+z^2}} \leq \frac{3\sqrt{3}}{2}.$$

Solution: Since $x+y+z = xyz$, there is a triangle with angles α, β, γ such that $x = \tan\alpha$, $y = \tan\beta$, $z = \tan\gamma$. Then the inequality reduces to

$$\sin\alpha+\sin\beta+\sin\gamma \leq \frac{3\sqrt{3}}{2}.$$

This follows from **3.4.5** on page 118. ■

2.6 Properties of quadratic polynomials

The fact that a quadratic equation $ax^2+bx+c = 0$ has real roots if and only if its discriminant is non-negative helps in establishing many inequalities. The classic case is the Cauchy-Schwarz inequality. Take two sequences $a_1, a_2, a_3, \ldots, a_n$ and $b_1, b_2, b_3, \ldots, b_n$ of real numbers. We associate the quadratic polynomial with these sequences by

$$f(t) = \sum_{j=1}^{n}\big(a_j+b_jt\big)^2 = A+2Ct+Bt^2,$$

where

$$A = \sum_{j=1}^{n} a_j^2, \quad B = \sum_{j=1}^{n} b_j^2, \quad C = \sum_{j=1}^{n} a_j b_j.$$

We may assume that $B \neq 0$, for, otherwise $b_j = 0$ for all j forcing $C = 0$, and in this case the inequality $C^2 \leq AB$ is trivially true. Thus $B > 0$. Since $f(t)$ is the sum of several squares, we have $f(t) \geq 0$ for all t. Hence its discriminant is non-positive. This implies that $C^2 \leq AB$ and gives the Cauchy-Schwarz inequality. We also observe that equality holds if and only if $f(t) = 0$ has two coincident real roots. This is equivalent to $a_j + \lambda b_j = 0$, for $1 \leq j \leq n$, where

$$\lambda = -\frac{\sum_{j=1}^{n} a_j b_j}{\sum_{j=1}^{n} b_j^2},$$

is the double root of $f(t) = 0$.

Example 2.24. Let (a_1, a_2), (b_1, b_2) and (c_1, c_2) be three pairs of real numbers. Prove that

$$\left(\sum a_1 b_2 + \sum a_2 b_1 - 2\sum a_1 a_2\right)^2 \leq 4\left(\sum a_1^2 - \sum a_1 b_1\right)\left(\sum a_2^2 - \sum a_2 b_2\right),$$

where the sum is cyclically over a, b, c.

Solution: We know that for all real x, y, z, the following inequality holds:

$$x^2 + y^2 + z^2 \geq xy + yz + zx.$$

In fact this is equivalent to $(x-y)^2 + (y-z)^2 + (z-x)^2 \geq 0$. Moreover equality holds if and only if $x = y = z$. Now consider the quadratic polynomial

$$P(t) = \sum_{\text{cyclic}} (a_1 + ta_2)^2 - \sum_{\text{cyclic}} (a_1 + ta_2)(b_1 + tb_2).$$

Using $x^2 + y^2 + z^2 \geq xy + yz + zx$, we observe that $P(t) \geq 0$ for all real t. Hence the discriminant of $P(t)$ must be non-positive. We can also write $P(t)$ in the form

$$\begin{aligned} P(t) \quad &= \quad \Big(\sum_{\text{cyclic}} a_2^2 - \sum_{\text{cyclic}} a_2 b_2\Big) t^2 \\ &\qquad + \Big(2\sum_{\text{cyclic}} a_1 a_2 - \sum_{\text{cyclic}} a_1 b_2 - \sum_{\text{cyclic}} a_2 b_1\Big) t \\ &\qquad + \Big(\sum_{\text{cyclic}} a_1^2 - \sum_{\text{cyclic}} a_1 b_1\Big). \end{aligned}$$

Thus we obtain

$$\Big(\sum_{\text{cyclic}} a_1 b_2 + \sum_{\text{cyclic}} a_2 b_1 - 2\sum_{\text{cyclic}} a_1 a_2\Big)^2$$
$$\le 4\Big(\sum_{\text{cyclic}} a_1^2 - \sum_{\text{cyclic}} a_1 b_1\Big)\Big(\sum_{\text{cyclic}} a_2^2 - \sum_{\text{cyclic}} a_2 b_2\Big).$$

Here equality holds if and only if

$$a_1 + ta_2 = b_1 + tb_2 = c_1 + tc_2,$$

where t is the coincident root

$$t = -\frac{2\sum_{\text{cyclic}} a_1 a_2 - \sum_{\text{cyclic}} a_1 b_2 - \sum_{\text{cyclic}} a_2 b_1}{\sum_{\text{cyclic}} a_2^2 - \sum_{\text{cyclic}} a_2 b_2}.$$

This reduces after simplification to

$$b_1 c_2 - b_2 c_1 + c_1 a_2 - c_2 a_1 + a_1 b_2 - a_2 b_1 = 0.$$

■

Example 2.25. Let $n > 2$ and $x_1, x_2, \dots, x_n$ be n real numbers. Put

$$p = \sum_{j=1}^{n} x_j, \quad q = \sum_{1 \le j < k \le n} x_j x_k.$$

Prove that

$$\frac{p}{n} - \frac{n-1}{n}\sqrt{p^2 - \frac{2nq}{n-1}} \le x_j \le \frac{p}{n} + \frac{n-1}{n}\sqrt{p^2 - \frac{2nq}{n-1}},$$

for all j.

Solution: Consider p and q:

$$\begin{aligned} p &= x_1 + (x_2 + x_3 + \cdots + x_n) \\ q &= x_1x_2 + x_1x_3 + \cdots + x_1x_n + \sum_{2 \le j < k \le n} x_j x_k. \end{aligned}$$

We see that

$$\begin{aligned} (p - x_1)^2 &= (x_2 + x_3 + \cdots + x_n)^2 \\ &\le (n-1)(x_2^2 + x_3^2 + \cdots + x_n^2) \\ &= (n-1)\Big\{(x_2 + x_3 + \cdots + x_n)^2 - 2\sum_{2 \le j < k \le n} x_j x_k\Big\} \\ &= (n-1)\Big\{(p - x_1)^2 - 2\big(q - x_1(p - x_1)\big)\Big\} \\ &= (n-1)(p^2 - 2q - x_1^2). \end{aligned}$$

Thus we get a quadratic inequality:

$$nx_1^2 - 2px_1 - (n-2)p^2 + 2(n-1)q \leq 0.$$

Solving this inequality, we get

$$\frac{p}{n} - \frac{n-1}{n}\sqrt{p^2 - \frac{2nq}{n-1}} \leq x_1 \leq \frac{p}{n} + \frac{n-1}{n}\sqrt{p^2 - \frac{2nq}{n-1}}.$$

Since we can choose any x_j in place of x_1, we get the desired inequality. ■

2.7 A useful transformation

Sometimes, an inequality involving the sides of a triangle can be converted to an inequality for positive real numbers and vice versa. This achieves a lot of simplification in solutions. In a triangle with sides a, b, c, we know that $a < b + c$, $b < c + a$ and $c < a + b$. Hence, if we introduce

$$2x = b + c - a, \quad 2y = c + a - b, \quad 2z = a + b - c,$$

then we see that x, y, z are positive reals and $a = y+z$, $b = z+x$, $c = x+y$. This transformation is often referred as *Ravi transformation* in the mathematical literature.

Example 2.26. (CRMO, 1999) Let a, b, c be the sides of a triangle. Prove that

$$\frac{a}{b+c-a} + \frac{b}{c+a-b} + \frac{c}{a+b-c} \geq 3.$$

Solution: Using the transformations described above, this reduces to

$$\frac{y+z}{x} + \frac{z+x}{y} + \frac{x+y}{z} \geq 6.$$

This follows by the AM-GM inequality:

$$\begin{aligned}
\frac{y+z}{x} + \frac{z+x}{y} + \frac{x+y}{z} &= \left(\frac{x}{y} + \frac{y}{x}\right) + \left(\frac{y}{z} + \frac{z}{y}\right) + \left(\frac{z}{x} + \frac{x}{z}\right) \\
&\geq 2 + 2 + 2 = 6.
\end{aligned}$$

■

Example 2.27. If a, b, c are the sides of a triangle, prove

$$(b+c-a)(c+a-b)(a+b-c) \leq abc.$$

Solution: Using x, y, z as described above, this reduces to the inequality

$$8xyz \le (x+y)(y+z)(z+x).$$

Using the AM-GM inequality, we see that

$$8xyz = 2\sqrt{xy} \cdot 2\sqrt{yz} \cdot 2\sqrt{zx} \le (x+y)(y+z)(z+x).$$

■

Example 2.28. (INMO-2003) Let a, b, c be the sides of a triangle ABC. Let $A'B'C'$ be the triangle whose sides are $a + \frac{b}{2}$, $b + \frac{c}{2}$, $c + \frac{a}{2}$. Prove that

$$[A'B'C'] \ge \frac{9}{4}[ABC].$$

Solution: It is easy to observe that there is a triangle with sides $a + \frac{b}{2}$, $b + \frac{c}{2}$, $c + \frac{a}{2}$. Using Heron's formula, we get

$$16[ABC]^2 = (a+b+c)(a+b-c)(b+c-a)(c+a-b),$$

and

$$16\left[A'B'C'\right]^2 = \frac{3}{16}(a+b+c)(-a+b+3c)(-b+c+3a)(-c+a+3b).$$

Since a, b, c are the sides of a triangle, there are positive real numbers x, y, z such that $a = y+z, b = z+x, c = x+y$. Using these, we obtain

$$\frac{[ABC]^2}{\left[A'B'C'\right]^2} = \frac{16xyz}{3(2x+y)(2y+z)(2z+x)}.$$

Thus it is sufficient to prove that

$$(2x+y)(2y+z)(2z+x) \ge 27xyz$$

for positive real numbers x, y, z. Using the AM-GM inequality, we get

$$2x + y \ge 3(x^2y)^{1/3}, 2y+z \ge 3(y^2z)^{1/3}, 2z + x \ge 3(z^2x)^{1/3}.$$

Multiplying these, we obtain the desired result. We also observe that equality holds if and only if $x = y = z$. This is equivalent to the statement that ABC is equilateral. ■

2.8 Schur's inequality

There is a nice inequality, due to *Schur*, which is often helpful to prove some results. It asserts that:

> *Let a, b, c be positive real numbers and let λ be any real number. Then*
>
> $$a^\lambda(a-b)(a-c)+b^\lambda(b-c)(b-a)+c^\lambda(c-a)(c-b) \geq 0,$$
>
> *with equality if and only if $a=b=c$.*

Proof: If any two of a, b, c are equal, the result is immediate. Hence we may assume that no two are equal. Because of the symmetry, we may assume $a>b>c$. If $\lambda \geq 0$, then

$$\begin{aligned} & a^\lambda(a-b)(a-c)+b^\lambda(b-c)(b-a)+c^\lambda(c-a)(c-b) \\ & \quad = \quad (a-b)\Big\{(a-c)a^\lambda-(b-c)b^\lambda\Big\}+(a-c)(b-c)c^\lambda \\ & \quad \geq \quad (a-b)(a-c)\Big\{a^\lambda-b^\lambda\Big\} \geq 0. \end{aligned}$$

If $\lambda<0$, then

$$\begin{aligned} & a^\lambda(a-b)(a-c)+b^\lambda(b-c)(b-a)+c^\lambda(c-a)(c-b) \\ & \quad = \quad a^\lambda(a-b)(a-c)+(b-c)\Big\{(a-c)c^\lambda-(a-b)b^\lambda\Big\} \\ & \quad \geq \quad (b-c)(a-c)\Big\{c^\lambda-b^\lambda\Big\} \geq 0. \end{aligned}$$

Example 2.29. (IMO, 2000) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) \leq 1.$$

Solution: Introduce $a=x/y$, $b=y/z$, $c=z/x$; this is possible since a, b, c are positive real numbers such that $abc=1$. The inequality reduces to

$$(z+x-y)(x+y-z)(y+z-x) \leq xyz.$$

Expanding the left hand side of the above inequality, we obtain

$$x(y-x)(x-z)+y(z-y)(y-x)+z(x-z)(z-y)+xyz.$$

Thus it is sufficient to prove that

$$x(y-x)(x-z)+y(z-y)(y-x)+z(x-z)(z-y) \leq 0.$$

This follows from **Schur's inequality**. (For different solutions refer to problem 3.6 on page 288.) ■

Example 2.30. (APMO, 2004) Let x, y, z be positive real numbers. Prove that

$$\left(x^2+2\right)\left(y^2+2\right)\left(z^2+2\right) \geq 9\left(xy+yz+zx\right).$$

Solution: Expanding, the inequality is

$$x^2y^2z^2 + 2\sum_{\text{cyclic}} x^2y^2 + 4\sum_{\text{cyclic}} x^2 + 8 \geq 9\sum_{\text{cyclic}} xy.$$

Note that

$$2\sum_{\text{cyclic}} x^2y^2 - 4\sum_{\text{cyclic}} xy + 6 = 2\sum_{\text{cyclic}} (xy-1)^2 \geq 0.$$

Moreover $\sum_{\text{cyclic}} x^2 \geq \sum_{\text{cyclic}} xy$. Thus it is sufficient to prove that

$$x^2y^2z^2 + \sum_{\text{cyclic}} x^2 + 2 \geq 2\sum_{\text{cyclic}} xy.$$

If a, b, c are positive reals, Schur's inequality gives

$$\begin{aligned} \sum_{\text{cyclic}} a^3 + 3abc &\geq \sum_{\text{cyclic}} a^2b + \sum_{\text{cyclic}} ab^2 \\ &= ab(a+b) + bc(b+c) + ca(c+a). \end{aligned}$$

Using $u + v \geq 2\sqrt{uv}$ for any positive reals u, v, we get

$$\sum_{\text{cyclic}} a^3 + 3abc \geq 2\sum_{\text{cyclic}} (ab)^{3/2}.$$

Taking $a = x^{2/3}$, $b = y^{2/3}$ and $z = c^{2/3}$, this reduces to

$$x^2 + y^2 + z^2 + 3\left(xyz\right)^{2/3} \geq 2(xy+yz+zx).$$

However, we observe that

$$x^2y^2z^2 + 2 \geq 3\left(xyz\right)^{2/3}.$$

In fact, this is equivalent to $t^3 + 2 \geq 3t$, where $t = \left(xyz\right)^{2/3}$; this follows from $t^3 - 3t + 2 = (t-1)^2(t+2) \geq 0$. Thus

$$\begin{aligned} x^2y^2z^2 + x^2 + y^2 + z^2 + 2 &\geq x^2 + y^2 + z^2 + 3\left(xyz\right)^{2/3} \\ &\geq 2(xy+yz+zx). \end{aligned}$$

■

Example 2.31. Let x, y, z be positive real numbers and define $p = x + y + z$, $q = xy + yz + zx$ and $r = xyz$. Then

$$(i)p^3 - 4pq + 9r \geq 0; \quad (ii)p^4 - 5p^2q + 4q^2 + 6pr \geq 0.$$

Solution: We use Schur's inequality that

$$x^t(x-y)(x-z)+y^t(y-x)(y-z)+z^t(z-x)(z-y)\geq 0$$

for any positive reals x, y, z and real number t. Taking $t=1$, we get

$$(x^3+y^3+z^3)-\sum_{\text{sym}} x^2y+3xyz\geq 0.$$

Using the known identities

$$(x+y+z)^3=x^3+y^3+z^3+3(x+y+z)(xy+yz+zx)-3xyz,$$

and

$$\sum_{\text{sym}} x^2y=(x+y+z)(xy+yz+zx)-3xyz,$$

we obtain

$$(x+y+z)^3-4(x+y+z)(xy+yz+zx)+9xyz\geq 0.$$

This proves (i). A similar proof works for (ii) with $t=2$ in Schur's inequality.

■

Example 2.32. Let a, b, c be non-negative real numbers such that $a+b+c=2$. Prove that

$$a^4+b^4+c^4+abc\geq a^3+b^3+c^3.$$

Solution: Using Schur's inequality, we have

$$a^2(a-b)(a-c)+b^2(b-c)(b-a)+c^2(c-a)(c-b)\geq 0.$$

This gives

$$\begin{aligned}a^4+b^4+c^4+abc(a+b+c) &\geq a^3(b+c)+b^3(c+a)c^3(a+b)\\ &= (a^3+b^3+c^3)(a+b+c)-a^4+b^4+c^4.\end{aligned}$$

This simplifies to

$$2(a^4+b^4+c^4)+abc(a+b+c)\geq(a^3+b^3+c^3)(a+b+c).$$

Since $a+b+c=2$, we obtain

$$a^4+b^4+c^4+abc\geq a^3+b^3+c^3.$$

■

Example 2.33. Suppose a, b, c are non-negative real numbers. Prove that

$$a^3+b^3+c^3+4(a+b+c)+abc\geq 8(ab+bc+ca).$$

Solution: Suppose we prove that

$$a^3 + b^3 + c^3 + 9abc \geq \frac{4(ab + bc + ca)^2}{a + b + c}.$$

Then we see that

$$\begin{aligned} a^3 + b^3 + c^3 + 4(a + b + c) + abc &\geq 4(a + b + c) + \frac{4(ab + bc + ca)^2}{a + b + c} \\ &\geq 8(ab + bc + ca), \end{aligned}$$

where we have used AM-GM inequality. Therefore it remains to prove

$$a^3 + b^3 + c^3 + 9abc \geq \frac{4(ab + bc + ca)^2}{a + b + c}.$$

This may be put in an equivalent form by clearing the denominators:

$$\begin{aligned} a^4 + b^4 + c^4 + 9abc(a + b + c) + \sum_{\text{cyclic}} a^3(b + c) \\ \geq 4\left(\sum_{\text{cyclic}} a^2b^2\right) + 8abc(a + b + c). \end{aligned}$$

Further reduction leads to

$$a^4 + b^4 + c^4 + abc(a + b + c) + \sum_{\text{cyclic}} ab(a^2 + b^2) \geq 4\left(\sum_{\text{cyclic}} a^2b^2\right).$$

We now use Schur's inequality:

$$a^2(a - b)(a - c) + b^2(b - c)(b - a) + c^2(c - a)(c - b) \geq 0.$$

Expansion gives

$$a^4 + b^4 + c^4 + abc(a + b + c) \geq \sum_{\text{cyclic}} a^3(b + c).$$

Therefore

$$\begin{aligned} a^4 + b^4 + c^4 + 9abc(a + b + c) + \sum_{\text{cyclic}} ab(a^2 + b^2) \\ \geq \sum_{\text{cyclic}} a^3(b + c) + \sum_{\text{cyclic}} ab(a^2 + b^2) \\ = 2\sum_{\text{cyclic}} ab(a^2 + b^2) \geq 2\sum_{\text{cyclic}} ab(2ab) = 4\sum_{\text{cyclic}} (a^2 + b^2). \end{aligned}$$

■

2.9 Majorisation technique

There is another useful technique, called the *majorisation technique*, often used to derive new inequalities. Let $\boldsymbol{x} = \big(x_1, x_2, \dots, x_n\big)$ and $\boldsymbol{y} = \big(y_1, y_2, \dots, y_n\big)$ be two vectors in $\mathbb{R}^n$ such that

$$x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n \quad \text{and} \quad y_1 \geq y_2 \geq y_3 \geq \cdots \geq y_n.$$

We say $\boldsymbol{x}$ is majorised by $\boldsymbol{y}$ and write $\boldsymbol{x} \prec \boldsymbol{y}$ if

(i) $x_1 + x_2 + x_3 + \cdots + x_k \leq y_1 + y_2 + y_3 + \cdots + y_k$ for $1 \leq k \leq n-1$;

(ii) $x_1 + x_2 + x_3 + \cdots + x_n = y_1 + y_2 + y_3 + \cdots + y_n$.

Let A be a subset of $\mathbb{R}^n$. A function $f : A \to \mathbb{R}$ is said to be Schur-convex if

$$\boldsymbol{x} \prec \boldsymbol{y} \Longrightarrow f\big(\boldsymbol{x}\big) \leq f\big(\boldsymbol{y}\big).$$

If the inequality is reversed, we say f is Schur-concave. It is clear that f is Schur-concave if and only if $-f$ is Schur-convex.

There is a useful criterion for checking whether a given function is Schur-convex, at least when the function is reasonably smooth. Let

$$\mathbf{I}^n = (a, b) \times (a, b) \times \cdots \times (a, b),$$

be an 'interval' in $\mathbb{R}^n$. Suppose $f : \mathbf{I}^n \to \mathbb{R}$ has partial derivatives of the first order. Then f is Schur-convex if and only if f is symmetric in the variables and

$$\left(x_j - x_k\right)\left(\frac{\partial f(\boldsymbol{x})}{\partial x_j} - \frac{\partial f(\boldsymbol{x})}{\partial x_k}\right) \geq 0,$$

on $\mathbf{I}^n$, for all $j \neq k$. Here are some useful classes of Schur-convex functions.

(i) If $g : (a, b) \to \mathbb{R}$ is a convex function, then

$$f\big(x_1, x_2, x_3, \dots, x_n\big) = \sum_{j=1}^{n} g\big(x_j\big)$$

is Schur-convex on $\mathbf{I}^n$. This leads to a standard majorisation inequality: if $f : (a, b) \to \mathbb{R}$ is convex, then

$$\big(x_1, x_2, x_3, \dots, x_n\big) \prec \big(y_1, y_2, y_3, \dots, y_n\big) \Longrightarrow \sum_{j=1}^{n} f\big(x_j\big) \leq \sum_{j=1}^{n} f\big(y_j\big).$$

(ii) A function $f : \mathbf{I}^n \to \mathbb{R}$ is said to be convex if

$$f\Big(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}\Big) \leq \lambda f\Big(\boldsymbol{x}\Big) + (1-\lambda) f\Big(\boldsymbol{y}\Big),$$

for all vectors $\boldsymbol{x}, \boldsymbol{y}$ in $\mathbf{I}^n$, and $\lambda \in [0, 1]$. Any convex function is Schur-convex.

(iii) A function $f : \mathbf{I}^n \to \mathbb{R}$ is said to be quasi-convex if

$$f\Big(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}\Big) \le \max\Big\{f(\boldsymbol{x}), f(\boldsymbol{y})\Big\},$$

for all vectors $\boldsymbol{x}, \boldsymbol{y}$ in $\mathbf{I}^n$, and $\lambda \in [0,1]$. Any quasi-convex function is Schur-convex.

Example 2.34. If a, b, c are the sides of a triangle, prove that

$$\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \le \sqrt{a} + \sqrt{b} + \sqrt{c}.$$

Solution: Suppose $a \ge b \ge c$. Then

$$\begin{aligned} 2(s-c) &\ge a, \\ 2(s-c) + 2(s-b) &\ge a+b, \\ 2(s-b) + 2(s-a) + 2(s-c) &= a+b+c. \end{aligned}$$

Thus

$$\big(a, b, c\big) \prec \big(2(s-c), 2(s-b), 2(s-a)\big).$$

Since $f(t) = \sqrt{t}$ is concave on $(0, \infty)$, using majorisation theorem,

$$\sqrt{2(s-c)} + \sqrt{2(s-b)} + \sqrt{2(s-a)} \le \sqrt{a} + \sqrt{b} + \sqrt{c}.$$

This reduces to

$$\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \le \sqrt{a} + \sqrt{b} + \sqrt{c}.$$

■

Example 2.35. Let $a_1, a_2, a_3, \ldots, a_n$ be n natural numbers such that

$$a_1 \le a_2 \le a_3 \le \cdots \le a_n, \text{ and } a_1 + a_2 + a_3 + \cdots + a_n = nk + m,$$

where k, m are non-negative integers and $0 \le m < n$. Prove that

$$a_1 a_2 a_3 \cdots a_n \le (k+1)^m k^{n-m}.$$

Solution: Consider the vectors

$$\big(a_n, a_{n-1}, a_{n-2}, \ldots, a_1\big) \text{ and } \big(k+1, k+1, \ldots, k+1, k, k, \ldots, k\big),$$

where $k+1$ appears m times and k appears $n-m$ times. If $m > 0$, then we observe that $a_n \ge k+1$. Otherwise $a_n \le k$ and hence $\sum a_j \le nk$, a contradiction. Similarly, it is easy to see that if $m > 1$, then $a_n + a_{n-1} \ge 2(k+1)$. An easy induction proves that

$$a_n + a_{n-1} + \cdots + a_{n-m+1} \ge m(k+1).$$

Similarly,

$$a_n + a_{n-1} + \cdots + a_{n-m+1} + a_{n-m} + \cdots + a_{n-m-j} \geq m(k+1) + (j+1)k,$$

for $0 \leq j \leq n-m-1$. It follows that

$$\big(k+1, k+1 \ldots, k+1, k, k, \ldots, k\big) \prec \big(a_n, a_{n-1}, a_{n-2}, \ldots, a_1\big).$$

Since $f(x) = \ln x$ is concave on $(0, \infty)$, we get

$$\sum_{j=1}^{n} \ln a_j \leq m \ln(k+1) + (n-m) \ln k.$$

This simplifies to

$$a_1 a_2 a_3 \cdots a_n \leq (k+1)^m k^{n-m}.$$

■

Example 2.36. Show that in any triangle with sides a, b, c,

$$(a+b-c)(b+c-a)(c+a-b) \leq abc.$$

Solution: If we take $a \geq b \geq c$, then it is easy to see that

$$(a, b, c) \prec \big(2(s-c), 2(s-b), 2(s-a)\big),$$

where $s = (a+b+c)/2$ is the semi-perimeter of the triangle. Using the concave function $f(x) = \ln x$, we get

$$\ln 2(s-c) + \ln 2(s-b) + \ln 2(s-a) \leq \ln a + \ln b + \ln c.$$

This simplifies to

$$(a+b-c)(b+c-a)(c+a-b) \leq abc.$$

■

2.10 Muirhead's theorem

There is a classical theorem which was discovered by Muirhead. Consider a vector $\boldsymbol{a} = \big(a_1, a_2, a_3, \ldots, a_n\big)$ in $\mathbb{R}^n$. For each positive vector $\boldsymbol{x} = \big(x_1, x_2, x_3, \ldots, x_n\big)$, i.e., vector with positive components, consider the sum

$$S\big(\boldsymbol{a}; \boldsymbol{x}\big) = \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)}^{a_1} x_{\sigma(2)}^{a_2} \cdots x_{\sigma(n)}^{a_n},$$

where S_n denotes the set of all permutations of $\{1, 2, 3, \ldots, n\}$. Then Muirhead's theorem asserts that

$$S\big(\boldsymbol{a}; \boldsymbol{x}\big) \leq S\big(\boldsymbol{b}; \boldsymbol{x}\big)$$

for all positive vectors $\boldsymbol{x}$ in $\mathbb{R}^n$ if and only if $\boldsymbol{a} \prec \boldsymbol{b}$. For a proof of this interesting theorem, please refer to [3].

For example $(2,1,1) \prec (3,1,0)$. Hence Muirhead's theorem gives

$$2\left(x^2yz + y^2zx + z^2xy\right) \le x^3y + y^3z + z^3x + xy^3 + yz^3 + zx^3,$$

for all non-negative numbers x, y, z. Similarly $(1,1,1) \prec (3,0,0)$ and hence $3xyz \le x^3 + y^3 + z^3$ for all non-negative reals x, y, z. In fact, it is easy to derive the AM-GM inequality. Observe that for any positive integer n

$$\left(1/n, 1/n, \ldots, 1/n\right) \prec \left(1, 0, 0, \ldots, 0\right).$$

Hence for any n non-negative real numbers a_1,a_2,a_3,...,a_n, we obtain

$$n\left(a_1a_2\cdots a_n\right)^{1/n} \le a_1 + a_2 + \cdots + a_n.$$

Example 2.37. Show that for any positive reals a, b, c, the inequality

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \ge a + b + c,$$

holds.

Solution: We observe that the inequality is equivalent to

$$a^2bc + ab^2c + abc^2 \le a^4 + b^4 + c^4.$$

However $(2,1,1) \prec (4,0,0)$ since $2 < 4$, $2+1 < 4+0$ and $2+1+1 = 4+0+0$. Hence Muirhead's theorem at once gives the result. ■

Example 2.38. Let a, b, c be non-negative real numbers. Prove that

$$a^3 + b^3 + c^3 + abc \ge \frac{1}{7}(a+b+c)^3.$$

Solution: Expanding the right side, the inequality may be written in an equivalent form:

$$6\left(a^3 + b^3 + c^3\right) + abc \ge 3\left(a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2\right).$$

Note that

$$2\left(a^3 + b^3 + c^3\right) \ge a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2$$

by Muirhead's theorem. Hence the result follows. ■

2.11 Homogenisation

There is a large class of inequalities which are non-homogeneous and come with constraints. In many cases, the given constraint may be used to put the inequality in a homogeneous form and then one may use results like Muirhead's theorem or Schur's inequality to prove them.

Example 2.39. Let x, y, z be positive real numbers such that $x + y + z = 1$. Prove that

$$2\left(x^2 + y^2 + z^2\right) + 9xyz \geq 1.$$

Solution: We put this in a homogeneous form using $x + y + z = 1$:

$$2\left(x + y + z\right)\left(x^2 + y^2 + z^2\right) + 9xyz \geq \left(x + y + z\right)^3.$$

Expanding, we get

$$2\sum_{\text{cyclic}} x^3 + 2\sum_{\text{cyclic}} xy^2 + 2\sum_{\text{cyclic}} x^2y + 9xyz \geq \sum_{\text{cyclic}} x^3 + 3\sum_{\text{cyclic}} xy^2 + 3\sum_{\text{cyclic}} x^2y + 6xyz.$$

This may be written in the form

$$\sum_{\text{cyclic}} x^3 - \sum_{\text{cyclic}} xy^2 - \sum_{\text{cyclic}} x^2y + 3xyz \geq 0.$$

Equivalently,

$$\sum_{\text{cyclic}} x(x - y)(x - z) \geq 0,$$

which follows from Schur's inequality. ■

Example 2.40. (Tournament of Towns, 1997) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a + b + 1} + \frac{1}{b + c + 1} + \frac{1}{c + a + 1} \leq 1.$$

Solution: We introduce $a = x^3$, $b = y^3$ and $c = z^3$ so that x, y, z are positive real numbers such that $xyz = 1$. We may now write the inequality in a homogeneous form:

$$\frac{1}{x^3 + y^3 + xyz} + \frac{1}{y^3 + z^3 + xyz} + \frac{1}{z^3 + x^3 + xyz} \leq \frac{1}{xyz}.$$

Equivalently, we have

$$xyz\sum_{\text{cyclic}}\left(x^3 + y^3 + xyz\right)\left(y^3 + z^3 + xyz\right) \leq \prod_{\text{cyclic}}\left(x^3 + y^3 + xyz\right).$$

But the left side reduces to

$$xyz\left(3\sum_{\text{cyclic}} x^3y^3 + 4\sum_{\text{cyclic}} x^4yz + \sum_{\text{cyclic}} x^6 + 3x^2y^2z^2\right).$$

And the right side is

$$3x^3y^3z^3 + 2x^2y^2z^2\sum_{\text{cyclic}} x^3 + xyz\sum_{\text{cyclic}} x^6 \\ + 3xyz\sum_{\text{cyclic}} x^3y^3 + \sum_{\text{cyclic}} x^6y^3 + \sum_{\text{cyclic}} x^3y^6.$$

The inequality is equivalent to

$$2\sum_{\text{cyclic}} x^5y^2z^2 \le \sum_{\text{cyclic}} x^6y^3 + \sum_{\text{cyclic}} x^3y^6.$$

This may be written as

$$S(5,2,2) \le S(6,3,0).$$

Since $(5,2,2) \prec (6,3,0)$, the result follows from Muirhead's theorem. ■

Example 2.41. (IMO, 1984) Let x, y, z be three non-negative real numbers such that $x + y + z = 1$. Prove that

$$0 \le xy + yz + zx - 2xyz \le \frac{2}{27}.$$

Solution: Here we give a proof using homogenisation. For a different proof, see solution to problem 3.6 on page 291.

Using $x + y + z = 1$, we put the inequality in a homogeneous form:

$$0 \le (x+y+z)(xy+yz+zx) - 2xyz \le \frac{7}{27}(x+y+z)^3.$$

Observe that

$$(x+y+z)(xy+yz+zx) - 2xyz = \sum_{\text{cyclic}} x^2y + \sum_{\text{cyclic}} xy^2 + xyz.$$

Hence the left-side inequality follows. On the other hand

$$(x+y+z)^3 = \sum_{\text{cyclic}} x^3 + 3\sum_{\text{cyclic}} x^2y + 3\sum_{\text{cyclic}} xy^2 + 6xyz.$$

Hence the inequality reduces to

$$6\sum_{\text{cyclic}} x^2y + \sum_{\text{cyclic}} xy^2 \le 7\sum_{\text{cyclic}} x^3 + 15xyz.$$

We may put it in the form

$$\left(2\sum_{\text{cyclic}} x^3 - \sum_{\text{cyclic}} x^2y - \sum_{\text{cyclic}} xy^2\right) + 5\left(\sum_{\text{cyclic}} x(x-y)(x-z)\right) \geq 0.$$

Since $(2,1,0) \prec (3,0,0)$, Muirhead's theorem gives

$$2\sum_{\text{cyclic}} x^3 - \sum_{\text{cyclic}} x^2y - \sum_{\text{cyclic}} xy^2 \geq 0.$$

By Schur's inequality, we get

$$\sum_{\text{cyclic}} x(x-y)(x-z) \geq 0.$$

Hence the result follows.

■

Example 2.42. (USSR, 1962) Let a, b, c, d be non-negative real numbers such that $abcd = 1$. Prove that

$$a^2 + b^2 + c^2 + d^2 + ab + ac + ad + bc + bd + cd \geq 10.$$

Solution: The result at once follows from the AM-GM inequality. But we give here a different solution using homogenisation. We may homogenise this to get an equivalent inequality:

$$\left(a^2 + b^2 + c^2 + d^2 + ab + ac + ad + bc + bd + cd\right)^2 \geq 100abcd.$$

For a given 4-tuple (m_1, m_2, m_3, m_4) of non-negative integers, we use the notation:

$$S\left(m_1, m_2, m_3, m_4\right) = \frac{1}{24}\sum_{\sigma \in S_4} a^{m_{\sigma(1)}} b^{m_{\sigma(2)}} c^{m_{\sigma(3)}} d^{m_{\sigma(4)}},$$

where S_4 is the set of all permutations of $\{1, 2, 3\}$. Expanding the left-side, we may write the inequality in the form

$$4S(4,0,0,0) + 24S(3,1,0,0) + 18S(2,2,0,0) + 48S(2,1,1,0) \geq 94S(1,1,1,1).$$

Now Muirhead's theorem gives

$$S(4,0,0,0) \geq S(3,1,0,0) \geq S(2,2,0,0) \geq S(2,1,1,0) \geq S(1,1,1,1).$$

Hence

$$\begin{aligned} 4S(4,0,0,0) + 24S(3,1,0,0) + 18S(2,2,0,0) + 48S(2,1,1,0) \\ \geq (4 + 24 + 18 + 48)S(1,1,1,1) = 94S(1,1,1,1). \end{aligned}$$

This gives the required inequality. ■

2.12 Normalisation

Yet another technique used while proving inequalities is normalisation. This is the reverse process to homogenisation. Many times homogeneous inequalities may be normalised to simplify the proofs. The standard application is the AM-GM inequality. We have to prove that

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{1/n},$$

for non-negative real numbers $a_1, a_2, \ldots, a_n$. Since it is homogeneous, we may normalise it by $a_1 a_2 a_3 \cdots a_n = 1$. Thus we have to prove that

$$a_1 + a_2 + \cdots + a_n \geq n,$$

for any n positive real numbers $a_1, a_2, \ldots, a_n$, under the additional condition that $a_1 a_2 a_3 \cdots a_n = 1$. This may be proved by the principle induction. For $n = 1$, it is immediate. If $n = 2$, then

$$a_1 + a_2 - 2 = a_1 + a_2 - 2\sqrt{a_1 a_2} = \left(\sqrt{a_1} - \sqrt{a_2}\right)^2 \geq 0.$$

Suppose it holds for any k positive real numbers whose product is 1, where $k < n$. Take n positive real numbers $a_1, a_2, \ldots, a_n$ such that $a_1 a_2 a_3 \cdots a_n = 1$. Among these n numbers, there must be some number ≥ 1 and there must be some number ≤ 1. Thus we may assume that $a_1 \geq 1 \geq a_2$. We may apply induction to $n - 1$ numbers $a_1 a_2, a_3, \ldots, a_n$:

$$a_1 a_2 + a_3 + \cdots a_n \geq n - 1.$$

Thus it is enough to prove that $a_1 + a_2 \geq 1 + a_1 a_2$. But this is equivalent to $(a_1 - 1)(a_2 - 1) \leq 0$ which is a consequence of $a_1 \geq 1 \geq a_2$. Hence the proof is complete.

Example 2.43. (INMO, 2007) Let x, y, z be positive real numbers. Prove that

$$(x + y + z)^2 \left(yz + zx + xy\right)^2 \leq 3\left(y^2 + yz + z^2\right)\left(z^2 + zx + x^2\right)\left(x^2 + xy + y^2\right).$$

Solution: The inequality is homogeneous of degree 6. Hence we may assume $x + y + z = 1$. Let $\alpha = xy + yz + zx$. We see that

$$\begin{aligned} x^2 + xy + y^2 &= (x + y)^2 - xy \\ &= (x + y)(1 - z) - xy \\ &= x + y - \alpha = 1 - z - \alpha. \end{aligned}$$

Thus

$$\begin{aligned} \prod (x^2 + xy + y^2) &= (1 - \alpha - z)(1 - \alpha - x)(1 - \alpha - y) \\ &= (1 - \alpha)^3 - (1 - \alpha)^2 + (1 - \alpha)\alpha - xyz \\ &= \alpha^2 - \alpha^3 - xyz. \end{aligned}$$

Thus we need to prove that $\alpha^2 \leq 3(\alpha^2 - \alpha^3 - xyz)$. This reduces to

$$3xyz \leq \alpha^2(2 - 3\alpha).$$

However

$$3\alpha = 3(xy + yz + zx) \leq (x + y + z)^2 = 1,$$

so that $2 - 3\alpha \geq 1$. Thus it suffices to prove that $3xyz \leq \alpha^2$. But

$$\begin{aligned} \alpha^2 - 3xyz &= (xy + yz + zx)^2 - 3xyz(x + y + z) \\ &= \sum_{\text{cyclic}} x^2y^2 - xyz(x + y + z) \\ &= \frac{1}{2} \sum_{\text{cyclic}} (xy - yz)^2 \geq 0. \end{aligned}$$

For different solutions, see solution to problem 3.6 on page 445. ■

Example 2.44. (IMO, 2001) Prove that

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1$$

for all positive real numbers a, b and c.

Solution: Here we give a solution using normalisation $a + b + c = 1$. We use the convexity of $f(t) = \frac{1}{\sqrt{t}}$. Using the weighted Jensen inequality, we get

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq \frac{1}{\sqrt{a(a^2 + 8bc) + b(b^2 + 8ca) + c(c^2 + 8ab)}}.$$

Since $f(t) = \frac{1}{\sqrt{t}}$ is strictly decreasing and $f(1) = 1$, it suffices to prove that

$$a(a^2 + 8bc) + b(b^2 + 8ca) + c(c^2 + 8ab) \leq 1.$$

For this we go back to homogenisation:

$$a(a^2 + 8bc) + b(b^2 + 8ca) + c(c^2 + 8ab) \leq (a + b + c)^3.$$

But this reduces to

$$3 \sum_{\text{cyclic}} a^2b + 3 \sum_{\text{cyclic}} ab^2 \geq 18abc.$$

This follows from the AM-GM inequality (or one can use Muirhead's theorem).

(For a different normalisation and a generalisation, refer to problem 3.6 on page 401.) ■

2.13 Stolarsky's theorem

Suppose we have a homogeneous polynomial of degree 3 in three variables. We have already encountered many such polynomials: for example, $x^3+y^3+z^3-3xyz$ or $2(x^3+y^3+z^3)-(x^2y+y^2z+z^2x+xy^2+yz^2+zx^2)$. There is a beautiful result which gives us conditions under which such polynomials take only non-negative values for all non-negative real numbers x, y, z.

Theorem 14. Let $P(x,y,z)$ be a homogeneous polynomial of degree 3 in three variables. Then the following statements are equivalent:

(i) $P(1,1,1) \geq 0$, $P(1,1,0) \geq 0$ and $P(1,0,0) \geq 0$;

(ii) $P(\lambda,\mu,\nu) \geq 0$ for all non-negative reals λ, μ, ν.

Proof: It is sufficient to prove that (i) implies (ii). Let us put

$$P(x,y,z) = A\sum_{\text{cyclic}} x^3 + B\Big(\sum_{\text{cyclic}} x^2y + \sum_{\text{cyclic}} xy^2\Big) + Cxyz.$$

We also introduce

$$\begin{aligned} p &= P(1,1,1) = 3A+6B+C, \\ 2q &= P(1,1,0) = 2A+2B, \\ r &= P(1,0,0) = A. \end{aligned}$$

It is easy to get

$$A = r, \quad B = q-r, \quad C = p-6q+3r.$$

We may write $P(x,y,z)$ in terms of p, q, r:

$$P(x,y,z) = r\sum_{\text{cyclic}} x^3 + (q-r)\Big(\sum_{\text{cyclic}} x^2y + \sum_{\text{cyclic}} xy^2\Big) + (p-6q+3r)xyz.$$

Here we consider two cases.

Case 1: Suppose $q \geq r$. In this case we write

$$\begin{aligned} P(x,y,z) &= r\Big(\sum_{\text{cyclic}} x^3 - 3xyz\Big) \\ &\quad + \Big(q-r\Big)\Big(\sum_{\text{cyclic}} x^2y + \sum_{\text{cyclic}} xy^2 - 6xyz\Big) + pxyz. \end{aligned}$$

Since $r \geq 0$, $p \geq 0$ and $q > r$, the result follows.

Case 2: Suppose $q \le r$. We write

$$P(x,y,z) = q\Big(\sum_{\text{cyclic}} x^3 - 3xyz\Big) + \Big(r-q\Big)\Big(\sum_{\text{cyclic}} x^3 - \sum_{\text{cyclic}} x^2y - \sum_{\text{cyclic}} xy^2 + 3xyz\Big) + pxyz.$$

Again the result follows from $q \ge 0$, $p \ge 0$, $r \ge q$, and Schur's inequality. ■

As a consequence of this, here is a striking result which is due to Stolarsky.

Theorem 15. (Stolarsky's theorem) Let $P(x,y,z)$ be a symmetric homogeneous form of degree 3:

$$P(x,y,z) = \sum_{\text{sym}} \big(px^3 + qx^2y + rxyz\big),$$

where p, q, r are real numbers. Suppose $P(1,1,1) \ge 0$, $P(1,1,0) \ge 0$ and $P(2,1,1) \ge 0$. Then $P(a,b,c) \ge 0$, whenever a, b, c are the sides of a triangle.

Proof: Let a, b, c be the sides of a triangle. Then we can find positive reals x, y, z such that

$$a = y+z, \quad b = z+x, \quad c = x+y.$$

It is easy to compute

$$\begin{aligned} P(1,1,1) &= 6p+6q+6r \ge 0, \\ P(1,1,0) &= 4p+2q \ge 0, \\ P(2,1,1) &= 20p+14q+12r \ge 0. \end{aligned}$$

Note that

$$\begin{aligned} P(x,y,z) &= 2p\big(x^3+y^3+z^3\big) \\ &\quad +q\big(x^2y+y^2z+z^2x+xy^2+yz^2+zx^2\big)+6rxyz. \end{aligned}$$

Hence some computation leads to

$$\begin{aligned} P(a,b,c) &= (4p+2q)\Big(\sum_{\text{cyclic}} x^3\Big) + (6p+5q+6r)\Big(\sum_{\text{cyclic}} x^2y + \sum_{\text{cyclic}} xy^2\Big) \\ &\quad +12(q+r)xyz \\ &= Q(x,y,z). \end{aligned}$$

Observe that

$$\begin{aligned} Q(1,1,1) &= 48(p+q+r) \ge 0 \\ Q(1,1,0) &= 20p+14q+12r \ge 0, \\ Q(1,0,0) &= 4p+2q \ge 0. \end{aligned}$$

Hence $Q(x, y, z) \geq 0$ for all non-negative real numbers x, y, z. It follows that $P(a, b, c) \geq 0$. ■

Example 2.45. Let a, b, c be the sides of a triangle. Prove that

$$3(a+b)(b+c)(c+a) \leq 8\big(a^3+b^3+c^3\big).$$

Solution: Consider the polynomial

$$P(a, b, c) = 8\big(a^3+b^3+c^3\big) - 3(a+b)(b+c)(c+a).$$

This is a homogeneous polynomial of degree 3 in the variables a, b, c. Observe that

$$\begin{aligned} P(1,1,1) &= 24-18-6 \geq 0, \\ P(1,1,0) &= 16-6 = 10 > 0, \\ P(2,1,1) &= 80-42-12 = 16 > 0. \end{aligned}$$

By Stolarsky's theorem, $P(a, b, c) \geq 0$, for all a, b, c which are the sides of a triangle. (In fact we can apply theorem 14 since $P(1, 0, 0) = 8 > 0$ and hence the result is true for positive a, b, c.) ■

Example 2.46. Let a, b, c be the sides of a triangle. Prove that

$$2(a+b+c)\big(a^2+b^2+c^2\big) \geq 3\big(a^3+b^3+c^3+3abc\big)$$

Solution: If we take

$$P(x, y, z) = 2(x+y+z)\big(x^2+y^2+z^2\big) - 3\big(x^3+y^3+z^3+3xyz\big),$$

then $P(x, y, z)$ is a homogeneous polynomial of degree 3 in three variables. Moreover, $P(1, 1, 1) = 0$, $P(1, 1, 0) = 2$ and $P(2, 1, 1) = 0$. Hence Stolarsky's theorem is applicable and we conclude that $P(a, b, c) \geq 0$ whenever a, b, c are the sides of a triangle. (Note that we cannot apply theorem 14 directly, since $P(1, 0, 0) = -1 < 0$.) ■

2.14 Methods for symmetric inequalities

In this section we deal with more inequalities connecting elementary symmetric functions and their application.

2.14.1 Use of elementary symmetric functions

For any non-negative real numbers x, y, z, we introduce their elementary symmetric polynomials:

$$p = x+y+z, \quad q = xy+yz+zx, \quad r = xyz.$$

We prove several identities involving symmetric functions.

1. $x^2+y^2+z^2 = (x+y+z)^2 - 2(xy+yz+zx) = p^2-2q.$

2. $$\begin{aligned} x^3+y^3+z^3 &= (x+y+z)^3 - 3(x+y+z)(xy+yz+zx) + 3xyz \\ &= p^3 - 3pq + 3r. \end{aligned}$$

3. $x^2y^2+y^2z^2+z^2x^2 = (xy+yz+zx)^2 - 2xyz(x+y+z) = q^2-2pr.$

4. $$\begin{aligned} x^4+y^4+z^4 &= (x^2+y^2+z^2)^2 - 2(x^2y^2+y^2z^2+z^2x^2) \\ &= \left(p^2-2q\right)^2 - 2\left(q^2-2pr\right) \\ &= p^4 - 2p^2q + 2q^2 + 4pr. \end{aligned}$$

5. $$\begin{aligned} &(x+y)(y+z)(z+x) \\ &\quad= (p-x)(p-y)(p-z) \\ &\quad= p^3 - p^2(x+y+z) + p(xy+yz+zx) - xyz \\ &\quad= pq+r. \end{aligned}$$

6. $$\begin{aligned} &(x+y)(y+z)+(y+z)(z+x)+(z+x)(z+y) \\ &\quad= (p-z)(p-x)+(p-x)(p-y)+(p-y)(p-z) \\ &\quad= 3p^2 - 2p(x+y+x) + (xy+yz+zx) \\ &\quad= p^2+q. \end{aligned}$$

7. $$\begin{aligned} &(x+y)^2(y+z)^2+(y+z)^2(z+x)^2+(z+x)^2(x+y)^2 \\ &\quad= \big((x+y)(y+z)+(y+z)(z+x)+(z+x)(z+y)\big)^2 \\ &\qquad\quad - 2(x+y)(y+z)(z+x)\big(2(x+y+z)\big) \\ &\quad= (p^2-q)^2 - 4p(pq-r). \end{aligned}$$

8. $$\begin{aligned} &xy(x+y)+yz(y+z)+zx(z+x) \\ &\quad= xy(p-z)+yz(p-x)+zx(p-y) \\ &\quad= p(xy+yz+zx) - 3xyz = pq-3r. \end{aligned}$$

9.

$$
\begin{aligned}
&x^2(y+z)+y^2(z+x)+z^2(x+y)\\
&\qquad = x^2(p-x)+y^2(p-y)+z^2(p-z)\\
&\qquad = p(x^2+y^2+z^2)-(x^3+y^3+z^3)\\
&\qquad = p(p^2-2q)-(p^3-3pq+3r)=pq-3r.
\end{aligned}
$$

10.

$$
\begin{aligned}
&x^3y^3+y^3z^3+z^3x^3\\
&\qquad = (xy+yz+zx)^3-3(xy+yz+zx)xyz(x+y+z)+3x^2y^2z^2\\
&\qquad = q^3-3pqr+3r^2.
\end{aligned}
$$

11.

$$
\begin{aligned}
&xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2)\\
&\qquad = xy(p^2-2q-z^2)+yz(p^2-2q-x^2)+zx(p^2-2q-y^2)\\
&\qquad = (p^2-2q)(xy+yz+zx)-xyz(x+y+z)\\
&\qquad = p^2q-2q^2-pr.
\end{aligned}
$$

Here are some inequalities connecting p, q, r.

Example 2.47. $p^2 \geq 3q$.

Solution: Follows from $(x+y+z)^2 \geq 3(xy+yz+zx)$. ■

Example 2.48. $p^3 \geq 27r$.

Solution: Follows from $(x+y+z)^3 \geq 27xyz$. ■

Example 2.49. $q^3 \geq 27r^2$.

Solution: Follows from $(xy+yz+zx)^3 \geq 27(xyz)^2$. ■

Example 2.50. $q^2 \geq 3pr$.

Solution: We have

$$(xy+yz+zx)^2 = (x^2y^2+y^2z^2+z^2x^2)+2xyz(xy+yz+zx).$$

But

$$x^2y^2+y^2z^2+z^2x^2 \geq (xy)(yz)+(yz)(zx)+(zx)(xy),$$

by C-S inequality. It follows that

$$q^3 \geq pr + 2pr = 3pr.$$

■

Example 2.51. $2p^3 + 9r \geq 7pq$.

Solution: This reduces to

$$2(x^3 + y^3 + z^3) \geq \sum_{\text{sym}} x^2 y.$$

Since $(2, 1, 0) \prec (3, 0, 0)$, the result follows from Muirhead's theorem. ■

Example 2.52. $p^2 q + 3pr \geq 4q^2$.

Solution: After expanding this reduces to

$$\sum_{\text{sym}} x^3 y \geq 2(x^2 y^2 + y^2 z^2 + z^2 x^2).$$

Since $(2, 2, 0) \prec (3, 1, 0)$, this inequality is a consequence of Muirhead's theorem. ■

Example 2.53. $q^3 + 9r^2 \geq 4pqr$.

Solution: Again, expansion reduces this to

$$x^3 y^3 + y^3 z^3 + z^3 x^3 + 3x^2 y^2 z^2 \geq \sum_{\text{sym}} x^3 y^2 z.$$

By Schur's inequality, we have for non-negative reals a, b, c

$$a^3 + b^3 + c^3 + 3abc \geq a^2 b + b^c + c^2 a + ab^2 + bc^2 + ca^2.$$

Take $a = xy$, $b = yz$ and $c = zx$. We obtain

$$x^3 y^3 + y^3 z^3 + z^3 x^3 + 3x^2 y^2 z^2 \geq \sum_{\text{sym}} x^3 y^2 z.$$

■

Example 2.54. $p^4 + 3q^2 \geq 4p^2 q$.

Solution: This factorises as $(3q - p^2)(q - p^2) \geq 0$. We know that $3q \leq p^2$. Hence $q \leq p^2$. Thus $(3q - p^2)$ and $(q - p^2)$ are both non-positive. This implies that their product is non-negative. ■

Example 2.55. $pq^2 \geq 2p^2 r + 3qr$.

Solution: We know that $3q \leq p^2$ and this gives $3qr \leq p^2r$. Therefore

$$3pr + 2p^2r \leq 3p^2r \leq q^2p,$$

since $q^2 \geq 3pr$. ■

Example 2.56. [10] Let a, b, c be positive real number such that $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 1$. Prove that

$$\frac{1}{8ab+1} + \frac{1}{8bc+1} + + \frac{1}{8ca+1} \geq 1.$$

Solution: Introducing $p = a+b+c$, $q = ab+bc+ca$ and $r = abc$, the given condition reduces to $q + 2r = 1$. The inequality to be proved reduces to

$$64pr + 16q + 3 \geq 512r^2 + 64pr + 8q + 1.$$

Hence it is sufficient to prove that

$$8q + 2 \geq 512r^2.$$

We know that $q^3 \geq 27r^2$. Using $q = 1 - 2r$, this is equivalent to

$$(8r-1)(r+1)^2 \leq 0.$$

Hence $8r - 1 \leq 0$. Now the inequality to be proved is

$$8(1-2r) + 2 \geq 512r^2.$$

This can be written as

$$(8r-1)(64r+10) \geq 0.$$

Since $8r - 1 \leq 0$, the result follows. ■

Example 2.57. Let a, b, c be positive real numbers such that $a+b+c = 1$. Prove that

$$6\left(a^3 + b^3 + c^3\right) + 1 \geq 5\left(a^2 + b^2 + c^2\right).$$

Solution: We use $p = a+b+c$, $q = ab+bc+ca$ and $r = abc$.Observe that $p = 1$. We also know

$$\begin{aligned} a^3 + b^3 + c^3 &= p(p^2 - 3q) + 3r = 1 - 3q + 3r, \\ a^2 + b^2 + c^2 &= p^2 - 2q = 1 - 2q. \end{aligned}$$

Hence we have to prove

$$6 - 18q + 18r + 1 \geq 5 - 10q.$$

This reduces to $9r + 1 \geq 4q$. But we know that $p^3 - 4pq + 9r \geq 0$. This follows from Schur's lemma. Since $p = 1$, this gives $9r + 1 \geq 4q$. ■

Example 2.58. Suppose x, y, z are positive real numbers such that $x^2+y^2+z^2 = 1$. Prove that

$$(1-xy)(1-yz)(1-zx) \geq \frac{8}{27}.$$

Solution: Introduce $p = x+y+z$, $q = xy+yz+zx$ and $r = xyz$. The inequality reduces to

$$1-q+pr-r^2 \geq \frac{8}{27}.$$

Using $p^3 - 4pq + 9r \geq 0$, we also get

$$9r \geq p(4q-p^2) = p(2q-1),$$

since $p^2 - 2q = x^2+y^2+z^2 = 1$. We also have $p^2 \geq 3q$. Hence $2q+1 = p^2 \geq 3q$, which shows that $q \leq 1$. Using $pq - 9r \geq 0$, we get

$$p \geq pq \geq 9r.$$

Therefore $9p - 9r \geq 8p$. This gives us $(p-r) \geq 8p/9$. Thus we obtain

$$r(p-r) \geq \frac{8}{9}pr \geq \frac{8}{9}p \cdot \frac{p(2q-1)}{9} = \frac{8}{81}p^2(2q-1) = \frac{8}{81}(2q-1)(2q+1).$$

Using this estimate, we obtain

$$1-q+pr-r^2 \geq 1-q+\frac{8}{81}(4q^2-1).$$

Therefore, we need to prove that

$$1-q+\frac{8}{81}(4q^2-1) \geq \frac{8}{27}.$$

Simplification leads to the inequality

$$(1-q)(49-32q) \geq 0.$$

Since $q \leq 1$, the result holds.

■

2.14.2 An alternate approach for symmetric inequalities

Some times, the introduction of elementary symmetric functions may not give sharp bounds the inequality may require sharp bounds. We use a slightly different approach in such cases. For real numbers a, b, c, we introduce

$$p = a+b+c, \quad q = \sqrt{a^2+b^2+c^2-ab-bc-ca}, \quad r = abc.$$

Note that $q \geq 0$ and $ab+bc+ca$ is related to p, q by

$$\frac{p^2-q^2}{3} = ab+bc+ca.$$

Then the following inequality holds:

Example 2.59.

$$\frac{(p+q)^2(p-2q)}{27} \leq r \leq \frac{(p-q)^2(p+2q)}{27}. \tag{2.9}$$

Equality holds if and only if some two of a, b, c are equal.

Solution: Consider the polynomial whose root are a, b, c:

$$f(x) = (x-a)(x-b)(x-c) = x^3 - px^2 + \frac{p^2-q^2}{3}x - r.$$

Its derivative is

$$f'(x) = 3x^2 - 2ux + \frac{p^2-q^2}{3}.$$

Hence $f'(x) = 0$ if and only if

$$x = \frac{p \pm q}{3}.$$

Let

$$x_1 = \frac{p-q}{3}, \quad x_2 = \frac{p+q}{3}.$$

Then $f'(x) < 0$ for $x_1 < x < x_2$. Hence $f'(x) > 0$ for $x < x_1$ and $x > x_2$. Next we compute the second derivative of $f(x)$:

$$f''(x) = 6x - 2p.$$

Hence

$$f''(x_2) = 6\left(\frac{p+q}{3}\right) - 2p = 2q > 0$$

Hence f has a local minimum at $x = x_2$. Similarly, we see that $f''(x_1) = -2q < 0$ and hence f has a local maximum at x_2. Thus we see that $f(x_2) \leq 0$ and $f(x_1) \geq 0$. But it is easy to compute that

$$f(x_2) = \frac{(p+q)^2(p-2q)}{27} - r,$$

and

$$f(x_1) = \frac{(p-q)^2(p+2q)}{27} - r.$$

Combining, we get

$$\frac{(p+q)^2(p-2q)}{27} \leq r \leq \frac{(p-q)^2(p+2q)}{27}.$$

Equality holds if and only if $f(x_1) = 0 = f(x_2)$. Since $f'(x_1) = 0 = f'(x_2)$, it follows that equality holds if and only if f has a double root. This means some two of a, b, c are equal. ■

We can use this in proving several inequalities.

Example 2.60. [10] Let a, b, c be real numbers. Prove that

$$a^4 + b^4 + c^4 \geq abc(a + b + c).$$

Solution: We may assume that $a + b + c = 1$, since the inequality is homogeneous. Observe that

$$a^2 + b^2 + c^2 = \frac{p^2 + 2q^2}{3},$$

and

$$a^4 + b^4 + c^4 = \frac{-p^4 + 8p^2q^2 + 2q^4}{9} + 4pr.$$

Using $p = 1$, the inequality reduces to

$$\frac{-1 + 8q^2 + 2q^4}{9} + 4r \geq r.$$

This can be written in the form

$$-1 + 8q^2 + 2q^4 + 27r \geq 0.$$

Using the inequality (2.9), it is enough to prove that

$$1 - 8q^2 + 2q^4 + (1 + q)^2(1 - 2q) \geq 0.$$

This simplifies to

$$q^2(2q^2 - 2q + 5) \geq 0.$$

However, we can write

$$q^2(2q^2 - 2q + 5) = \frac{q^2\left((2q - 1)^2 + 9\right)}{2},$$

which is non-negative.

■

Example 2.61. [10] Let a, b, c be real numbers such that $a^2 + b^2 + c^2 = 9$. Prove that

$$2(a + b + c) - abc \leq 10.$$

Solution: Using the neqw variables p, q, r as earlier, we get

$$9 = a^2 + b^2 + c^2 = \frac{p^2 + 2q^2}{3}.$$

Hence $p^2 + 2q^2 = 27$. Using (2.9), we get

$$2(a+b+c)-abc = 2p - r \le 2p - \frac{(p+q)^2(p-2q)}{27}$$
$$= \frac{54p - p^3 + 3pq^2 + 2q^3}{27}.$$

However, we see that

$$\begin{aligned} 54p - p^3 + 3pq^2 + 2q^3 &= 54p - p(p^2+2q^2) + 5pq^2 + 2q^3 \\ &= p(27+5q^2) + 2q^3. \end{aligned}$$

Hence we have to prove that

$$p(27+5q^2) \le 270 - 2q^3.$$

However,

$$\left(270-2q^3\right)^2 - \left(p(27+5q^2)\right)^2$$
$$= 27(q-3)^2\left(2q^4 + 12q^3 + 49q^2 + 146q + 219\right) \ge 0.$$

This implies the required result. ■

Example 2.62. [10] Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + +48(ab+bc+ca) \ge 25.$$

Solution: We begin with $ab+bc+ca = (1-q^2)/3$ and $r = abc$. Observe that $q \in [0,1]$and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + +48(ab+bc+ca) = \frac{1-q^2}{3r} + 16(1-q^2).$$

Thus we need to prove that

$$\frac{1-q^2}{3r} + 16(1-q^2) \ge 25.$$

Using (2.9), we have

$$\begin{aligned} \frac{1-q^2}{3r} + 16(1-q^2) &\ge 27\frac{1-q^2}{3(1-q)^2(1+2q)} + 16(1-q^2) \\ &= 9\frac{1+q}{(1-q)(1+2q)} + 16(1-q^2) \\ &= \frac{2q^2(4q-1)^2}{(1-q)(1+2q)} + 25 \ge 25. \end{aligned}$$

Equality holds if and only if $a=b=c=1/3$ or $(a,b,c) = (1/2, 1/4, 1/4)$ and permutations thereof.

■

Example 2.63. Prove Schur's inequality: for non-negative real numbers a, b, c, we have

$$a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)c-b) \geq 0.$$

Solution: We write the inequality in the form

$$a^3+b^3+c^3+3abc \geq ab(a+b)+bc(b+c)+ca(c+a).$$

Introduce p, q, r by

$$p=a+b+c, \quad q=\sqrt{a^2+b^2+c^2-ab-bc-ca}, \quad r=abc.$$

Since the inequality is homogeneous, we may assume $p=a+b+c=1$. The inequality now changes to

$$27r+4q^2-1 \geq 0.$$

Observe that

$$0 \leq q=(a+b+c)^2-3(ab+bc+ca) \leq 1.$$

Hence $q \in [0,1]$. If $q \geq 1/2$, the inequality is obvious. If $q \leq 1/2$, then (2.9) shows that

$$27r+4q^2- \geq (1+q)^2(1-2q)+4q^2-1=q^2(1-2q) \geq 0.$$

■

2.14.3 Squares in handling symmetric inequalities

Another useful method for proving symmetrical inequality is to transform it in to a sum of squares with positive coefficients. Just to give an idea, consider the standard inequality $a^3+b^3+c^3 \geq 3abc$ for non-negative real numbers a, b, c. We write this as

$$a^3+b^3+c^3-3abc=s(a-b)^2+s(b-c)^2+s(c-a)^2,$$

where $s=(a+b+c)/2$. Since $s \geq 0$, it follows that $a^3+b^3+c^3 \geq 3abc$. In general, we try to write the given inequality in the form

$$f_a(b-c)^2+f_b(c-a)^2+f_c(a-b)^2 \geq 0$$

where f_a, f_b, f_c are functions of a, b, c. The proof of the required inequality now depends on the analysis of these functions f_a, f_b, f_c.

Example 2.64. [10] Let a, b, c be positive real numbers. Prove that

$$\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{8abc}{(a+b)(b+c)(c+a)} \geq 2.$$

Solution: We write the inequality in the form

$$\frac{a^2+b^2+c^2}{ab+bc+ca}-1+\frac{8abc}{(a+b)(b+c)(c+a)}-1\geq 0.$$

This reduces to

$$f_a(b-c)^2+f_b(c-a)^2+f_c(a-b)^2\geq 0,$$

where

$$f_a=b+c-a-\frac{abc}{ab+bc+ca}$$

and similar expressions for f_b, f_c. We may assume $a\geq b\geq c$ Hence

$$(a-c)^2=(a-b+b-c)^2=(a-b)^2+(b-c)^2+2(a-b)(b-c)\geq (a-b)^2+(b-c)^2.$$

Thus we have

$$\begin{aligned} &f_a(b-c)^2+f_b(c-a)^2+f_c(a-b)^2\\ &\qquad\geq f_a(b-c)^2+f_b\big((a-b)^2+(b-c)^2\big)+f_c(a-b)^2\\ &\qquad\qquad=\big(f_a+f_b\big)(b-c)^2+\big(f_c+f_b\big)(a-b)^2. \end{aligned}$$

But

$$f_a+f_b=2c-\frac{2abc}{ab+bc+ca}=\frac{4c(bc+ca)}{ab+bc+ca}\geq 0.$$

Similarly, $f_c+f_b\geq 0$. Hence we obtain

$$f_a(b-c)^2+f_b(c-a)^2+f_c(a-b)^2\geq 0.$$

■

Example 2.65. [10] Let a, b, c be real numbers. Prove that

$$3(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2)\geq a^3b^3+b^3c^3+c^3a^3.$$

Solution: We may assume $a, b, c\geq 0$. Expanding the left side, we get

$$3(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2)=\sum_{\text{sym}}a^4b^2-\sum_{\text{cyclic}}a^3b^3-\sum_{\text{cyclic}}a^4bc+a^2b^2c^2.$$

Therefore the inequality can be written in an equivalent form

$$3\sum_{\text{sym}}a^4b^2-4\sum_{\text{cyclic}}a^3b^3-3\sum_{\text{cyclic}}a^4bc+3a^2b^2c^2\geq 0.$$

This further reduces to the form

$$f_a(b-c)^2+f_b(c-a)^2+f_c(a-b)^2\geq 0, \tag{2.10}$$

where

$$
\begin{aligned}
f_a &= 2a^4 + 3b^2c^2 - abc(a+b+c),\\
f_b &= 2b^4 + 3c^2a^2 - abc(a+b+c),\\
f_c &= 2c^4 + 3a^2b^2 - abc(a+b+c).
\end{aligned}
$$

We may assume $a \geq b \geq c$. We see that

$$f_a = 2a^4 + 3b^2c^2 - abc(a+b+c) \geq a^4 + 2a^2bc - abc(a+b+c),$$

since

$$a^4 + 3b^2c^2 - 2a^2bc = (a^2 - bc)^2 + 2b^2c^2 \geq 0.$$

Hence $f_a \geq 0$ using $a \geq b \geq c$. Similarly

$$f_c = 2c^4 + 3a^2b^2 - abc(a+b+c) \geq 3a^2b^2 - abc(a+b+c) \geq 0.$$

If $f_b \geq 0$, then (2.10) holds. Suppose $f_b < 0$. It is easy to check that

$$(a-c)^2 \leq 2(a-b)^2 + 2(b-c)^2.$$

Therefore

$$
\begin{aligned}
&f_a(b-c)^2 + f_b(c-a)^2 + f_c(a-b)^2\\
&\quad\geq\ f_a(b-c)^2 + f_b\big(2(a-b)^2 + 2(b-c)^2\big) + f_c(a-b)^2\\
&\quad=\ \big(f_a + 2f_b\big)(b-c)^2 + \big(f_c + 2f_b\big)(a-b)^2.
\end{aligned}
$$

However

$$
\begin{aligned}
f_a + 2f_b &= 2a^4 + 3b^2c^2 + 4b^4 + 6a^2c^2 - 3abc(a+b+c)\\
&\geq a^4 + 2a^2bc + 8b^2ca - 3abc(a+b+c) \geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
f_c + 2f_b &= 2c^4 + 3a^2b^2 + 4b^4 + 6a^2c^2 - 3abc(a+b+c)\\
&\geq (3a^2b^2 + 3a^2c^2) + 3a^2c^2 - 3abc(a+b+c) \geq 0.
\end{aligned}
$$

If $a \leq b \leq c$, we do the same type of analysis. It follows that (2.10) holds. This completes the solution.

■

2.14.4 Strong mixing of variables

Another useful method in proving inequalities is a method known as *strong mixing of variables* or SMV method for short. The crucial result of the method is the following theorem:

Theorem 16. If $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function which is symmetric and satisfies $f(a_1, a_2, a_3, \ldots, a_n) \geq f(b_1, b_2, b_3, \ldots, b_n)$ where $(b_1, b_2, \ldots, b_n)$ is obtained from $(a_1, a_2, a_3, \ldots, a_n)$ by replacing a_j, a_k by their average $(a_j + a_k)/2$, where

$$a_j = \min\{a_1, a_2, \ldots, a_n\}, \quad a_k = \max\{a_1, a_2, \ldots, a_n\},$$

then

$$f(a_1, a_2, \ldots, a_n) \geq f(a, a, \ldots, a)$$

where $a = (a_1 + a_2 + \cdots + a_n)/n$.

For a proof of this, refer [11]. It is not necessary, we should take the average $(a_j + a_k)/2$. We can also take geometric mean $\sqrt{a_j a_k}$ or root-mean-square $\sqrt{\dfrac{a_j^2 + a_k^2}{2}}$, whichever is convenient. The method is extremely useful in four variables inequalities.

The method can be described as follows. Suppose we take some numbers a_p and a_q among $a_1, a_2, \ldots, a_n$ and replace them by $(a_p + a_q)/2$. Let the new sequence be $(b_1, b_2, \ldots, b_n)$. Check that $f(a_1, a_2, \ldots, a_n) \geq f(b_1, b_2, \ldots, b_n)$. Suppose we check that $f(a, a, \ldots, x) \geq 0$ for all a and x. Then it follows that $f(a_1, a_2, \ldots, a_n) \geq 0$. This is because replacing a_p, a_q by $(a_p + a_q)/2$ infinitely many times leads to taking all equal to the average $(a_1 + a_2 + \cdots + a_n)/n$. We explain this by examples.

Example 2.66. (Short-list, IMO-1997) Suppose that a, b, c, d are non-negative real numbers such that $a + b + c + d = 1$. Prove that

$$abc + bcd + cda + dab \leq \frac{1}{27} + \frac{176}{27}abcd.$$

Solution: Let $f(a, b, c, d) = abc + bcd + cda + dab - (176/27)abcd$. We may assume that $a \leq b \leq c \leq d$. In all other cases, the same method works. We write $f(a, b, c, d)$ as

$$f(a, b, c, d) = ac(b + d) + bd\left(a + c - \frac{176}{27}\right).$$

Using $a \leq b \leq c \leq d$, we obtain

$$a + c \leq \frac{1}{2}(a + b + c + d) = \frac{1}{2}.$$

Using AM-HM inequality, we see that

$$\frac{1}{a} + \frac{1}{c} \geq \frac{4}{a + c} \geq 8 \geq \frac{176}{27}.$$

Therefore,

$$f(a, b, c, d) \geq f\left(a, \frac{b + d}{2}, c, \frac{b + d}{2}\right).$$

In order to apply the theorem 16, it is enough to check that $f(a,x,x,x) \leq 1/27$ whenever $a+3x=1$. But we have

$$f(a,x,x,x) = 3ax^2 + x^3 - \frac{176}{27}ax^3.$$

Hence we have to show that

$$3ax^2 + x^3 - \frac{176}{27}ax^3 \leq \frac{1}{27}$$

under the condition $3x+a=1$. Replacing a by $(1-3x)$, this can be written as

$$(1-3x)(4x-1)^2(11x+1) \geq 0.$$

Since $x \leq 1/3$, the result follows. Equality holds whenever $a=b=c=d=1/4$ and $a=b=c=1/3$, $d=0$ and permutations thereof.

■

Example 2.67. [10] Let a,b,c,d be non-negative real numbers such that $a+b+c+d=4$. Prove that

$$(1+3a)(1+3b)(1+3c)(1+3d) \leq 125 + 131abcd.$$

Solution: We consider the expression

$$f(a,b,c,d) = (1+3a)(1+3b)(1+3c)(1+3d) - 131abcd.$$

Again it is enough to consider the case $a \geq b \geq c \geq d$. Consider the difference

$$f(a,b,c,d) - f\left(\frac{a+c}{2}, b, \frac{a+c}{2}, d\right).$$

Some computation shows that this expression is equal to

$$9(1+3b)(1=3d)\left(ac - \frac{(a+c)^2}{4}\right) - 131bd\left(ac - \frac{(a+c)^2}{4}\right).$$

Observe that

$$9(1+3b)(1+3d) \geq 131bd \text{ if and only if } 9++27(b+d) \geq 50bd.$$

We also observe that $2(b+d) \leq a+b+c+d = 4$ so that $b+d \leq 2$ and $bd \leq 1$. Therefore

$$9+27(b+d) \geq 54\sqrt{bd} \geq 54bd \geq 50bd.$$

This shows that

$$f(a,b,c,d) \leq f\left(\frac{a+c}{2}, b, \frac{a+c}{2}, d\right).$$

Therefore it is enough to prove $f(a,a,a,x) \le 125$, whenever $3a + x = 4$. But

$$f(a,a,a,x) = (1+3a)^3(1+3x) - 131a^3x \le 125$$

if and only if

$$(a-1)^2(3a-4)(5a+28) \le 0.$$

Since $3a + x = 4$, we have $3a \le 4$ and the result follows. ■

Example 2.68. (Pham King Hung) Let $a_1, a_2, \dots, a_n$ be positive real numbers such that $a_1a_2a_3\cdots a_n = 1$. Prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n} + \frac{3n}{a_1 + a_2 + a_3 + \cdots + a_n} \ge n+3$$

for all $n \ge 4$.

Solution: As earlier, we assume $a_1 \ge a_2 \ge \cdots \ge a_n$. We show that

$$f(a_1, a_2, a_3, \dots, a_n) \ge f(a_1, \sqrt{a_2a_n}, a_3, \dots, a_{n-1}, \sqrt{a_2a_n}).$$

The difference can be written as

$$\left(\frac{1}{\sqrt{a_2}} - \frac{1}{\sqrt{a_n}}\right)^2 - \frac{3n(\sqrt{a_2} - \sqrt{a_n})^2}{(a_1 + a_2 + \cdots + a_n)(a_1 + 2\sqrt{a_2a_n} + a_3 + \cdots + a_{n-1})}.$$

Hence we have to prove

$$(a_1 + a_2 + \cdots + a_n)(a_1 + 2\sqrt{a_2a_n} + a_3 + \cdots + a_{n-1}) \ge 3na_2a_n.$$

Using $a_1 \ge a_2 \ge \cdots \ge a_n$, we get

$$\begin{aligned}
&(a_1 + a_2 + \cdots + a_n)(a_1 + 2\sqrt{a_2a_n} + a_3 + \cdots + a_{n-1}) \\
&\quad \ge \quad (2a_2 + (n-2)a_n)(a_2 + 2\sqrt{a_2a_n} + (n-3)a_n) \\
&\quad \ge \quad 2\sqrt{2(n-2)}\big(2 + 2\sqrt{n-3}\big)a_2a_n \ge 3na_2a_n,
\end{aligned}$$

since $n \ge 4$. Hence it is enough to prove that

$$f(a_1, a, a, a, \dots, a) \ge n+3$$

where $a = \sqrt[n-1]{a_2a_3\cdots a_n}$ and $a_1 = 1/a^{n-1}$. Therefore we get

$$f(a_1, a, a, \dots, a) = a^{n-1} + \frac{n-1}{a} + + \frac{3na^{n-1}}{1 + (n-1)a^n}.$$

Consider the new function $g(a)$ obtained after replacing n by $n+1$:

$$g(a) = a^n + \frac{n}{a} + \frac{3(n+1)a^n}{1 + na^{n+1}}.$$

We show that $g(a) \geq n+4$. We first prove that g is a decreasing function for $a \leq 1$. Its derivative is

$$g'(a) = na^{n-1} - \frac{n}{a^2} + \frac{3n(n+1)}{(na^{n+1}+1)^2}\left(a^{n-1} - a^{2n}\right).$$

Consider the numerator obtained by clearing the denominators:

$$\left(a^{n+1} - 1\right)\left((na^{n+1}+1)^2 - 3(n+1)a^{n+1}\right).$$

Using AM-GM inequality, we have

$$\left(na^{n+1}+1\right)^2 \geq 4na^{n+1} \geq 3(n+1)a^{n+1}.$$

Hence $g'(a) \leq 0$ for all $0 \leq a \leq 1$. Therefore g is decreasing on [0,1]. Since $g(1) = n+4$, we get $g(a) \geq g(1) = n+4$.

■

Chapter 3

Geometric inequalities

3.1 Introduction

Many of the inequalities we have studied and the techniques we have learnt have their direct implications in a class of inequalities known as *geometric inequalities*. These inequalities explore relations among various geometric elements. For example, when we consider a triangle, we can associate many things with it: angles, sides, area, medians, altitudes, circum-radius, in-radius, ex-radii and so on. We have already some inequalities, viz., *triangle inequalities* associated with the sides: $a < b + c$, $b < c + a$, $c < a + b$, where a, b, c are the sides of a triangle; these conditions are necessary and sufficient for the existence of a triangle with sides a, b, c. We can derive various relations among these geometric elements. The classic example is Euler's inequality: $R \geq 2r$, where R is the circum-radius and r is the in-radius. This chapter provides several inequalities of this kind, but the list is not exhaustive. For an excellent and a fairly exhaustive collection of geometric inequalities, please refer to [5] and [6].

3.2 Notations

For a triangle ABC, we use the following standard notations:

- $a = |BC|$, $b = |CA|$, $c = |AB|$;
- $\alpha = \angle BAC$, $\beta = \angle CBA$, $\gamma = \angle ACB$;
- m_a, m_b, m_c are respectively the lengths of the medians from A, B, C on to BC, CA, AB;
- h_a, h_b, h_c are respectively the lengths of the altitudes from A, B, C on to BC, CA, AB;
- w_a, w_b, w_c are respectively the lengths of the angle bisectors of $\angle BAC$, $\angle CBA$, $\angle ACB$;
- R is the circum-radius; r is the in-radius; r_a, r_b, r_c are the ex-radii;
- Δ is the area of ABC;
- s is the semi-perimeter of ABC: $s = (a + b + c)/2$;
- O is the circum-centre; H is the ortho-centre; I is the in-centre; G is the centre of gravity; N is the nine-point centre;

- Ω_1 denotes the first Brocard point of ABC; i.e., the unique point inside ABC such that $\angle CA\Omega_1 = \angle AB\Omega_1 = \angle BC\Omega_1 = \omega$, the Brocard angle of ABC. It is known that $\omega \leq \pi/6$.

We also use a large number of results:

1. the sine rule: $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c} = 2R$;
2. the cosine rule: $a^2 = b^2 + c^2 - 2bc\cos\alpha$, $b^2 = c^2 + a^2 - 2ca\cos\beta$, $c^2 = a^2 + b^2 - 2ab\cos\gamma$;
3. the half-angle rule:
$$\sin\alpha/2 = \sqrt{\frac{(s-b)(s-c)}{bc}}, \cos\alpha/2 = \sqrt{\frac{s(s-a)}{bc}},$$
etc.;
4. $R = \dfrac{abc}{4\Delta}$;
5. $r = 4R\sin(\alpha/2)\sin(\beta/2)\sin(\gamma/2) = (s-a)\tan(\alpha/2)$
$= (s-b)\tan(\beta/2) = (s-c)\tan(\gamma/2)$;
6. $\Delta = rs = (1/2)bc\sin\alpha = 2R^2\sin\alpha\sin\beta\sin\gamma$;
7. $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ (Heron's formula);
8. $16\Delta^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4$ (another form of Heron's formula);
9. (Stewart's theorem) If D is a point on the side BC of a triangle ABC such that $BD : DC = \lambda : \mu$, then
$$\lambda b^2 + \mu c^2 = (\lambda + \mu)(AD^2 + BD \cdot DC).$$
Note that Appolonius' theorem is a special case of this result: $b^2 + c^2 = 2AD^2 + 2BD^2$, where D is the mid-point of BC;
10. $m_a = (1/2)\sqrt{2b^2 + 2c^2 - a^2}$, etc.;
11. $w_a = \Big(2bc/(b+c)\Big)\cos(\alpha/2) = \Big(2\sqrt{bc}/(b+c)\Big)\sqrt{s(s-a)}$, etc.;
12. $r_a = \sqrt{\big(s(s-b)(s-c)\big)/(s-a)} = \Delta/(s-a)$, etc.;
13. $OI^2 = R^2 - 2Rr = R^2\Big(1 - 8\sin(\alpha/2)\sin(\beta/2)\sin(\gamma/2)\Big)$;
14. $OH^2 = R^2\Big(1 - \cos\alpha\cos\beta\cos\gamma\Big) = 9R^2 - \big(a^2 + b^2 + c^2\big)$;

15. $IH^2 = 2r^2 - 4R^2 \cos\alpha \cos\beta \cos\gamma$;

16. $\sum \cos\alpha = 1 + 4 \prod \sin(\alpha/2)$. (Remark: the notation $\sum$ always denote in this chapter the cyclical sum and $\prod$ denotes the cyclical product unless otherwise stated. For example, $\sum a = a + b + c$, $\sum a^2 b = a^2 b + b^2 c + c^2 a$, $\sum \sin\alpha = \sin\alpha + \sin\beta + \sin\gamma$, $\prod \cos\alpha = \cos\alpha \cos\beta \cos\gamma$ etc.)

3.3 Some identities involving elements of a triangle

In this section we develop some identities involving various elements of a triangle: sides, angles, circum-radius, in-radius, ex-radii, altitudes, semi-perimeter. etc. In fact we identify some polynomial equations satisfied by these elements and use these equations for further explorations.

3.3.1 Equation for the sides

We start with the relation

$$a = 2R\sin\alpha = 4R\sin\big(\alpha/2\big)\cos\big(\alpha/2\big), \quad s - a = r\cot\big(\alpha/2\big).$$

Using these, we obtain

$$\sin^2\big(\alpha/2\big) = \frac{ar}{4R(s-a)}, \quad \cos^2\big(\alpha/2\big) = \frac{a(s-a)}{4Rr}.$$

We thus get

$$\begin{aligned} 1 &= \sin^2\big(\alpha/2\big) + \cos^2\big(\alpha/2\big) \\ &= \frac{a}{4R}\left(\frac{r}{(s-a)} + \frac{(s-a)}{r}\right). \end{aligned}$$

This simplifies to the relation

$$a^3 - 2sa^2 + a\big(s^2 + r^2 + 4Rr\big) - 4Rrs = 0.$$

Thus, a is a root of the cubic equation

$$t^3 - 2st^2 + \big(s^2 + r^2 + 4Rr\big)t - 4Rrs = 0. \tag{3.1}$$

Similarly, we can prove that b, c also satisfy the cubic equation (3.1). Now using the relations between the roots and coefficients of a cubic equation, we obtain

$$\sum a = 2s, \quad \sum ab = s^2 + r^2 + 4Rr, \quad abc = 4Rrs.$$

Using (3.1), we obtain the equation for the reciprocals of the sides of a triangle. Thus, $1/a, 1/b, 1/c$ satisfy the equation:

$$4Rrst^3 - \big(s^2 + r^2 + 4Rr\big)t^2 + 2st - 1 = 0. \tag{3.2}$$

This gives

$$\sum \frac{1}{a} = \frac{s^2 + r^2 + 4Rr}{4Rrs}, \quad \sum \frac{1}{ab} = \frac{1}{2Rr}.$$

3.3.2 Equation for Altitudes

Using (3.2), we have

$$4Rrs\frac{1}{a^3} - \left(s^2 + r^2 + 4Rr\right)\frac{1}{a^2} + 2s\frac{1}{a} - 1 = 0.$$

Using $a = 2\Delta/h_a$, this reduces to

$$2Rh_a^3 - \left(s^2 + r^2 + 4Rr\right)h_a^2 + 4rs^2 h_a - 4r^2 s^2 = 0.$$

We get similar expressions for h_b and h_c. Thus, h_a, h_b, h_c are the roots of the cubic equation

$$2Rt^3 - \left(s^2 + r^2 + 4Rr\right)t^2 + 4rs^2 t - 4r^2 s^2 = 0. \tag{3.3}$$

We hence obtain

$$\sum h_a = \frac{s^2 + r^2 + 4Rr}{2R}, \quad \sum h_a h_b = \frac{2s^2 r}{R}, \quad \prod h_a = \frac{2r^2 s^2}{R}.$$

Using the reciprocal equation

$$4r^2 s^2 t^3 - 4rs^2 t^2 + \left(s^2 + r^2 + 4Rr\right)t - 2R = 0,$$

we also obtain

$$\sum \frac{1}{h_a} = \frac{1}{r}, \quad \sum \frac{1}{h_a h_b} = \frac{s^2 + r^2 + 4Rr}{4r^2 s^2}.$$

3.3.3 Equation for $s - a, s - b, s - c$

The equation (3.1) may also be written in the form

$$(s - t)^3 - s(s - t)^2 + \left(r^2 + 4Rr\right)(s - t) - sr^2 = 0.$$

However we know that a, b, c are the roots of the equation (3.1). Thus, we see that $s - a$, $s - b$, $s - c$ are the roots of the equation

$$t^3 - st^2 + \left(r^2 + 4Rr\right)t - sr^2 = 0. \tag{3.4}$$

We thus obtain the relations

$$\sum (s - a) = s, \quad \sum (s - a)(s - b) = r(r + 4R), \quad \prod (s - a) = sr^2.$$

Going to the reciprocal equation, we see that $1/(s - a)$, $1/(s - b)$, $1/(s - c)$ are the roots of the equation

$$sr^2 t^3 - r(r + 4R)t^2 + st - 1 = 0. \tag{3.5}$$

We also obtain

$$\sum \frac{1}{s - a} = \frac{r + 4R}{sr}, \quad \sum \frac{1}{(s - a)(s - b)} = \frac{1}{r^2}.$$

3.3.4 Equation for ex-radii

Now $1/(s-a)$ is a root of (3.5). Hence, we have

$$sr^2\frac{1}{(s-a)^3} - r(r+4R)\frac{1}{(s-a)^2} + s\frac{1}{(s-a)} - 1 = 0.$$

This can be written in the form

$$r^2\left(\frac{s}{s-a}\right)^3 - r(r+4R)\left(\frac{s}{s-a}\right)^2 + s^2\left(\frac{s}{s-a}\right) - s^2 = 0.$$

However we know that $rs = r_a(s-a) = \Delta$. Substituting $s/(s-a) = r_a/r$, we get the relation

$$r_a^3 - (r+4R)r_a^2 + s^2 r_a - s^2 r = 0.$$

Similar relations for r_b and r_c may be obtained. Thus, r_a, r_b, r_c are the roots of the equation

$$t^3 - (r+4R)t^2 + s^2 t - s^2 r = 0. \tag{3.6}$$

Consequently, we have the relations

$$\sum r_a = r + 4R, \quad \sum r_a r_b = s^2, \quad r_a r_b r_c = s^2 r.$$

Again the reciprocal equation of (3.6) is

$$s^2 r t^3 - s^2 t^2 + (r+4R)t - 1 = 0, \tag{3.7}$$

whose roots are $1/r_a$, $1/r_b$, $1/r_c$. We hence obtain the relations

$$\sum \frac{1}{r_a} = \frac{1}{r}, \quad \sum \frac{1}{r_a r_b} = \frac{r+4R}{s^2 r}.$$

3.4 Some geometric inequalities

In this section, we prove several inequalities among various elements of a triangle. Again the readers are reminded that the following section only gives a sample of geometric inequalities, but not an exhaustive list of these inequalities. The excellent collection of examples in [5] and [6] gives an idea of the innumerable possibilities one can have in the class of geometric inequalities. The techniques used are essentially what were developed earlier; we occasionally resort to different ideas.

3.4.1. $abc \geq 8(s-a)(s-b)(s-c)$.

Proof: We have $a^2 - (b-c)^2 \le a^2$ and equality holds if and only if $b = c$. Similar inequalities hold: $b^2 - (c-a)^2 \le b^2$, $c^2 - (a-b)^2 \le c^2$. Hence,

$$\begin{aligned} abc &\ge \sqrt{a^2-(b-c)^2}\sqrt{b^2-(c-a)^2}\sqrt{c^2-(a-b)^2} \\ &= (a+b-c)(b+c-a)(c+a-b) \\ &= 8(s-a)(s-b)(s-c). \end{aligned}$$

Equality holds if and only if $a = b = c$.

Alternatively, Stolarsky's theorem (see theorem 15 on page 94) may be used. Considering the polynomial

$$P(x,y,z) = xyz - (x+y-z)(y+z-x)(z+x-y),$$

we have a homogeneous polynomial of degree 3 in the variables x, y, z. Moreover $P(1,1,1) = 0$, $P(1,1,0) = 0$ and $P(2,1,1) = 2 > 0$. Hence,

$$abc - (a+b-c)(b+c-a)(c+a-b) \ge 0,$$

and the result follows. ■

3.4.2. $abc < \sum a^2(s-a) \le \dfrac{3}{2}abc.$

Proof: We have

$$\begin{aligned} 2\sum a^2(s-a) &= a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) \\ &= \sum a^2b + \sum ab^2 - \sum a^3. \end{aligned}$$

On the other hand, we also see that

$$\begin{aligned} (b+c-a)(c+a-b)(a+b-c) &= \left(c^2-a^2-b^2+2ab\right)\left(a+b-c\right) \\ &= a^2b+ab^2+b^2c+bc^2+c^2a+ca^2-a^3-b^3-c^3-2abc. \end{aligned}$$

Thus, we obtain

$$2\sum a^2(s-a) = (b+c-a)(c+a-b)(a+b-c) + 2abc.$$

Since a, b, c are the sides of a triangle, we know that $b+c-a > 0$, $c+a-b > 0$ and $a+b-c > 0$. Hence, $abc < \sum a^2(s-a)$. Now using **3.4.1**, we get

$$2\sum a^2(s-a) \le abc + 2abc = 3abc,$$

which proves the right hand side inequality.

Again, we may use Stolarsky's theorem(theorem 15 on page 94). Considering

$$P(x, y, z) = \sum_{\text{cyclic}} x^2(y + z - x) - 2xyz,$$

we see that it is a homogeneous polynomial of degree 3 and $P(1, 1, 1) = 1$, $P(1, 1, 0) = 0$, $P(2, 1, 1) = 0$. Hence, $P(a, b, c) > 0$, giving the left-side inequality. On the other hand, the polynomial

$$Q(x, y, z) = 3xyz - \sum_{\text{cyclic}} x^2(y + z - x),$$

gives $Q(1, 1, 1) = 0$, $Q(1, 1, 0) = 0$ and $Q(2, 1, 1) = 2$. Thus $Q(a, b, c) > 0$, and we get the right-side inequality. ∎

3.4.3. $\frac{3}{2} \leq \sum \frac{a}{b + c} < 2$. Equality holds on the left if and only if $a = b = c$.

Proof: We have proved this in chapter 1; refer to (1.4). The left hand side of the above inequality is generally known as *Nesbitt's inequality*. There are a variety of ways of proving this. We give two such proofs.

(i) Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} (a + b + c)^2 &= \left(\sum \sqrt{\frac{a}{b + c}}\sqrt{a(b + c)}\right)^2 \\ &\leq \left(\sum \frac{a}{b + c}\right)\left(\sum a(b + c)\right). \end{aligned}$$

This gives

$$\sum \frac{a}{b + c} \geq \frac{(a + b + c)^2}{2(ab + bc + ca)} \geq \frac{3(ab + bc + ca)}{2(ab + bc + ca)} = \frac{3}{2},$$

since $(a + b + c)^2 \geq 3(ab + bc + ca)$.

(ii) We may assume $a \leq b \leq c$, since the inequality is symmetric in a, b, c. This implies that

$$\frac{1}{b + c} \leq \frac{1}{c + a} \leq \frac{1}{a + b}.$$

Using rearrangement inequality, we obtain

$$\begin{aligned} \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} &\geq \frac{a}{c + a} + \frac{b}{a + b} + \frac{c}{b + c}, \\ \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} &\geq \frac{a}{a + b} + \frac{b}{b + c} + \frac{c}{c + a}. \end{aligned}$$

Adding these two, we obtain

$$2\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)\geq 3.$$

This gives the desired inequality.

■

3.4.4. $\sqrt{s}<\sqrt{s-a}+\sqrt{s-b}+\sqrt{s-c}\leq\sqrt{3s}.$

Proof: The first inequality follows from the fact that for any positive reals x, y, z, we have

$$\sqrt{x+y+z}<\sqrt{x}+\sqrt{y}+\sqrt{z}.$$

On the other hand $f(x)=\sqrt{x}$ is a concave function on $(0,\infty)$. Hence,

$$\sum\frac{1}{3}\sqrt{s-a}\leq\sqrt{\sum\frac{1}{3}(s-a)},$$

which gives

$$\sqrt{s-a}+\sqrt{s-b}+\sqrt{s-c}\leq\sqrt{3s}.$$

■

3.4.5. $0<\sin\alpha+\sin\beta+\sin\gamma\leq\dfrac{3\sqrt{3}}{2}.$

Proof: We know that $OH^2=9R^2-\left(a^2+b^2+c^2\right)$, which gives

$$a^2+b^2+c^2\leq 9R^2.$$

Hence, we obtain

$$(a+b+c)^2\leq 3\left(a^2+b^2+c^2\right)\leq 27R^2.$$

This gives

$$0<a+b+c\leq 3\sqrt{3}R.$$

Using $a=2R\sin\alpha$, etc., we get

$$0<\sin\alpha+\sin\beta+\sin\gamma\leq\frac{3\sqrt{3}}{2}.$$

■

3.4.6. $\sin 2\alpha+\sin 2\beta+\sin 2\gamma\leq\sin\alpha+\sin\beta+\sin\gamma.$

Proof: Consider $f(x) = -\sin x$ on $(0, \pi)$. This is a convex function. Suppose $\alpha \le \beta \le \gamma$. Using Theorem 10, we see that for any permutation $(\alpha', \beta', \gamma')$ of (α, β, γ), we have

$$\sum -\sin\left(\alpha' + \alpha\right) \le \sum -\sin\left(\alpha + \alpha\right) = -\sum \sin 2\alpha.$$

This reduces to

$$\sum \sin 2\alpha \le \sum \sin\left(\alpha' + \alpha\right).$$

Taking $\alpha' = \beta$, $\beta' = \gamma$ and $\gamma' = \alpha$, and using $\sin(\alpha + \beta) = \sin \gamma$, etc., we get

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma \le \sin \alpha + \sin \beta + \sin \gamma.$$

■

3.4.7. $\displaystyle \prod \sin \alpha \le \frac{3\sqrt{3}}{8}.$

Proof: Using the AM-GM inequality and **3.4.5**, we have

$$\prod \sin \alpha \le \left(\frac{\sum \sin \alpha}{3}\right)^3 \le \left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{8}.$$

■

3.4.8. $\displaystyle 0 < \prod \sin\left(\alpha/2\right) \le \frac{1}{8}.$

Proof: We know that $OI^2 = R^2\left(1 - 8\prod \sin\left(\alpha/2\right)\right)$. Hence, it follows that $1 - 8\prod \sin\left(\alpha/2\right) \ge 0$. The first inequality is obvious. Equality holds if and only if the triangle is equilateral. ■

3.4.9. $\displaystyle 1 < \sum \cos \alpha \le \frac{3}{2}.$

Proof: We use the known identity: $\sum \cos \alpha = 1 + 4\prod \sin\left(\alpha/2\right)$. Since $\prod \sin\left(\alpha/2\right) > 0$, we get the left side inequality. Now using the inequality **3.4.8**, we get the right side inequality. ■

3.4.10. $\displaystyle 1 < \sum \sin(\alpha/2) \le \frac{3}{2}.$

Proof: Whenever α, β, γ are the angles of a triangle, $(\pi-\alpha)/2$, $(\pi-\beta)/2$, $(\pi-\gamma)/2$ are also the angles of some triangle. Applying **3.4.9** to this triangle, we get these inequalities. ■

3.4.11. $\prod \cos\alpha \le \frac{1}{8}$.

Proof: Using **3.4.9** and the AM-GM inequality, we get

$$\prod \cos\alpha \le \left(\frac{\sum \cos\alpha}{3}\right)^3 \le \frac{1}{8}.$$

■

3.4.12. If x, y, z are real numbers such that $xyz > 0$, then

$$x\cos\alpha + y\cos\beta + z\cos\gamma \le \frac{1}{2}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right).$$

Proof: We start with the obvious inequality

$$\big(xz\cos\alpha + yz\cos\beta - xy\big)^2 + \big(xz\sin\alpha - yz\sin\beta\big)^2 \ge 0.$$

This simplifies to

$$2x^2yz\cos\alpha + 2xy^2z\cos\beta - 2xyz^2\cos(\alpha+\beta) \le \sum y^2z^2.$$

Using $\cos(\alpha+\beta) = -\cos\gamma$, we obtain the desired inequality. Equality holds if and only if $xz\cos\alpha + yz\cos\beta - xy = 0$ and $xz\sin\alpha - yz\sin\beta = 0$. This takes the form

$$\frac{1}{x} : \frac{1}{y} : \frac{1}{z} = \sin\alpha : \sin\beta : \sin\gamma.$$

We easily infer that the right side of **3.4.9** is a consequence of the present inequality. ■

3.4.13. In an obtuse-angled triangle, $\sum \cos^2\alpha > 1$ and $\sum \sin^2\alpha < 2$.

Proof: We have

$$IH^2 = R^2\left(1 - \prod \cos\alpha\right) = 9R^2 - \sum a^2.$$

Since the triangle is obtuse, $\prod \cos\alpha < 0$. Hence, $IH^2 > R^2$. This gives

$$R^2 < IH^2 = 9R^2 - \sum a^2.$$

Thus, $\sum a^2 < 8R^2$. This may be put in the form

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma < 2.$$

Equivalently

$$\sum \cos^2\alpha > 1.$$

$\blacksquare$

3.4.14. $\sum \tan\alpha \geq 3\sqrt{3}$ if the triangle is acute, and $\sum \tan\alpha < 0$ if the triangle is obtuse.

Proof: Suppose the triangle is acute. In this case $\tan\alpha$, $\tan\beta$ and $\tan\gamma$ are all positive. We have

$$\prod \tan\alpha = \sum \tan\alpha \geq 3\Big(\prod \tan\alpha\Big)^{1/3}.$$

This gives $\prod \tan\alpha \geq 3\sqrt{3}$. Using $\prod \tan\alpha = \sum \tan\alpha$, we get the desired inequality for acute-angled triangles. Alternatively, we can also use the fact that $f(x) = \tan x$ is a convex function on $\big(0, \pi/2\big)$.

If the triangle is obtuse, say $\alpha > 90°$, then $\tan\alpha < 0$ and $\tan\beta$, $\tan\gamma$ are positive. Hence, $\prod \tan\alpha < 0$ and this gives $\sum \tan\alpha < 0$. $\blacksquare$

3.4.15. $\sum \cot\alpha \geq \sqrt{3}$.

Proof: We have

$$\begin{aligned}
\cot\alpha + \cot\beta &= \frac{\cos\alpha}{\sin\alpha} + \frac{\cos\beta}{\sin\beta} \\
&= \frac{\sin\big(\alpha+\beta\big)}{\sin\alpha\sin\beta} \\
&= \frac{2\sin\gamma}{\cos(\alpha-\beta) + \cos\gamma} \\
&\geq \frac{2\sin\gamma}{1+\cos\gamma} = 2\tan\big(\gamma/2\big).
\end{aligned}$$

Similarly, we may prove that

$$\cot\beta + \cot\gamma \geq 2\tan\big(\alpha/2\big), \cot\gamma + \cot\alpha \geq 2\tan\big(\beta/2\big).$$

Adding, we obtain

$$2\Big(\cot\alpha + \cot\beta + \cot\gamma\Big) \geq 2\Big(\tan\big(\alpha/2\big) + \tan\big(\beta/2\big) + \tan\big(\gamma/2\big)\Big).$$

However using the convexity of $f(x) = \tan x$ on $(0, \pi/2)$, we obtain

$$\begin{aligned}\frac{1}{3}\Big(\tan(\alpha/2) + \tan(\beta/2) + \tan(\gamma/2)\Big) &\geq \tan\left(\frac{\alpha+\beta+\gamma}{6}\right)\\ &= \tan 30^\circ = \frac{1}{\sqrt{3}}.\end{aligned}$$

It follows that

$$\tan(\alpha/2) + \tan(\beta/2) + \tan(\gamma/2) \geq \sqrt{3}.$$

This gives

$$\cot\alpha + \cot\beta + \cot\gamma \geq \tan(\alpha/2) + \tan(\beta/2) + \tan(\gamma/2) \geq \sqrt{3}.$$

■

3.4.16. $\sum \cot(\alpha/2) \geq 3\sqrt{3}$.

Proof: Using the convexity of the function $f(x) = \cot x$ on $(0, \pi/2)$, we get the desired inequality. The convexity part can be proved using the second derivative of $f(x)$. ■

3.4.17. $\sum \cot^2(\alpha/2) \geq \Big(\sum \cot(\alpha/2)\Big)\Big(\sum \cot\alpha\Big)$.

Proof: Let us put $\cot(\alpha/2) = x$, $\cot(\beta/2) = y$, $\cot(\gamma/2) = z$. Then we know that $x + y + z = xyz$. We have

$$(x+y+z)^2 = (x+y+z)xyz = \sum (x^2-1)yz + \sum xy.$$

This gives

$$x^2 + y^2 + z^2 + \sum xy = \sum (x^2-1)yz.$$

It follows that

$$2(x^2+y^2+z^2) \geq x^2 + y^2 + z^2 + \sum xy = \sum (x^2-1)yz.$$

We write this in the form

$$x^2 + y^2 + z^2 \geq xyz\left(\sum \frac{x^2-1}{2x}\right).$$

However, observe that

$$\frac{x^2-1}{2x} = \cot\alpha.$$

Thus, we obtain

$$\begin{aligned}\sum \cot^2(\alpha/2) &\geq \Big(\prod \cot(\alpha/2)\Big)\Big(\sum \cot\alpha\Big)\\ &= \Big(\sum \cot(\alpha/2)\Big)\Big(\sum \cot\alpha\Big).\end{aligned}$$

■

3.4.18. $\prod \cot\alpha \leq 1/(3\sqrt{3})$ in an acute-angled triangle and $\prod \cot\alpha < 0$ for an obtuse-angled triangle.

Proof: This easily follows from the fact that

$$\prod \tan\alpha \geq (3\sqrt{3}),$$

in an acute-angled triangle and this product is negative for an obtuse-angled triangle (see the proof of **3.4.13**). ■

3.4.19. $\sum \cot^2\alpha \geq \sum \cot\alpha \cot\beta = 1.$

Proof: This follows from the rearrangement inequality. ■

3.4.20. $\sum \sec\alpha \geq 6.$

Proof: Using $\sum \cos\alpha \leq 3/2$ (see **3.4.9**) and AM-HM inequality, we get

$$\sum \sec\alpha \geq \frac{9}{\sum \cos\alpha} \geq 9 \times \frac{2}{3} = 6.$$

■

3.4.21. $\sum \operatorname{cosec}\alpha \geq 2\sqrt{3}.$

Proof: This follows from the convexity of $f(x) = \operatorname{cosec} x$ on $(0,\pi)$. In fact

$$f''(x) = \operatorname{cosec} x\big(\operatorname{cosec}^2 x + 2\cot^2 x\big) \geq 0,$$

on $(0,\pi)$. Hence $f(x) = \operatorname{cosec} x$ is convex on $(0,\pi)$. This gives

$$\frac{1}{3}\sum \operatorname{cosec}\alpha \geq \operatorname{cosec}\left(\frac{\alpha+\beta+\gamma}{3}\right) = \operatorname{cosec} 60^\circ = \frac{2}{\sqrt{3}}.$$

Hence, we get

$$\sum \operatorname{cosec}\alpha \geq 2\sqrt{3}.$$

■

3.4.22. $\sum \operatorname{cosec}^2\alpha \geq 4.$

Proof: Using the rearrangement inequality, we have

$$\begin{aligned}\sum \operatorname{cosec}^2\alpha &\geq \sum \operatorname{cosec}\alpha \operatorname{cosec}\beta \\ &= \frac{\sum \sin\alpha}{\prod \sin\alpha} \\ &\geq \frac{3}{\left(\prod \sin\alpha\right)^{2/3}}.\end{aligned}$$

We have used the AM-GM inequality at the end. However we know that $\prod \sin\alpha \leq 3\sqrt{3}/8$ (see **3.4.7**). It follows that

$$\sum \operatorname{cosec}^2\alpha \geq 3 \times \frac{4}{3} = 4.$$

■

3.4.23. $\left(\sum \sin(\alpha/2)\right)^2 \leq \sum \cos^2(\alpha/2).$

Proof: We may arrange the angles such that either $\alpha \geq 60^\circ \geq \beta \geq \gamma$ or $\alpha \leq 60^\circ \leq \beta \leq \gamma$. This implies that

$$\begin{aligned}&\left(\sin\frac{\beta}{2} - \frac{1}{2}\right)\left(\sin\frac{\gamma}{2} - \frac{1}{2}\right) \geq 0 \\ &\Rightarrow\quad 4\sin\frac{\beta}{2}\sin\frac{\gamma}{2} \geq 2\left(\sin\frac{\beta}{2} + \sin\frac{\gamma}{2}\right) - 1 \\ &\Rightarrow\quad 1 + 4\prod \sin\frac{\alpha}{2} \geq 2\sin\frac{\alpha}{2}\left(\sin\frac{\beta}{2} + \sin\frac{\gamma}{2}\right) + 1 - \sin\frac{\alpha}{2}.\end{aligned}$$

However we know that

$$\begin{aligned}2\sin\frac{\beta}{2}\sin\frac{\gamma}{2} &= \cos\left(\frac{\beta}{2} - \frac{\gamma}{2}\right) - \sin\frac{\alpha}{2} \\ &\leq 1 - \sin\frac{\alpha}{2}.\end{aligned}$$

This gives

$$\begin{aligned}&1 + 4\prod \sin\frac{\alpha}{2} \geq 2\sin\frac{\alpha}{2}\left(\sin\frac{\beta}{2} + \sin\frac{\gamma}{2}\right) + 2\sin\frac{\beta}{2}\sin\frac{\gamma}{2} \\ &\Rightarrow\quad \sum \cos\alpha \geq 2\sum \sin\frac{\alpha}{2}\sin\frac{\beta}{2} \\ &\Rightarrow\quad \sum \cos^2\frac{\alpha}{2} \geq \sum \sin^2\frac{\alpha}{2} + 2\sum \sin\frac{\alpha}{2}\sin\frac{\beta}{2}.\end{aligned}$$

This simplifies to

$$\left(\sum \sin(\alpha/2)\right)^2 \leq \sum \cos^2(\alpha/2).$$

■

3.4.24. $a^2 + b^2 + c^2 \geq 4\sqrt{3}\Delta$ (Weitzenböck's inequality).

Proof: Using Heron's formula for the area of a triangle, we have

$$\begin{aligned} 16\Delta^2 &= (a+b+c)(a+b-c)(b+c-a)(c+a-b) \\ &\leq \left(a+b+c\right)\left(\frac{a+b+c}{3}\right)^3 \\ &= \frac{(a+b+c)^4}{27}. \end{aligned}$$

This gives

$$4\sqrt{3}\Delta \leq \frac{(a+b+c)^2}{3}.$$

However we know from the Cauchy-Schwarz inequality that

$$(a+b+c)^2 \leq 3\left(a^2+b^2+c^2\right).$$

It follows that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}\Delta.$$

Solution 2: We prove a stronger version of the above inequality which is known as the *Hadwiger-Finsler* inequality. We show that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}\Delta + \sum (a-b)^2.$$

Using the *cosine rule*, we have

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\alpha \\ &= (b-c)^2 + 2bc(1-\cos\alpha) \\ &= (b-c)^2 + 4\Delta\tan(\alpha/2), \end{aligned}$$

since

$$2bc(1-\cos\alpha) = 4\Delta\left(\frac{1-\cos\alpha}{\sin\alpha}\right) = 4\Delta\tan(\alpha/2).$$

Thus, we get

$$a^2 + b^2 + c^2 = 4\Delta\Big(\tan(\alpha/2) + \tan(\beta/2) + \tan(\gamma/2)\Big) + \sum (a-b)^2.$$

However, we know that $f(x) = \tan x$ is a convex function on $(0, \pi/2)$. Hence, it follows that

$$\frac{1}{3}\Big(\tan(\alpha/2) + \tan(\beta/2) + \tan(\gamma/2)\Big) \geq \tan\left(\frac{\alpha+\beta+\gamma}{6}\right) = \tan 30^\circ = \frac{1}{\sqrt{3}}.$$

This shows that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}\Delta + \sum (a-b)^2.$$

We observe that equality holds if and only if $a = b = c$. ∎

3.4.25. $ab + bc + ca \geq 4\sqrt{3}\Delta.$

Proof: We know that

$$\Delta = \frac{1}{2}ab\sin\gamma = \frac{1}{2}bc\sin\alpha = \frac{1}{2}ca\sin\beta.$$

Hence, we get

$$\sum ab = 2\Delta \sum \frac{1}{\sin\alpha} \geq 6\Delta\left(\prod \frac{1}{\sin\alpha}\right)^{1/3}.$$

However we know from **3.4.7** that $\prod \frac{1}{\sin\alpha} \geq \left(8/3\sqrt{3}\right)$. This gives

$$\sum ab \geq 6\Delta\left(\frac{8}{3\sqrt{3}}\right)^{1/3} = 4\sqrt{3}\Delta.$$

∎

3.4.26. $\sum a^4 \geq 16\Delta^2.$

Proof: Using **3.4.24** and the Cauchy-Schwarz inequality, we get

$$\sum a^4 \geq \frac{1}{3}\left(\sum a^2\right)^2 \geq 16\Delta^2.$$

∎

3.4.27. $\sum a^2b^2 \geq 16\Delta^2.$

Proof: This follows from Heron's formula and **3.4.26**. ∎

3.4.28. $(abc)^2 \geq \left(\frac{4\Delta}{\sqrt{3}}\right)^3.$

Proof: We use $a+b+c = 2R\big(\sin\alpha+\sin\beta+\sin\gamma\big) \le (3\sqrt{3})R$ (see **3.4.5**) and $abc = 4R\Delta$. Thus,

$$\frac{4\Delta}{\sqrt{3}} = \frac{abc}{R\sqrt{3}} \ge \frac{3abc}{a+b+c} \le (abc)^{2/3},$$

where we have used the AM-GM inequality in the last step. The result follows by taking cubes on both the sides. ■

3.4.29. (Euler's inequality) $2r \le R$ and equality holds if and only if the triangle is equilateral.

Proof: We use $OI^2 = R(R-2r)$. ■

3.4.30. $9r(r+4R) \le 3s^2 \le (r+4R)^2$.

Proof: We know that a, b, c are the roots of the cubic equation (see section **3.3.1**) $t^3 - 2st^2 + \big(s^2+r^2+4Rr\big)t - 4Rrs = 0$. Since all its roots are real, the derivative polynomial $3t^2 - 4st + \big(s^2+r^2+4Rr\big) = 0$ has only real roots. This imposes a condition on the discriminant, namely, $s^2 \ge 3r(r+4R)$. This gives the left inequality.

Again we know that r_a, r_b, r_c are the roots of the equation (see section **3.3.4**) $t^3 - (r+4R)t^2 + s^2t - s^2r = 0$. We look at the derivative polynomial: $3t^2 - 2(r+4R)t + s^2 = 0$. This again has only real roots. Hence, using the condition on the discriminant, we obtain $3s^2 \le (r+4R)^2$, which is the other inequality. ■

3.4.31. $s^2 \ge 27r^2$.

Proof: We have

$$\begin{aligned}\frac{s}{3} = \frac{(s-a)+(s-b)+(s-c)}{3} &\ge \big\{(s-a)(s-b)(s-c)\big\}^{1/3}\\ &= \left(\frac{\Delta^2}{s}\right)^{1/3}\\ &= \big(r^2s\big)^{1/3}.\end{aligned}$$

On cubing the above relation and rearranging the terms, we get the result. ■

3.4.32. $36r^2 \le \sum a^2 \le 9R^2$.

Proof: Using **3.4.31**, we have

$$27r^2 \le \left(\frac{a+b+c}{2}\right)^2 \le \frac{3}{4}\sum a^2,$$

where we have used the Cauchy-Schwarz inequality. This gives the left side inequality. On the other hand, using $OH^2 = 9R^2 - \left(a^2+b^2+c^2\right)$, we get the right side inequality. ■

3.4.33. $\displaystyle\sum \frac{a^2}{r_b r_c} \ge 4.$

Proof: We know (see **3.3.4**) that $\sum r_b r_c = s^2$. Using the Cauchy-Schwarz inequality, we get

$$\sum \frac{a^2}{r_b r_c} \ge \frac{\left(\sum a\right)^2}{\sum r_b r_c} = \frac{4s^2}{s^2} = 4.$$

■

3.4.34. $5R - r \ge \sqrt{3}s.$

Proof: We use $\sum r_a = 4R + r$ (**3.3.4**) and $R \ge 2r$ (**3.4.29**). Thus,

$$5R - r = 4R + r + R - 2r \ge 4R + r = \sum r_a.$$

Using $r_a = \Delta/(s-a) = rs/(s-a)$ and similar expressions, we obtain

$$\begin{aligned} 5R - r \ge \sum r_a &= \Delta\left(\sum \frac{1}{s-a}\right) \\ &= \frac{s}{4\Delta}\left\{2\sum ab - \sum a^2\right\}. \end{aligned}$$

Thus, it is sufficient to prove that

$$\frac{s}{4\Delta}\left\{2\sum ab - \sum a^2\right\} \ge \sqrt{3}s.$$

This is equivalent to

$$2\sum ab - \sum a^2 \ge 4\sqrt{3}\Delta.$$

This follows from the *Hadwiger-Finsler* inequality. (See **3.4.24**.)

■

3.4.35. $\sum a(s-a) \le 9rR.$

Proof: We know that $r = (s-a)\tan\left(\alpha/2\right)$, etc. Hence,

$$\sum a(s-a) = \sum \frac{ar}{\tan\left(\alpha/2\right)} = 4Rr \sum \cos^2\left(\alpha/2\right).$$

Thus, it is sufficient to prove that $2\sum \cos^2\left(\alpha/2\right) \le \frac{9}{2}$. This is equivalent to $\sum \cos\alpha \le 3/2$. This follows from **3.4.9**. ■

3.4.36. $\sum h_a \le \sqrt{3}s$.

Proof: We have

$$(a+b+c)^2 \ge 3(ab+bc+ca) = \frac{3abc}{2\Delta}(h_a+h_b+h_c) = 6R\sum h_a.$$

Thus, we obtain

$$\sum h_a \le \frac{(a+b+c)^2}{6R} = \frac{s(a+b+c)}{3R}.$$

But we know that $a+b+c \le 3\sqrt{3}R$. It follows that $\sum h_a \le \sqrt{3}s$. ■

3.4.37. $\sum \frac{1}{a} \le \frac{\sqrt{3}}{2r}$.

Proof: We have

$$\sum \frac{1}{a} = \frac{\sum h_a}{2\Delta}.$$

Thus, it is sufficient to prove that

$$\sum h_a \le \frac{\sqrt{3}\Delta}{r}.$$

This follows from $\Delta = rs$ and **3.4.36**.

■

3.4.38. $\sum \frac{1}{a} \ge \frac{3\sqrt{3}}{2(r+R)}$.

Proof: We know that

$$r = 4R\prod \sin\left(\alpha/2\right), \quad 1 + 4\prod \sin\left(\alpha/2\right) = \sum \cos\alpha.$$

Using $a = 2R\sin\alpha$, etc., we get an equivalent inequality:

$$\sum \frac{1}{\sin\alpha} \ge \frac{3\sqrt{3}}{\sum \cos\alpha}.$$

This reduces, after cross multiplication, to

$$\sum \cot\alpha + \sum\left(\frac{\cos\alpha}{\sin\beta} + \frac{\cos\beta}{\sin\alpha}\right) \geq 3\sqrt{3}.$$

Since $\sum \cot\alpha \geq \sqrt{3}$ (**3.4.15**), it is sufficient to prove

$$\sum\left(\frac{\cos\alpha}{\sin\beta} + \frac{\cos\beta}{\sin\alpha}\right) \geq 2\sqrt{3}.$$

This is equivalent to

$$\sum \frac{\sin\gamma}{\sin\alpha\sin\beta} \geq 2\sqrt{3}.$$

By the AM-GM inequality, we have

$$\sum \frac{\sin\gamma}{\sin\alpha\sin\beta} \geq 3\left(\prod \sin\alpha\right)^{-1/3}.$$

Using $\prod \sin\alpha \leq 3\sqrt{3}/8$ (**3.4.7**), we get the desired inequality. ■

3.4.39. $\quad 9r \leq \sum r_a \leq \dfrac{9}{2}R.$

Proof: We know from the section **3.3.4** that $\sum r_a = 4R + r$. Using $2r \leq R$, we obtain $9r \leq 4R + r \leq (9/2)R$. ■

3.4.40. $\quad 9r \leq \sum h_a \leq \sqrt{3}\, s.$

Proof: We have

$$\sum h_a = 2\Delta \sum \frac{1}{a} \leq 2\Delta\frac{\sqrt{3}}{2r} = \sqrt{3}\, s,$$

and

$$\sum h_a \geq \frac{9}{\sum(1/h_a)} = 9r.$$

(See **3.3.2**.) ■

3.4.41. $\quad \sum h_a^2 \leq (3/4)\sum a^2.$

Proof: We know that $h_a \leq m_a$, $4m_a^2 = 2b^2 + 2c^2 - a^2$ and similar results for h_b, h_c, m_b, m_c. Thus, it follows that

$$\sum h_a^2 \leq \sum m_a^2 = \frac{3}{4}\left(a^2 + b^2 + c^2\right).$$

■

3.4.42. $\sum \frac{a^2}{h_b^2 + h_c^2} \geq 2.$

Proof: We write

$$\sum \frac{a^2}{h_b^2 + h_c^2} = \sum \frac{a^2 b^2 c^2}{4\Delta^2 \left(b^2 + c^2\right)}.$$

Hence, it is sufficient to prove that

$$\sum \frac{1}{b^2 + c^2} \geq \frac{1}{2R^2}.$$

Using the AM-HM inequality, we get

$$\sum \frac{1}{b^2 + c^2} \geq \frac{9}{2\sum a^2} \geq \frac{1}{2R^2},$$

since $\sum a^2 \leq 9R^2$. (This follows from $OH^2 = 9R^2 - \sum a^2$.) ■

3.4.43. $\sum h_a \leq \sum w_a \leq 3(R + r).$

Proof: The first part is obvious. For the second inequality, we begin with the expression

$$w_a = \frac{2bc}{b + c} \cos\left(\alpha/2\right),$$

and similar expressions for w_b and w_c. Since $2bc \leq (b + c)^2$, we get $w_a^2 \leq bc \cos^2\left(\alpha/2\right)$, and similar estimates hold for w_b^2 and w_c^2. Now the Cauchy-Schwarz inequality gives,

$$\begin{aligned} \left(\sum w_a\right)^2 &\leq \left(\sum \sqrt{bc} \cos\left(\alpha/2\right)\right)^2 \\ &\leq \left(\sum bc\right)\left(\sum \cos^2\left(\alpha/2\right)\right) \\ &\leq \frac{9}{4} \sum bc\,; \end{aligned}$$

here we have used the estimate $\sum \cos^2\left(\alpha/2\right) \leq 9/4$. Hence, it is sufficient to prove that

$$\begin{aligned} &\sum bc \leq 4(R + r)^2 \\ \iff\quad & \sum \sin\alpha \sin\beta \leq \left(1 + \frac{r}{R}\right)^2 = \left(\sum \cos\alpha\right)^2 \\ \iff\quad & \sum \sin\alpha \sin\beta \leq \sum \cos^2\alpha + 2\sum \cos\alpha \cos\beta \\ \iff\quad & \sum \cos\alpha \leq \sum \cos^2\alpha + \sum \cos\alpha \cos\beta \\ \iff\quad & \sum 2\cos\alpha \leq \left(\sum \cos\alpha\right)^2 + \sum \cos^2\alpha. \end{aligned}$$

Thus, it is sufficient to establish the quadratic inequality

$$\left(\sum \cos\alpha\right)^2 - 2\sum \cos\alpha + \sum \cos^2\alpha \geq 0.$$

If the triangle is obtuse, then $\sum \cos^2\alpha > 1$ (3.4.13) and hence the inequality follows from the fact that the discriminant is negative. Hence, we may assume that $\sum \cos^2\alpha \leq 1$. The above quadratic inequality is true if either $\sum \cos\alpha \leq 1 - \sqrt{1 - \sum \cos^2\alpha}$ or $\sum \cos\alpha \geq 1 + \sqrt{1 - \sum \cos^2\alpha}$. However, we know that $\sum \cos\alpha > 1$ and hence the first alternative does not occur. It is sufficient to prove that for an acute-angled triangle

$$\sum \cos\alpha \geq 1 + \sqrt{1 - \sum \cos^2\alpha}.$$

We use

$$1 - \sum \cos^2\alpha = 2\prod \cos\alpha, \quad \sum \cos\alpha = 1 + 4\prod \sin\left(\alpha/2\right).$$

Thus, we have to show that

$$8\prod \sin^2\left(\alpha/2\right) \geq \prod \cos\alpha.$$

However we know that $IH^2 = 2r^2 - 4R^2\prod \cos\alpha$. This gives

$$4R^2\prod \cos\alpha \leq 2r^2,$$

and hence

$$\prod \cos\alpha \leq \frac{r^2}{2R^2} = \frac{16R^2\prod \sin^2\left(\alpha/2\right)}{2R^2} = 8\prod \sin^2\left(\alpha/2\right).$$

This completes the proof of the desired inequality.

■

3.4.44. $\sum h_a h_b \leq 3\sqrt{3}\Delta$.

Proof: We know (see **3.3.2**) that

$$\sum h_a h_b = \frac{2s^2 r}{R} = \frac{2\Delta s}{R}.$$

However we also have

$$s = \frac{1}{2}(a+b+c) = R\left(\sum \sin\alpha\right) \leq \frac{3\sqrt{3}R}{2};$$

(see **3.4.7**). Using this estimate, we obtain the required inequality. ■

3.4.45. $\prod h_a \geq 27r^3$.

Proof: We begin with the known identity

$$\sum \frac{1}{h_a} = \frac{1}{r}.$$

Hence, the AM-GM inequality gives

$$\prod \frac{1}{h_a} \leq \left(\frac{1}{3} \sum \frac{1}{h_a}\right)^3 = \frac{1}{27r^3}.$$

Taking the reciprocals, we obtain

$$\prod h_a \geq 27r^3.$$

■

3.4.46. $\displaystyle \sum \frac{1}{h_a - 2r} \geq \frac{3}{r}.$

Proof: Using $\displaystyle \sum \frac{1}{h_a} = \frac{1}{r}$, we get

$$1 = 3 - 2 = \sum \left(\frac{h_a}{h_a} - 2\frac{r}{h_a}\right) = \sum \frac{h_a - 2r}{h_a}.$$

Using the AM-GM inequality, we obtain

$$\left(\sum \frac{h_a - 2r}{h_a}\right)\left(\sum \frac{h_a}{h_a - 2r}\right) \geq 9.$$

This gives

$$\sum \frac{h_a}{h_a - 2r} \geq 9.$$

But the left hand side is equal to

$$3 + \sum \frac{2r}{h_a - 2r}.$$

Simplification gives

$$\sum \frac{1}{h_a - 2r} \geq \frac{3}{r}.$$

■

3.4.47. $\sqrt{3}\Delta \leq r(4R + r).$

Proof: We start with the identity, $IH^2 = 2r^2 - 4R^2 \prod \cos\alpha$. However

$$\begin{aligned} 4R^2 \prod \cos\alpha &= 2R^2 \left(\sum \sin^2\alpha - 2\right) \\ &= \frac{1}{2}\sum a^2 - 4R^2 \\ &= \frac{1}{2}\left(\sum a\right)^2 - \sum ab - 4R^2 \\ &= \frac{1}{2}\left(\sum a\right)^2 - \left(s^2 + r(4R+r)\right) - 4R^2 \\ &= \frac{1}{4}\left(\sum a\right)^2 - (2R+r)^2. \end{aligned}$$

Here we have used $\sum ab = s^2 + r(4R+r)$; see **3.3.1**. Thus, it follows from $IH^2 \geq 0$ that

$$s^2 \leq 4\left(4R^2 + 4Rr + 3r^2\right).$$

This gives

$$\begin{aligned} \Delta^2 &\leq 4r^2\left(4R^2 + 4Rr + 3r^2\right) \\ &= r^2\left\{\left(4R^2 + (8/3)Rr + (1/3)r^2\right) + \frac{1}{3}\left(4Rr + 8r^2\right)\right\}. \end{aligned}$$

Using $4r^2 \leq 2Rr \leq R^2$, we obtain

$$\begin{aligned} \Delta^2 &\leq r^2\left\{\left(4R^2 + (8/3)Rr + (1/3)r^2\right) + \frac{4}{3}R^2\right\} \\ &= \frac{1}{3}r^2(4R+r)^2. \end{aligned}$$

This implies that $\sqrt{3}\Delta \leq r(4R+r)$. ■

3.4.48. $\dfrac{3}{2}s < \sum m_a < 2s.$

Proof: Let D, E, F be the mid-points of BC, CA, AB respectively. Let A' be the reflection of A in D. Similarly define B' and C'. Then $BA'CA$ is a parallelogram. (See Fig. 3.1.)

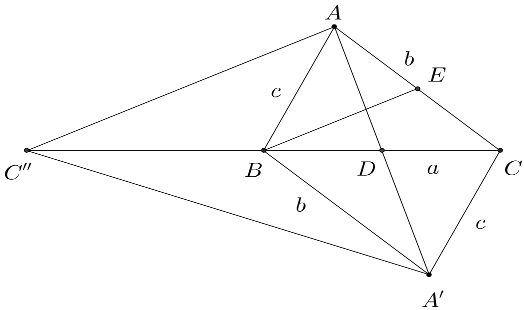

Fig 3.1

Using $2m_a = AA' < AB + BA' = b + c$ and similar inequalities for $2m_b$, $2m_c$, we get

$$m_a + m_b + m_c < \frac{1}{2}\Big(b + c + c + a + a + b\Big) = 2s.$$

Let C'' be the point of intersection of BC with a line through A and parallel to the median BE. Then the sides of $AA'C''$ are $2m_a$, $2m_b$ and $2m_c$. The lengths of its medians are $3a/2$, $3b/2$ and $3c/2$. Applying the inequality we have just proved to this triangle, we get

$$\frac{3}{2}\big(a + b + c\big) < 2\big(m_a + m_b + m_c\big).$$

This gives the left side inequality. ■

3.4.49. $\sum m_a \leq 4R + r$.

Proof: Let D, E, F be the mid-points of BC, CA, AB respectively. Join the circum-centre O to D, E, F. In the triangle ADO, we see that $AD \leq AO + OD = R + OD$, and equality holds if and only if A, O, D are collinear. Thus, we obtain $m_a \leq R + OD$. Similarly we have $m_b \leq R + OE$ and $m_c \leq R + OF$. Adding, we get

$$\sum m_a \leq 3R + OD + OE + OF.$$

Observe that $OD = R\cos\alpha$, $OE = R\cos\beta$ and $OF = R\cos\gamma$. Using

$$\sum \cos\alpha = 1 + 4\prod \sin\big(\alpha/2\big) = 1 + \frac{r}{R},$$

we get

$$\sum m_a \leq 3R + R\left(1 + \frac{r}{R}\right) = 4R + r.$$

■

3.4.50. $s < \sum w_a \leq \sqrt{s}\Big(\sum\sqrt{s-a}\Big) \leq \sqrt{3}s.$

Proof: Let AD $(= w_a)$ denote the bisector of α. Then $BD + AD > AB = c$ and $CD + AD > AC = b$. Adding we get

$$2w_a + (BD + CD) > b + c.$$

This shows that $w_a > (s - a)$ and hence $s < \sum w_a$. We also have

$$w_a = \frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)} \leq \sqrt{s(s-a)}.$$

Hence, we get

$$\sum w_a \leq \sqrt{s}\Big(\sum \sqrt{s-a}\Big).$$

Now using the Cauchy-Schwarz inequality, we have

$$\sum \sqrt{s-a} \leq \sqrt{3}\sqrt{s}.$$

Combining all these, we have the required inequalities.

■

3.4.51. $\sum w_a^2 \geq 3\sqrt{3}\Delta$.

Proof: Using Stewart's theorem or otherwise it is easy to compute that

$$w_a^2 = bc - \frac{a^2bc}{(b+c)^2}.$$

Since $bc/(b+c)^2 \leq 1/4$, we obtain

$$w_a^2 \geq bc - \frac{a^2}{4}.$$

This gives

$$\sum w_a^2 \geq \sum ab - \frac{1}{4}\sum a^2.$$

But we know that

$$\sum ab \geq 4\sqrt{3}\Delta + \sum (a-b)^2;$$

see **3.4.24**. Hence,

$$\begin{aligned}
\sum ab - \frac{1}{4}\sum a^2 &= \frac{1}{8}\Big(6\sum ab - \sum (a-b)^2\Big) \\
&= \frac{1}{8}\Big(5\sum ab + \sum ab - \sum (a-b)^2\Big) \\
&\geq \frac{1}{8}\Big(5\sum ab + 4\sqrt{3}\Delta\Big) \\
&\geq \frac{1}{8}\Big(20\sqrt{3}\Delta + 4\sqrt{3}\Delta\Big) \\
&= 3\sqrt{3}\Delta.
\end{aligned}$$

■

3.4.52. $R + r \leq \max\{h_a, h_b, h_c\}$.

Proof: This inequality is equivalent to

$$R + r \leq \max\left\{\frac{2\Delta}{a}, \frac{2\Delta}{b}, \frac{2\Delta}{c}\right\} = \frac{2\Delta}{\min\{a,b,c\}}.$$

Hence, it is sufficient to prove that $\min\{a,b,c\}(R+r) \leq 2\Delta$. However $R+r = R\left(\sum \cos\alpha\right) = OD+OE+OF$, where D, E, F are the midpoints of BC, CA, AB respectively and O is the circum-centre. Thus, the inequality reduces to the form

$$\min\{a,b,c\}(OD+OE+OF) \leq 2\Delta.$$

However we observe that

$$\begin{aligned} \min\{a,b,c\}(OD+OE+OF) &\leq a\cdot OD + b\cdot OE + c\cdot OF \\ &= 2[BOC] + 2[COA] + 2[AOB] \\ &= 2\Delta. \end{aligned}$$

■

3.4.53. $\sum a \sin(\alpha/2) \geq s$.

Proof: We first prove that in a triangle ABC, the following inequality holds:

$$\prod \sin(\alpha/2) \leq \prod \left(1 - \sin(\alpha/2)\right). \qquad (\star)$$

We observe that α, β, γ are the angles of a triangle if and only if $(\pi-\alpha)/2$, $(\pi-\beta)/2$, $(\pi-\gamma)/2$ are angles of another triangle. Hence, it is sufficient to prove

$$\prod \cos\alpha \leq \prod (1 - \cos\alpha)$$

But we have

$$(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma) = 8\sin^2(\alpha/2)\sin^2(\beta/2)\sin^2(\gamma/2) = \frac{r^2}{2R^2},$$

where R and r are circum-radius and in-radius of the triangle whose angles are α, β, γ. Since

$$4R^2\cos\alpha\cos\beta\cos\gamma = 2r^2 - IH^2 \leq 2r^2,$$

where I and H are the in-centre and ortho-centre of the triangle, we get the result.

Now, we may write the inequality $(\star)$ in the form

$$\frac{\prod \sin(\alpha/2)}{\prod \cos(\alpha/2)} \leq \frac{\prod \sin\left(\frac{\pi-\alpha}{4}\right)}{\prod \cos\left(\frac{\pi-\alpha}{4}\right)}.$$

Using the identities

$$\begin{aligned}\sum \sin(\alpha/2) &= 1+\prod \sin\left(\frac{\pi-\alpha}{4}\right),\\ \sum \cos(\alpha/2) &= \prod \cos\left(\frac{\pi-\alpha}{4}\right),\end{aligned}$$

we obtain

$$\frac{\prod \sin(\alpha/2)}{\prod \cos(\alpha/2)} \leq \frac{-1+\sum \sin(\alpha/2)}{\sum \cos(\alpha/2)}.$$

However using $r = 4R\prod \sin(\alpha/2)$ and $s = 4R\prod \cos(\alpha/2)$, the left side is simply r/s. Thus, the inequality takes the form

$$\frac{r}{s} \leq \frac{-1+\sum \sin(\alpha/2)}{\sum \cos(\alpha/2)}.$$

But observe that $(s-a)\sin(\alpha/2) = r\cos(\alpha/2)$. Thus, we get

$$\sum (s-a)\sin(\alpha/2) \leq s\Big(-1+\sum \sin(\alpha/2)\Big).$$

This reduces to

$$\sum a\sin(\alpha/2) \geq s.$$

■

3.4.54. $3\sum \cos\alpha \geq 2\sum \sin\alpha\sin\beta.$

Proof: We begin with the inequality

$$\sum \big(\cos\alpha - \cos\beta\big)^2 \geq 0.$$

Expanding this we get

$$\sum \cos^2\alpha \geq \sum \cos\alpha\cos\beta.$$

But then

$$\begin{aligned}\sum \cos^2\alpha \geq \sum \cos\alpha\cos\beta &\Longrightarrow \Big(\sum \cos\alpha\Big)^2 \geq 3\sum \cos\alpha\cos\beta\\ &\Longrightarrow 3\sum \cos\alpha\cos\beta \leq \frac{3}{2}\sum \cos\alpha\\ &\Longrightarrow 2\sum \cos\alpha\cos\beta \leq \sum \cos\alpha\end{aligned}$$

$$\Longrightarrow \sum \cos(\alpha-\beta) + \sum \cos(\alpha+\beta) \le \sum \cos\alpha$$
$$\Longrightarrow \sum \cos(\alpha-\beta) \le 2\sum \cos\alpha$$
$$\Longrightarrow \sum \cos(\alpha-\beta) - \sum \cos(\alpha+\beta) \le 3\sum \cos\alpha$$
$$\Longrightarrow 2\sum \sin\alpha \sin\beta \le 3\sum \cos\alpha.$$

We have used the well known result: $\sum \cos\alpha \le 3/2$. (**3.4.9**) ■

3.4.55. $\max\{r_a, r_b, r_c\} \ge \dfrac{3}{2}R.$

Proof: The required inequality is equivalent to

$$\max\left\{\frac{1}{s-a}, \frac{1}{s-b}, \frac{1}{s-c}\right\} \ge \frac{3R}{2\Delta}.$$

This is further equivalent to

$$\max\Big\{\tan(\alpha/2), \tan(\beta/2), \tan(\gamma/2)\Big\} \ge \frac{3Rr}{2\Delta} = \frac{3R}{2s}.$$

This may also be written in the form

$$\Big(\sum \sin\alpha\Big) \max\Big\{\tan(\alpha/2), \tan(\beta/2), \tan(\gamma/2)\Big\} \ge \frac{3}{2}.$$

However, we have

$$\Big(\sum \sin\alpha\Big) \max\Big\{\tan(\alpha/2), \tan(\beta/2), \tan(\gamma/2)\Big\}$$
$$\ge \sum \sin\alpha \tan(\alpha/2) = 2\sum \sin^2(\alpha/2).$$

On the other hand,

$$\begin{aligned}\sum \sin^2(\alpha/2) &= \frac{3}{2} - \frac{1}{2}\sum \cos\alpha \\ &= \frac{3}{2} - \frac{1}{2}\Big(1 + 4\prod \sin(\alpha/2)\Big) \\ &= \frac{3}{2} - \frac{1}{2}\Big(1 + \frac{r}{R}\Big) \\ &= 1 - \frac{r}{2R}.\end{aligned}$$

Since $2r \le R$, the result follows. In fact we have proved more:

$$\max\{r_a, r_b, r_c\} \ge 2R - r.$$

■

3.4.56. $R - 2r \ge w_a - h_a.$

Proof: We use

$$r = 4R\prod \sin(\alpha/2), \quad w_a = \frac{4R\sin\beta\sin\gamma}{\sin\beta+\sin\gamma}\cos(\alpha/2),$$

and

$$h_a = 2R\sin\beta\sin\gamma.$$

The inequality to be proved is

$$1 - 8\prod\sin(\alpha/2) \geq \frac{2\sin\beta\sin\gamma}{\cos\big((\beta-\gamma)/2\big)}\left(1-\cos\big((\beta-\gamma)/2\big)\right).$$

Let us put $t = \cos\big((\beta-\gamma)/2\big)$ and $x = \sin(\alpha/2)$. Then $t \in (0,1]$ and $x \in (0,1)$. We have to prove

$$f(x;t) = 2\big(1+t\big)x^2 - 4t^2x + \big(t+2t^2-t^3\big) \geq 0.$$

Consider this as a function of x, say $g(x)$. We observe that

$$g'(x) = 4(1+t)x - 4t^2, \quad g''(x) = 4(1+t) > 0.$$

Hence, g has the minimum at $x = t^2/(1+t)$. But observe that

$$f\left(\frac{t^2}{1+t};t\right) = \frac{t(1-t)(t+3)}{t+1} \geq 0.$$

Hence, $\min_x f(x;t) \geq 0$. Thus, $f(x;t) \geq 0$ for all $x \in (0,1)$ and $t \in (0,1]$. Equality holds if and only if $t = 1$. Equivalently $x = 1/2$. This is equivalent to $\alpha = \pi/3$ and $\beta = \gamma$, which corresponds to the case of an equilateral triangle. ■

3.4.57. $\sqrt{s(s-a)} + \sqrt{s(s-b)} + m_c \leq \sqrt{3}\, s.$

Proof: Using the AM-GM inequality, observe that

$$2\sqrt{(s-a)(s-b)} \leq s-a+s-b = c.$$

Equality holds if and only if $a = b$. Thus,

$$\begin{aligned}
4m_c^2 &= 2a^2+2b^2-c^2 \\
&= (a+b)^2+(a-b)^2-c^2 \\
&= (a+b)^2-(b+c-a)(a+c-b) \\
&= (a+b)^2-4(s-a)(s-b) \\
&= \left(a+b+2\sqrt{(s-a)(s-b)}\right)\left(a+b-2\sqrt{(s-a)(s-b)}\right) \\
&= \left(a+b+2\sqrt{(s-a)(s-b)}\right)\left(2s-\left(\sqrt{s-a}+\sqrt{s-b}\right)^2\right) \\
&\leq 2s\left(2s-\left(\sqrt{s-a}+\sqrt{s-b}\right)^2\right).
\end{aligned}$$

This implies that

$$\sqrt{s(s-a)} + \sqrt{s(s-b)} \leq \sqrt{2}\sqrt{s^2 - m_c^2}.$$

On the other hand

$$\left(\sqrt{2}\sqrt{s^2 - m_c^2} + m_c\right)^2 \leq 3s^2,$$

as this is equivalent to the inequality $\left(s^2 - 3m_c^2\right)^2 \geq 0$. Hence, the result follows. ∎

3.5 Two triangles one inscribed in the other

3.5.1. Let D, E, F be points in the interior of the segments BC, CA, AB of a triangle ABC. Prove that

$$[DEF] \geq \min\left\{[BDF], [CED], [AFE]\right\}.$$

Here equality holds if and only if D, E, F are the mid-points of the respective line segments on which they lie.

Proof: Let α, β, γ, be the areas of the three corner triangles so arranged that $0 < \alpha \leq \beta \leq \gamma$ and let $\delta = [DEF]$. We prove the stronger statement: $\delta \geq \sqrt{\alpha\beta}$. We normalise the areas such that $[ABC] = 1$. Let

$$\frac{BD}{a} = x,\ \frac{DC}{a} = x',\ \frac{CE}{b} = y,\ \frac{EA}{b} = y'\ ,\ \frac{AF}{c} = z,\ \frac{FB}{c} = z'.$$

Let AX and FY be the altitudes drawn respectively from A and F on to BC.

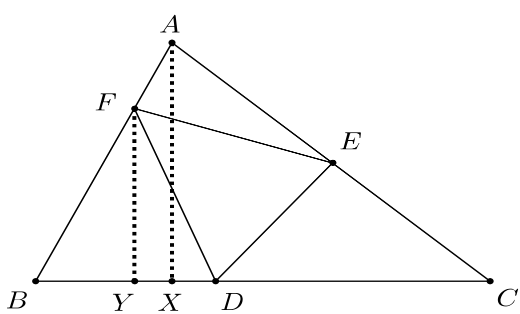

Fig 3.2

Then we see that

$$\frac{[BDF]}{[ABC]} = \frac{BD \cdot FY}{BC \cdot AX} = \frac{BD}{BC} \cdot \frac{FB}{AB} = xz'.$$

This gives $[BDF] = xz'$. Similarly, we obtain $[CED] = yx'$, $[AFE] = zy'$. Using these we get

$$\begin{aligned}
\delta &= 1 - \big(xz' + yx' + zy'\big) \\
&= 1 - x\big(1-z\big) + y\big(1-x\big) + z\big(1-y\big) \\
&= 1 - \big(x+y+z\big) + \big(xy+yz+zx\big) \\
&= \big(1-x\big)\big(1-y\big)\big(1-z\big) + xyz \\
&= x'y'z' + xyz.
\end{aligned}$$

Now we consider two cases : $\gamma < 1/4$ and $\gamma \geq 1/4$. Suppose $\gamma < 1/4$. If $\delta < \sqrt{\alpha\beta}$, then $\delta < \sqrt{\gamma \cdot \gamma} = \gamma < 1/4$ and hence

$$1 = \alpha + \beta + \gamma + \delta < \alpha + \beta + \frac{1}{2}.$$

This gives $\alpha + \beta > 1/2$ and in turn we obtain

$$\frac{1}{2} < \alpha + \beta \leq 2\gamma < \frac{1}{2}.$$

This contradiction proves that $\delta \geq \sqrt{\alpha\beta}$, when $\gamma < 1/4$.

Suppose on the other hand that $\gamma \geq 1/4$. In this case

$$\begin{aligned}
\delta &= x'y'z' + xyz \\
&\geq 2\Big(xx'yy'zz'\Big)^{1/2} \\
&= 2\Big(\alpha\beta\gamma\Big)^{1/2} \\
&= \sqrt{\alpha\beta} \cdot 2\sqrt{\gamma} \\
&\geq \sqrt{\alpha\beta}.
\end{aligned}$$

This completes the solution. ■

3.5.2. Let D, E, F be points in the interior of the segments BC, CA, AB of a triangle ABC. Prove that

$$\mathrm{p}(DEF) \geq \min\big\{\mathrm{p}(BDF), \mathrm{p}(CED), \mathrm{p}(AFE)\big\},$$

where $\mathrm{p}(XYZ)$ denotes the perimeter of the triangle XYZ. Here again equality holds if and only if D, E, F are the midpoints of the respective line segments on which they lie.

Proof: For angles x, y, z, let $u = \tan\left(x/2\right)$, $v = \tan\left(v/2\right)$, $w = \tan\left(w/2\right)$. Then, we have the following results.

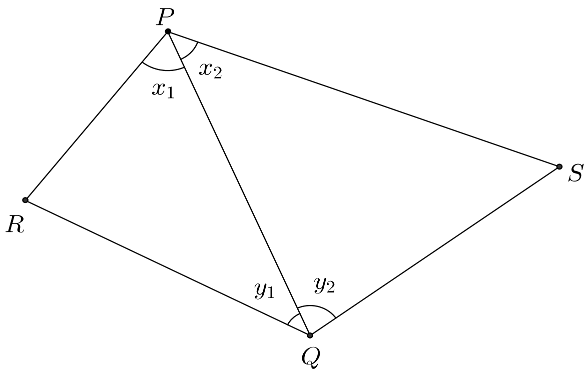

Fig. 3.3

(a) If $x+y+z=\pi$, then $uv+vw+wu=1$. This follows immediately from the property of tan function.

(b) If PQR and PQS are two triangles with the common side PQ and if $\angle RPQ = x_1$, $\angle RQP = y_1$, $\angle SPQ = x_2$, $\angle SQP = y_2$, then $\mathrm{p}(PQR) < \mathrm{p}(PQS)$ or $\mathrm{p}(PQR) = \mathrm{p}(PQS)$ or $\mathrm{p}(PQR) > \mathrm{p}(PQS)$ according as $u_1v_1 < u_2v_2$ or $u_1v_1 = u_2v_2$ or $u_1v_1 > u_2v_2$ respectively. .

proof: It is easy to prove that

$$\frac{\mathrm{p}(PQR)}{PQ} = \frac{2}{1-u_1v_1}, \quad \frac{\mathrm{p}(PQS)}{PQ} = \frac{2}{1-u_2v_2}.$$

Observe that $u_1v_1 < 1$ and $u_2v_2 < 1$. The result follows from comparison.

Suppose $\mathrm{p}(AFE) \geq \mathrm{p}(DEF)$ and $\mathrm{p}(FBD) \geq \mathrm{p}(DEF)$. We move A and B towards F, if necessary, while keeping D, E, F fixed until we get $\mathrm{p}(AFE) = \mathrm{p}(DEF) = \mathrm{p}(FBD)$. This only increases the perimeter of the triangle CED. Thus, it is sufficient to prove that $\mathrm{p}(DEF) \geq \mathrm{p}(CED)$ under the assumption that $\mathrm{p}(AFE) = \mathrm{p}(DEF) = \mathrm{p}(FBD)$.

Using (a) and (b), we get

$$\begin{aligned} uu_1 + u_1u_2 + u_2u &= 1, \\ vv_1 + v_1v_2 + v_2v &= 1, \\ ww_1 + w_1w_2 + w_2w &= 1, \\ u_2u_1 = uw, \quad v_1w_2 = vw, \quad v_2u_1 &= kvu. \end{aligned}$$

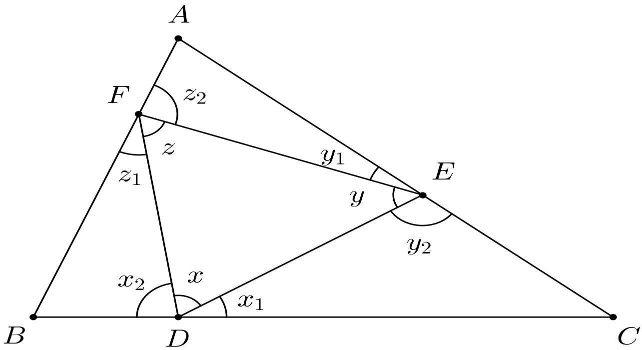

Fig. 3.4

Further we also have $uv + vw + wu = 1$. We have to prove that $k \leq 1$. We can get the following relations:

$$u_1 = \frac{kuv}{v_2}, \quad v_1 = \frac{vw}{w_2}, \quad w_1 = \frac{uw}{u_2}.$$

Substituting these in the first three relations, we obtain

$$\begin{aligned} u \cdot \frac{kuv}{v_2} + \frac{kuv}{v_2} \cdot u_2 + u_2 u &= 1, \\ v \cdot \frac{vw}{w_2} + \frac{vw}{w_2} \cdot v_2 + v_2 v &= 1, \\ w \cdot \frac{uw}{u_2} + \frac{uw}{u_2} \cdot w_2 + w_2 w &= 1. \end{aligned}$$

Solving for u_2 and v_2 from the last two relations, we get

$$v_2 = \frac{w_2 - v^2 w}{v(w_2 + w)}, \quad u_2 = \frac{uw(w + w_2)}{1 - ww_2}.$$

Substituting this in the first relation above, we obtain

$$ku^2v^2\left(1 + w^2\right)\left(w + w_2\right) + \left(w_2 - v^2 w\right)\left(u^2w^2 + u^2ww_2 - 1 + ww_2\right) = 0.$$

We show that

$$\begin{aligned} &\left(w_2 - v^2 w\right)\left(u^2w^2 + u^2ww_2 - 1 + ww_2\right) + u^2v^2\left(1 + w^2\right)\left(w + w_2\right) \\ &\qquad = w\left(1 + u^2\right)\left(w_2 - v\right)^2. \end{aligned}$$

We have

$$\begin{aligned}
\big(w_2 - v^2 w\big)\big(u^2w^2 + u^2ww_2 - 1 + ww_2\big) &+ u^2v^2\big(1+w^2\big)\big(w+w_2\big)\\
&= u^2v^2\big(w+w_2\big) + u^2w^2w_2 + u^2ww_2^2 - w_2\\
&\qquad\qquad + ww_2^2 + v^2w - v^2w^2w_2\\
&= u^2v^2w + w_2\big(u^2v^2 + v^2w^2 + w^2u^2\big) - 2v^2w^2w_2\\
&\qquad\qquad + u^2ww_2^2 - w_2 + ww_2^2 + v^2w\\
&= w\Big\{u^2v^2 - 2vw_2\big(uv+vw+wu\big)\\
&\qquad + \big(u^2+1\big)\big(v^2+w_2^2-2vw_2\big) + 2vw_2\Big\}\\
&= w\big(1+u^2\big)\big(w_2-v\big)^2;
\end{aligned}$$

we have used $uv + vw + wu = 1$. It follows that

$$\big(k-1\big)u^2v^2\big(1+w^2\big)\big(w+w_2\big) + w\big(1+u^2\big)\big(w_2-v\big)^2 = 0.$$

The expression shows that $k \leq 1$. This implies that $v_2u_1 \leq vu$. This is equivalent to $\mathrm{p}(DEF) \geq \mathrm{p}(CED)$ in view of (b). Equality corresponds to $k = 1$. This forces $w_2 = v$ and hence $v_1 = w$. Using the symmetry we conclude $u_2 = w$, $w_1 = u$ and $v_2 = u$, $u_1 = v$. Looking at the expressions involving $u, v, w,\ u_1, v_1, w_1,\ u_2, v_2, w_2$, we conclude that D, E, F are the mid-points of BC, CA, AB respectively. ■

3.5.3. Let ABC be an acute-angled triangle and DEF be an acute-angled triangle inscribed in ABC. Show that

$$\lambda \leq \mathrm{R}(DEF) \leq \mu,$$

where

$$\begin{aligned}
\lambda &= \min\big\{\mathrm{R}(AFE), \mathrm{R}(BDF), \mathrm{R}(CED)\big\},\\
\mu &= \max\big\{\mathrm{R}(AFE), \mathrm{R}(BDF), \mathrm{R}(CED)\big\}.
\end{aligned}$$

(Here $\mathrm{R}(XYZ)$ denotes the circum-radius of the triangle XYZ.)

Proof: The segment EF is given by $EF = 2\mathrm{R}(AFE)\sin\alpha$ which is equal to $2\mathrm{R}(DEF)\sin\angle FDE$. This implies that

$$\frac{\mathrm{R}(DEF)}{\mathrm{R}(AFE)} = \frac{\sin\alpha}{\sin\angle FDE}.$$

Similarly, one can obtain

$$\frac{\mathrm{R}(DEF)}{\mathrm{R}(BDF)} = \frac{\sin\beta}{\sin\angle DEF},$$

$$\frac{\mathrm{R}(DEF)}{\mathrm{R}(CED)} = \frac{\sin\gamma}{\sin\angle EFD}.$$

However, $\alpha+\beta+\gamma = \pi = \angle FDE + \angle DEF + \angle EFD$ and hence at least one of the ratios $\alpha : \angle FDE$, $\beta : \angle DEF$, $\gamma : \angle EFD$ is ≥ 1, and at least one of these ratios is also ≤ 1. Since all the angles are acute, at least one of the ratios $\mathrm{R}(DEF) : \mathrm{R}(AFE)$, $\mathrm{R}(DEF) : \mathrm{R}(BDF)$, $\mathrm{R}(DEF) : \mathrm{R}(CED)$ is greater than or equal to 1, and at least one of these ratios is less than or equal to 1. This proves that

$$\lambda \leq \mathrm{R}(DEF) \leq \mu.$$

Remark: If $\mathrm{r}(XYZ)$ denotes the in-radius of triangle XYZ, it is true that

$$\mathrm{r}(DEF) \geq \min\left\{\mathrm{r}(AFE), \mathrm{r}(BDF), \mathrm{r}(CED)\right\}.$$

See [5]. ■

3.5.4. Let ABC be an acute-angled triangle and D, E, F be arbitrary points on the segments BC, CA, AB respectively. Show that

$$2\sum a'\cos\alpha \geq \sum a\cos\alpha,$$

where a', b', c' are the sides of of DEF (i.e., $EF = a'$, $FD = b'$, $DE = c'$).

Proof: Let $BD = x_1$, $DC = y_1$, $CE = x_2$, $EA = y_2$, $AF = x_3$, $FB = y_3$. Then

$$\begin{aligned} a' &\geq a - y_3\cos\beta - x_2\cos\gamma, \\ b' &\geq b - y_1\cos\gamma - x_3\cos\alpha, \\ c' &\geq c - y_2\cos\alpha - x_1\cos\beta. \end{aligned}$$

Since $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are positive, the above estimates give

$$\sum a'\cos\alpha \geq \sum a\cos\alpha - \sum a\cos\beta\cos\gamma.$$

However,

$$\sum a\cos\alpha = \sum 2R\sin\alpha\cos\alpha = R\sum \sin 2\alpha = 4R\sin\alpha\sin\beta\sin\gamma,$$

and

$$\begin{aligned} \sum a\cos\beta\cos\gamma &= 2R\sum \sin\alpha\cos\beta\cos\gamma \\ &= 2R\cos\gamma\sin(\alpha+\beta) + 2R\cos\alpha\cos\beta\sin\gamma \\ &= 2R\sin\gamma\cos\gamma + 2R\cos\alpha\cos\beta\sin\gamma \\ &= 2R\sin\gamma\big(\cos\gamma + \cos\alpha\cos\beta\big) \\ &= 2R\sin\alpha\sin\beta\sin\gamma. \end{aligned}$$

Thus, it follows that

$$\begin{aligned}
\sum a' \cos\alpha &\geq \sum a\cos\alpha - \sum a\cos\beta\cos\gamma \\
&= 4R\sin\alpha\sin\beta\sin\gamma - 2R\sin\alpha\sin\beta\sin\gamma \\
&= 2R\sin\alpha\sin\beta\sin\gamma \\
&= \frac{1}{2}\sum a\cos\alpha.
\end{aligned}$$

Hence, $2\sum a'\cos\alpha \geq \sum a\cos A$, as required. It may be seen that equality holds if and only if D, E, F are mid-points of BC, CA, AB respectively. ■

3.6 Let P be a point ...

In this section, we consider inequalities involving a point inside a triangle and the quantities associated with such a point. We have already seen some such inequalities. For example, we can take the point P to be the circum-centre of ABC and then the distance of P from each vertex is equal to R. We have studied inequalities involving R and other elements of a triangle. Similarly, we have gone through several inequalities involving I, H or G.

Let us start with a point P inside a triangle ABC. We use the following notations for elements involving P: $R_1 = AP$, $R_2 = BP$, $R_3 = CP$; similarly r_1, r_2, r_3 denote respectively the distances of P from BC, CA, AB. Now we can develop a large number of inequalities involving $R_1, R_2, R_3, r_1, r_2, r_3$ and other elements of the triangle. We explore some of them below.

3.6.1. Let P be any point in a triangle ABC and s be its semi perimeter. Prove that

$$s < R_1 + R_2 + R_3 < 2s.$$

Proof: Since $AB < PA + PB$, $BC < PB + PC$ and $CA < PC + PA$, we get by adding $s < R_1 + R_2 + R_3$.

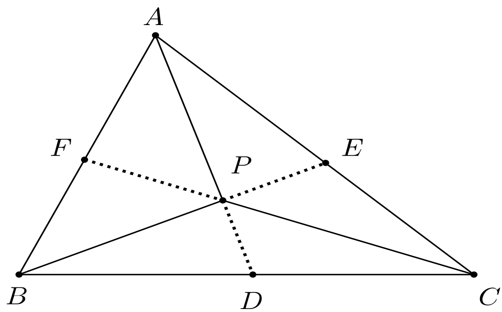

Fig. 3.5

Extend AP, BP, CP to meet opposite sides BC, CA, AB respectively in D, E, F. Using the triangle inequality, we have $BE < BC + CE$. Adding AP both sides and using $BE = BP + PE$, $AP < PE + AE$, we obtain $BP + PE + AP < BC + CE + AP < BC + CE + PE + AE$ so that $BP + AP < BC + CE + AE = BC + CA$. Similarly, we obtain $AP + CP < BC + AB$ and $BP + CP < AB + AC$. Adding these three inequalities, we get $R_1 + R_2 + R_3 < AB + BC + CA = 2s$.

Here is a beautiful extension of this result (It was short-listed in the 40-th IMO held in Romania during July 2000). The right side inequality can be strengthened to $\min\{PA, PB, PC\} + PA + PB + PC < 2s$. This depends on the following property: If P is any interior point of a convex quadrilateral $ABCD$, then $PA + PB < AD + DC + CB$. The proof is easy. Extend AP to meet the quadrilateral in Q and suppose, for example, Q lies on DC. Then $PA + PB < PA + PQ + QB \le AQ + QB \le AQ + QC + CB \le AD + DQ + QC + CB = AD + DC + CB$.

Let DEF be the medial triangle of ABC. This divides ABC into 4 regions. Each region is covered by at least two of the three convex quadrilaterals $ABDE$, $BCEF$, and $CAFD$. If, for example, P lies in quadrilaterals $ABDE$ and $BCEF$, then we have $PA + PB < AE + ED + DB$ and $PB + PC < BF + FE + EC$. Adding these two we get $PB + (PA + PB + PC) < AE + ED + DB + BF + FE + EC = AB + BC + CA$.

As an interesting corollary to this, consider an acute triangle and let P be its circum-centre (which lies inside ABC). Then $PA = PB = PC = R$, the circum-radius of ABC. The inequality takes the form $4R < a + b + c$. Using $a = 2R \sin A$ and similar expressions, we get $\sin\alpha + \sin\beta + \sin\gamma > 2$.

■

3.6.2. Let ABC be a triangle in which $a > b > c$. Let P be any interior

point and suppose AP, BP, CP meet the opposite sides in D, E, F respectively. Prove that

$$PD + PE + PF < a.$$

Proof: Draw PX parallel to AB, PY parallel to AC, XK parallel to CF and YL parallel to PE. Observe that a is larger than each of AD, BE, CF. Since XPY is similar to BAC, we get $XY > PD$. Similarly, we obtain $BX > XK = PF$ and $CY > YL = PE$. Adding these we get

$$PD + PE + PF < XY + CY + BX = BC = a.$$

■

3.6.3. Let P be a point inside a triangle ABC and let D, E, F be feet of perpendiculars from P on to the sides BC, CA, AB respectively. Prove that

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3).$$

(This is known as the Erdös-Mordell inequality.)

Proof: Suppose L and M are arbitrary points chosen on the sides AB and AC. Complete the parallelograms $LAPS$ and $MAPT$. Then $LMTS$ is also a parallelogram and $[LAPS] + [MAPT] = [LMTS]$.

Observe that $[LAPS] = LA \cdot PF$, $[MAPT] = MA \cdot PE$,$[LMTS] \leq LM \cdot MT = LM \cdot PA$. Here, the points L, M can be chosen on the extended lines AB and AC. Choosing L, M such that $AM = AB$ and $AL = AC$, we get

$$AC \cdot PF + AB \cdot PE \leq LM \cdot PA.$$

But, the triangle ALM is similar to ACB and hence $LM = CB$. Thus, we get $bPF + cPE \leq aPA$. It follows that

$$PA \geq \frac{b}{a}PF + \frac{c}{a}PE.$$

Similarly we get

$$PB \geq \frac{c}{b}PD + \frac{a}{b}PF, \quad PC \geq \frac{a}{c}PE + \frac{b}{c}PD.$$

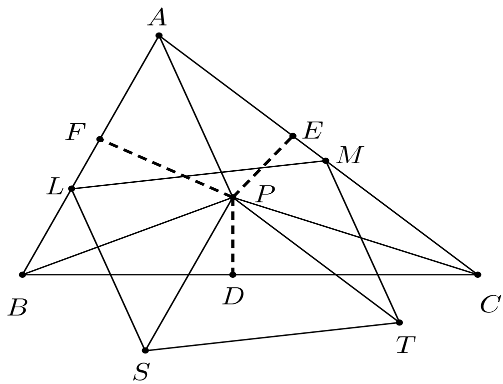

Fig. 3.6

Adding these inequalities we get

$$\begin{aligned} PA+PB+PC &\geq \left(\frac{c}{b}+\frac{b}{c}\right)PD+\left(\frac{a}{c}+\frac{c}{a}\right)PE+\left(\frac{a}{b}+\frac{b}{a}\right)PF \\ &\geq 2(PD+PE+PF). \end{aligned}$$

Alternate Proof: Observe that $AFPE$ is a cyclic quadrilateral. This gives $FE = AP\sin A$. Using cosine rule, we also get

$$\begin{aligned} FE^2 &= PE^2+PF^2-2PE\cot PF\cdot\cos\angle FPE \\ &= PE^2+PF^2+2PE\cot PF\cdot\cos\alpha \\ &= (PE\sin\gamma+PF\sin\beta)^2+(PE\cos\gamma-PF\cos\beta)^2 \\ &\geq (PE\sin\gamma+PF\sin\beta)^2. \end{aligned}$$

It follows that

$$PA=\frac{FE}{\sin\alpha}\geq\frac{PE\sin\gamma+PF\sin\beta}{\sin\alpha}.$$

Similarly we get

$$PB\geq\frac{PD\sin\gamma+PF\sin\alpha}{\sin\beta},\quad PC\geq\frac{PE\sin\alpha+PD\sin\beta}{\sin\gamma}.$$

Adding these we get

$$\begin{aligned} PA+PB+PC &\geq PD\left(\frac{\sin\gamma}{\sin\beta}+\frac{\sin\beta}{\sin\gamma}\right)+PE\left(\frac{\sin\gamma}{\sin\alpha}+\frac{\sin\alpha}{\sin\gamma}\right) \\ &\qquad +PF\left(\frac{\sin\alpha}{\sin\beta}+\frac{\sin\beta}{\sin\alpha}\right) \\ &\geq 2(PD+PE+PF); \end{aligned}$$

we have used the AM-GM inequality at the end.

Here are some interesting consequences of the Erdös-Mordell inequality.

(i) Take $P = O$, the circum-centre of ABC. Then $PA = PB = PC = R$ and $PD = R\cos A$, $PE = R\cos\beta$ and $PF = R\cos\gamma$. Thus, we get (see **3.4.9**)

$$\sum \cos\alpha \leq \frac{3}{2}.$$

(ii) Taking $P = H$, the ortho-centre of ABC, we get $PA = 2R\cos\alpha$, $PB = 2R\cos\beta$, $PC = 2R\cos\gamma$. Hence, we obtain $PD = 2R\cos\beta\cos\gamma$, $PE = 2R\cos\gamma\cos\alpha$, and $PF = 2R\cos\alpha\cos\beta$. Thus, we get

$$2(\cos\alpha\cos\beta + \cos\beta\cos\gamma + \cos\gamma\cos\alpha) \leq (\cos\alpha + \cos\beta + \cos\gamma).$$

Using the inequality in (i), we obtain

$$\cos\alpha\cos\beta + \cos\beta\cos\gamma + \cos\gamma\cos\alpha \leq \frac{3}{4}.$$

(iii) Taking $P = I$, the in-centre, the inequality takes the form

$$AI + BI + CI \geq 6.$$

Using $AI = r\text{cosec}\,(\alpha/2)$, etc., we get

$$\sum \text{cosec}\,(\alpha/2) \geq 6.$$

(iv) Let $P = K$, the symmedian point. If AK meets BC in D', then it is known that $BD'/D'C = c^2/b^2$ and similar ratios for other intersection points. Using this and Stewart's theorem, we can compute the symmedian length:

$$AD' = \frac{2bc}{b^2 + c^2} m_a.$$

This implies that

$$AK = \frac{2bc}{a^2 + b^2 + c^2} m_a,$$

and similar expressions for BK, CK. Using these we can also get the lengths of perpendiculars KL, KM, KN on to BC, CA, AB:

$$\begin{aligned} KL &= \frac{2a}{a^2 + b^2 + c^2}[ABC], \\ KM &= \frac{2b}{a^2 + b^2 + c^2}[ABC], \\ KN &= \frac{2c}{a^2 + b^2 + c^2}[ABC]. \end{aligned}$$

In this case, the Erdös-Mordell inequality takes the form

$$\frac{m_a}{a} + \frac{m_b}{b} + \frac{m_c}{c} \geq \frac{s}{R}.$$

■

3.6.4. $\sum \frac{1}{r_1 r_2} \geq 2\sum \frac{1}{R_1 r_1} \geq 4\sum \frac{1}{R_1 R_2}$.

Proof: Reflect P in the sides BC, CA, AB to get points A', B', C'. Consider the triangle $A'B'C'$ and its interior point P. If we denote the distances of P from the vertices A', B', C' respectively by R_1', R_2', R_3' and the distances of P from the sides $B'C'$, $C'A'$, $A'B'$ by r_1', r_2', r_3', then the Erdös-Mordell inequality applied to triangle $A'B'C'$ gives

$$R_1' + R_2' + R_3' \geq 2\left(r_1' + r_2' + r_3'\right).$$

If D, E are the feet of perpendiculars drawn from P to BC, CA respectively, then $DE = R_3 \sin\gamma$. Using the fact that D and E are the mid-points of PA' and PB', we get $A'B' = 2R_3 \sin\gamma$. Now let F' be the foot of perpendicular from P on to $A'B'$. We then have

$$PF' = \frac{2\left[A'PB'\right]}{A'B'}.$$

On the other hand

$$\begin{aligned} \left[A'PB'\right] &= \frac{1}{2}PA' \cdot PB' \cdot \sin\angle A'PB' \\ &= \frac{1}{2}(2r_1)(2r_2)\sin(\pi - \gamma) \\ &= 2r_1 r_2 \sin\gamma. \end{aligned}$$

This gives $r_3' = 2r_1 r_2 / R_3$. Similarly, we may obtain

$$r_1' = 2r_2 r_3 / R_1, \quad r_2' = 2r_3 r_1 / R_2.$$

Moreover

$$R_1' = 2r_1, \quad R_2' = 2r_2, \quad R_3' = 2r_3.$$

Using these we get

$$\sum \frac{1}{r_1 r_2} \geq 2\sum \frac{1}{R_1 r_1}.$$

The same process adopted to the point P and the triangle $A'B'C'$ gives

$$\sum \frac{1}{R_1 r_1} \geq 2\sum \frac{1}{R_1 R_2}.$$

■

3.6.5. $\sum R_1 r_1 \geq 2\sum r_1 r_2$.

Proof: We refer to Fig. 3.6. Taking $L = B$ and $M = C$, there, we obtain

$$BA \cdot PF + CA \cdot PE \leq BC \cdot PA.$$

This reduces to $cr_3 + br_2 \leq aR_1$. This may be written in the form

$$R_1 r_1 \geq \frac{c}{a} r_3 r_1 + \frac{b}{a} r_2 r_1.$$

Similarly, we obtain

$$\begin{aligned} R_2 r_2 &\geq \frac{a}{b} r_1 r_2 + \frac{c}{b} r_3 r_2 \\ R_3 r_3 &\geq \frac{b}{c} r_2 r_3 + \frac{a}{c} r_1 r_3. \end{aligned}$$

Adding these three inequalities, we get

$$\begin{aligned} \sum R_1 r_1 &\geq \sum r_1 r_2 \left(\frac{b}{a} + \frac{a}{b}\right) \\ &\geq 2 \sum r_1 r_2; \end{aligned}$$

we have used the AM-GM inequality. Equality holds if and only if ABC is equilateral and P is its centre.

Consequences:

(i) Taking $P = I$, the in-centre of ABC, we have $r_1 = r_2 = r_3 = r$ and

$$R_1 = r \operatorname{cosec}\big(\alpha/2\big), R_2 = r \operatorname{cosec}\big(\beta/2\big), R_3 = r \operatorname{cosec}\big(\gamma/2\big).$$

We obtain

$$\sum \operatorname{cosec}\big(\alpha/2\big) \geq 6.$$

(ii) Taking $P = O$, the circum-centre of ABC, we get

$$\sum \cos\alpha \geq 2 \sum \cos\alpha \cos\beta.$$

(iii) If $P = H$, the ortho-centre, then

$$R_1 = 2R\cos\alpha, \quad R_2 = 2R\cos\beta, \quad R_3 = 2R\cos\gamma,$$

and

$$r_1 = 2R\cos\beta\cos\gamma,\ r_2 = 2R\cos\gamma\cos\alpha,\ r_3 = 2R\cos\alpha\cos\beta.$$

The inequality we obtain is $\sum \cos\alpha \leq \frac{3}{2}$. (See **3.4.9**.)

(iv) If we take $P = G$, the centroid, then the inequality takes the form

$$\sum m_a h_a \geq \sum h_a h_b.$$

Adopting the method developed in **3.6.4**, we obtain

$$\sum R_1 R_2 \geq 2 \sum R_1 r_1.$$

Repeating this process once more, we also get

$$\sum \frac{1}{r_1} \geq 2 \sum \frac{1}{R_1}.$$

■

3.6.6. Let P be a point inside a triangle ABC and suppose AP, BP, CP extended meet BC, CA, AB respectively in D, E, F. Prove that

1. $$\frac{AP}{PD} + \frac{BP}{PE} + \frac{CP}{PF} \geq 6;$$
2. $$\frac{AP}{PD} \cdot \frac{BP}{PE} \cdot \frac{CP}{PF} \geq 8;$$
3. $$\frac{AD}{AP} + \frac{BE}{BP} + \frac{CF}{CP} \geq \frac{9}{2};$$
4. $$\frac{AD}{PD} \cdot \frac{BE}{PE} \cdot \frac{CF}{PF} \geq 27;$$
5. $$\frac{PD}{PA} + \frac{PE}{PB} + \frac{PF}{PC} \geq \frac{3}{2}$$

Proof: Let us take

$$\frac{BD}{DC} = \frac{x}{y}, \quad \frac{CE}{EA} = \frac{z}{x}, \quad \frac{AF}{FB} = \frac{y}{z}.$$

Then we see that $BD = \dfrac{ax}{x+y}$, $CD = \dfrac{ay}{x+y}$, $AF = \dfrac{cy}{z+y}$ and $FB = \dfrac{cz}{y+z}$. Using Menelaus' theorem, we obtain

$$\frac{AP}{PD} = \frac{BC}{CD} \cdot \frac{AF}{FB} = \frac{x+y}{z},$$

and similar expressions: $\dfrac{BP}{PE} = \dfrac{z+x}{y}$, $\dfrac{CP}{PF} = \dfrac{y+z}{x}$. Adding these we get

$$\begin{aligned}\frac{AP}{PD} + \frac{BP}{PE} + \frac{CP}{PF} &= \left(\frac{x}{z} + \frac{z}{x}\right) + \left(\frac{y}{x} + \frac{x}{y}\right) + \left(\frac{z}{y} + \frac{y}{z}\right) \\ &\geq 2 + 2 + 2 = 6.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{AP}{PD}\cdot\frac{BP}{PE}\cdot\frac{CP}{PF} &= \frac{(x+y)(y+z)(z+x)}{xyz}\\ &\geq \frac{2\sqrt{xy}\cdot 2\sqrt{yz}\cdot 2\sqrt{zx}}{xyz}=8.\end{aligned}$$

We also observe that

$$\frac{AD}{PD}=1+\frac{AP}{PD}=\frac{x+y+z}{z},\qquad \frac{AD}{AP}=\frac{x+y+z}{x+y}.$$

Thus,

$$\begin{aligned}\frac{AD}{AP}+\frac{BE}{BP}+\frac{CF}{CP} &= (x+y+z)\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right)\\ &\geq \frac{9}{2}.\end{aligned}$$

And

$$\frac{AD}{PD}\cdot\frac{BE}{PE}\cdot\frac{CF}{PF}=\frac{(x+y+z)^3}{xyz}\geq 27.$$

Finally

$$\begin{aligned}&\frac{PD}{PA}+\frac{PE}{PB}+\frac{PF}{PC}\\ &\qquad= \frac{z}{x+y}+\frac{x}{y+z}+\frac{x}{y+z}\\ &\qquad= (x+y+z)\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right)-3\\ &\qquad\geq \frac{9}{2}-3=\frac{3}{2}.\end{aligned}$$

■

3.6.7. Prove

(i) $\sum \frac{h_a}{r_1}\geq 9$;

(ii) $h_ah_bh_c\geq 27r_1r_2r_3$;

(iii) $\big(h_a-r_1\big)\big(h_b-r_2\big)\big(h_c-r_3\big)\geq 8r_1r_2r_3$;

(iv) $\min\big\{h_a,h_b,h_c\big\}\leq r_1+r_2+r_3\leq \max\{h_a,h_b,h_c\}$;

(v) $\sum \frac{r_1}{h_a-r_1}\geq \frac{3}{2}$.

Proof: Let D, E, F respectively be the points where AP, BP, CP meet BC, CA, AB. Put $BD : DC = z : y$, $CE : EA = x : z$ and $AF : FB = y : x$.

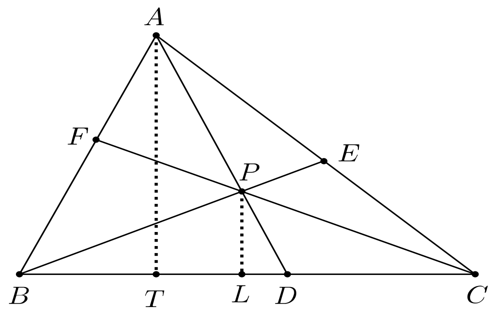

Fig. 3.7

We easily compute

$$\frac{AP}{PD} = \frac{y+z}{x}, \quad \frac{BP}{PE} = \frac{z+x}{y}, \quad \frac{CP}{PF} = \frac{y+x}{z}.$$

Thus, we obtain

$$\frac{AD}{PD} = \frac{x+y+z}{x}, \quad \frac{BE}{PE} = \frac{x+y+z}{y}, \quad \frac{CF}{PF} = \frac{x+y+z}{z}.$$

Using similar triangles PDL and ADT, we have

$$\frac{h_a}{r_1} = \frac{AT}{PL} = \frac{AD}{PD} = \frac{x+y+z}{x}.$$

Similarly, we may obtain

$$\frac{h_b}{r_2} = \frac{x+y+z}{y}, \text{ and } \frac{h_c}{r_3} = \frac{x+y+z}{z}.$$

Now

$$\begin{aligned} \sum \frac{h_a}{r_1} &= \left(x+y+z\right)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \\ &\geq 9. \end{aligned}$$

This proves (i). Similarly,

$$\frac{h_a}{r_1} \cdot \frac{h_b}{r_2} \cdot \frac{h_c}{r_3} = \frac{(x+y+z)^3}{xyz} \geq 27,$$

giving (ii). We also observe

$$\begin{aligned}\frac{(h_a-r_1)(h_b-r_2)(h_c-r_3)}{r_1r_2r_3} &= \left(\frac{h_a}{r_1}-1\right)\left(\frac{h_b}{r_2}-1\right)\left(\frac{h_c}{r_3}-1\right)\\ &= \left(\frac{z+y}{x}\right)\left(\frac{x+z}{y}\right)\left(\frac{y+x}{z}\right)\\ &= \frac{z+y}{\sqrt{yz}}\cdot\frac{x+z}{\sqrt{xz}}\cdot\frac{y+x}{\sqrt{yx}}\\ &\geq 2\cdot 2\cdot 2=8,\end{aligned}$$

proving (iii). Using the same expressions, we also get

$$\sum\frac{r_1}{h_a}=1.$$

This implies (iv). Finally, we have

$$\frac{r_1}{h_a-r_1}=\frac{PD}{AP}=\frac{x}{y+z},$$

and similar expressions for $\dfrac{r_2}{h_b-r_2}$, $\dfrac{r_3}{h_c-r_3}$. Thus, we obtain

$$\sum\frac{r_1}{h_a-r_1}=\sum\frac{x}{y+z}=\left(x+y+z\right)\left(\frac{1}{y+z}+\frac{1}{z+x}+\frac{1}{x+y}\right)-3\geq\frac{9}{2}-3=\frac{3}{2}.$$

We thus obtain (v). ■

3.6.8. Let P be an interior point of a triangle ABC. Prove

$$\frac{1}{r_A}+\frac{1}{r_B}+\frac{1}{r_C}\geq\frac{3(3+2\sqrt{3})}{R+r},$$

where r_A, r_B, r_C denote the in-radii of triangles PBC, PCA, PAB respectively.

Proof: We refer to the figure Fig. 3.7. We have

$$r_A=\frac{a\cdot PL}{a+PB+PC},\qquad\frac{PL}{h_a}=\frac{PD}{AD}.$$

Thus, we get

$$\frac{1}{r_A}=\frac{a+PB+PC}{a\cdot h_a}\cdot\frac{AD}{PD}.$$

Suppose $BD:DC=z:y$, $CE:EA=x:z$ and $AF:FB=y:x$. Then

$$\frac{AD}{PD}=\frac{x+y+z}{x},\ \frac{BE}{PE}=\frac{x+y+z}{y},\ \frac{CF}{PF}=\frac{x+y+z}{z}.$$

We hence obtain

$$PA^2 = \frac{(y+z)^2}{(x+y+z)^2} AD^2.$$

Using Stewart's theorem, we calculate AD;

$$AD^2 = \frac{(y+z)yb^2 + (y+z)zc^2 - yza^2}{(y+z)^2}.$$

Thus, we obtain

$$PA^2 = \frac{(y+z)yb^2 + (y+z)zc^2 - yza^2}{(x+y+z)^2}.$$

If we use the standard formula $a^2 = b^2 + c^2 - 2bc\cos\alpha$, this expression reduces to

$$PA^2 = \frac{y^2b^2 + z^2c^2 + 2bcyz\cos\alpha}{(x+y+z)^2}.$$

This may be written in the form

$$PA^2 = \frac{h_a^2}{(x+y+z)^2}\left\{\left(z\cot\gamma - y\cot\beta\right)^2 + \left(z+y\right)^2\right\}.$$

Similar expressions may be obtained for PB^2 and PC^2:

$$PB^2 = \frac{h_b^2}{(x+y+z)^2}\left\{\left(x\cot\alpha - z\cot\gamma\right)^2 + \left(x+z\right)^2\right\},$$

$$PC^2 = \frac{h_c^2}{(x+y+z)^2}\left\{\left(y\cot\beta - x\cot\alpha\right)^2 + \left(y+x\right)^2\right\}.$$

Observe that

$$PB \geq \frac{h_b}{x+y+z}\left(x+z\right), \quad PC \geq \frac{h_c}{x+y+z}\left(y+x\right).$$

Thus, we obtain

$$\begin{aligned} \frac{1}{r_A} &= \frac{x+y+z}{2\Delta}\left\{\frac{a+PB+PC}{x}\right\} \\ &\geq \frac{(x+y+z)a}{2\Delta x} + \frac{h_b(x+z)}{2\Delta x} + \frac{h_c(y+x)}{2\Delta x}. \end{aligned}$$

Similarly, we may obtain

$$\frac{1}{r_B} \geq \frac{(x+y+z)b}{2\Delta y} + \frac{h_c(y+x)}{2\Delta y} + \frac{h_a(z+y)}{2\Delta y},$$

and

$$\frac{1}{r_C} \geq \frac{(x+y+z)c}{2\Delta z} + \frac{h_a(z+y)}{2\Delta z} + \frac{h_b(x+z)}{2\Delta z}.$$

Adding, we have

$$\begin{aligned}\frac{1}{r_A}+\frac{1}{r_B}+\frac{1}{r_C} &\geq \frac{x+y+z}{2\Delta}\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)+\frac{1}{2\Delta}\sum h_a\left(z+y\right)\left(\frac{1}{z}+\frac{1}{y}\right)\\ &\geq \frac{x+y+z}{2\Delta}\sum\frac{a}{x}+\frac{2}{\Delta}\sum h_a.\end{aligned}$$

However, we know that (see **3.4.38**)

$$\frac{h_a+h_b+h_c}{\Delta}=2\sum\frac{1}{a}\geq\frac{3\sqrt{3}}{R+r}.$$

We also observe that

$$\frac{x+y+z}{2\Delta}\sum\frac{a}{x}\geq\frac{9}{2\Delta}(abc)^{1/3}$$

and (**3.4.28**)

$$\left(abc\right)^2\geq\left(\frac{4\Delta}{\sqrt{3}}\right)^3.$$

Using **3.4.47**, we can prove that $\sqrt{3}\Delta\leq\left(R+r\right)^2$. Combining all these, we get

$$\frac{1}{r_A}+\frac{1}{r_B}+\frac{1}{r_C}\geq\frac{3\left(3+2\sqrt{3}\right)}{R+r},$$

■

3.6.9. Let P be a point in a triangle ABC. Then

$$\sum\frac{R_1R_2}{ab}\geq 1.$$

Equality holds if and only if either P is one of the vertices or $P=H$, the ortho-centre of ABC.

Proof: Let us consider the vertices of ABC to be points in the complex plane; let us denote A,B,C by complex numbers z_1,z_2,z_3 respectively. Let us write z for P. Now consider the function

$$g(z)=\sum\frac{\left(z-z_1\right)\left(z-z_2\right)}{\left(z_3-z_1\right)\left(z_3-z_2\right)}.$$

This is a quadratic polynomial and $g\left(z_1\right)=g\left(z_2\right)=g\left(z_3\right)=1$. Hence, $g(z)=1$ for all z. Thus, we have

$$1=\left|g(z)\right|\leq\sum\frac{\left|z-z_1\right|\left|z-z_2\right|}{\left|z_3-z_1\right|\left|z_3-z_2\right|}=\sum\frac{R_1R_2}{ab}.$$

Equality holds if and only if $z = z_1$ or $z = z_2$ or $z = z_3$ or

$$\begin{aligned}\arg \frac{(z-z_1)(z-z_2)}{(z_3-z_1)(z_3-z_2)} &= \arg \frac{(z-z_2)(z-z_3)}{(z_1-z_2)(z_1-z_3)} \\ &= \arg \frac{(z-z_3)(z-z_1)}{(z_2-z_3)(z_2-z_1)}.\end{aligned}$$

The last case corresponds to $P = H$.

Consequences:
(i) Let us take $P = I$, the in-centre. Then

$$R_1 = r\operatorname{cosec}(\alpha/2), \quad R_2 = r\operatorname{cosec}(\beta/2), \quad R_3 = r\operatorname{cosec}(\gamma/2).$$

Then it is easy to calculate

$$\sum \frac{R_1 R_2}{ab} = \frac{1}{s} \sum a \sin(\alpha/2).$$

We hence obtain

$$\sum a \sin(\alpha/2) \geq s.$$

(For a direct proof see **3.4.53**.)

(ii) Taking $P = G$, the centre of gravity, we have $R_1 = 2m_a/3$, $R_2 = 2m_b/3$ and $R_3 = 2m_c/3$. Thus, we obtain the inequality

$$\sum \frac{m_a m_b}{ab} \geq \frac{9}{4}.$$

(iii) Taking $P = O$, the circum-centre, we see that $R_1 = R_2 = R_3 = R$ and the inequality is

$$\frac{1}{ab} \geq \frac{1}{R^2},$$

or $a + b + c \geq 4\Delta/R$.

(iv) Let us take $P = K$, the symmedian point. If D, E, F are respectively the points where AP, BP, CP meet the sides BC, CA, AB, then it is easy to check that

$$\frac{BD}{DC} = \frac{c^2}{b^2}, \quad \frac{CE}{EA} = \frac{a^2}{c^2}, \quad \frac{AF}{FB} = \frac{b^2}{a^2}.$$

Using Stewart's theorem, we can calculate R_1; we get

$$R_1 = \frac{2bc}{\sum a^2} m_a,$$

and similar expressions for R_2 and R_3. The inequality in this case is

$$\sum c^2 m_a m_b \geq \frac{\left(a^2 + b^2 + c^2\right)^2}{4}.$$

(v) When $P = \Omega_1$, the first Brocard point, the sine rule gives

$$R_1 = \frac{b}{\sin\alpha}\sin\omega, R_2 = \frac{c}{\sin\beta}\sin\omega, R_3 = \frac{c}{\sin\gamma}\sin\omega,$$

where ω is the Brocard angle. We then get

$$\sum \frac{a\sin\alpha}{b} \geq \frac{\prod \sin\alpha}{\sin^2\omega}.$$

Using the known fact that $\omega \leq \pi/6$, we have $\sin\omega \leq 1/2$ and hence

$$\sum \frac{a\sin\alpha}{b} \geq 4\prod \sin\alpha.$$

■

3.6.10. Let P be a point in a triangle ABC. Then

$$\sum \frac{r_1 r_2}{ab} \leq \frac{1}{4}.$$

Equality holds if and only if P is the circum-centre of ABC.

Proof: As in **3.6.8**, we produce AP, BP, CP to meet BC, CA, AB respectively in points D, E, F. Let $BD : DC = z : y$, $CE : EA = x : z$ and $AF : FB = y : x$. An easy computation shows that

$$r_1 = \frac{xh_a}{x+y+z}, \quad r_2 = \frac{yh_b}{x+y+z}, \quad r_3 = \frac{zh_c}{x+y+z}.$$

Thus, we get

$$\frac{r_1 r_2}{ab} = \frac{xy\sin^2\gamma}{(x+y+z)^2}.$$

The required inequality is

$$\frac{1}{(x+y+z)^2}\left\{xy\sin^2\gamma + yzsin^2\alpha + zx\sin^2\beta\right\} \leq \frac{1}{4}.$$

This simplifies to

$$\sum x^2 + \sum 2xy\cos 2\gamma \geq 0.$$

Writing

$$\begin{aligned} \cos 2\alpha &= \cos\left(2\beta + 2\gamma\right) \\ &= \cos 2\beta\cos 2\gamma - \sin 2\beta\sin 2\gamma, \end{aligned}$$

the inequality reduces to

$$\left(x + y\cos 2\gamma + z\cos 2\beta\right)^2 + \left(y\sin 2\gamma - z\sin 2\beta\right)^2 \geq 0,$$

which is trivially true. Equality holds if and only if

$$\frac{x}{\sin 2\alpha} = \frac{y}{\sin 2\beta} = \frac{z}{\sin 2\gamma}.$$

This corresponds to $P = O$, the circum-centre.

Consequences:

(i) Take $P = G$. Then $r_1 = h_a/3$, $r_2 = h_b/3$ and $r_3 = h_c/3$. Then we have

$$\sum \frac{r_1 r_2}{ab} = \frac{1}{9} \sum \frac{h_a h_b}{ab} = \frac{1}{9} \sum \sin^2 \alpha.$$

Thus, the inequality is

$$\sum \sin^2 \alpha \leq \frac{9}{4};$$

or equivalently $a^2 + b^2 + c^2 \leq 9R^2$.

(ii) Let $P = I$. Then $r_1 = r_2 = r_3$ and the inequality takes the form

$$\frac{1}{ab} \leq \frac{1}{4r^2}.$$

This is equivalent to $2r \leq R$.

(iii) Consider $P = H$, the ortho-centre. Then

$$r_1 = 2R \cos\beta \cos\gamma, \quad r_2 = 2R \cos\gamma \cos\alpha, \quad r_3 = \cos\alpha \cos\beta.$$

Using these values we may write the inequality in the form

$$\sum \sin 2\alpha \leq \frac{1}{2} \prod \tan\alpha = \frac{1}{2} \sum \tan\alpha.$$

(iv) If P is the symmedian point, then

$$\frac{r_1}{h_a} = \frac{a^2}{a^2 + b^2 + c^2},$$

and similar expressions for r_2 and r_3. We thus get

$$\sum \frac{r_1 r_2}{ab} = \frac{12\Delta^2}{(\sum a^2)^2}.$$

The inequality now is

$$\sum a^2 \geq 4\sqrt{3}\Delta.$$

(v) Suppose $P = \Omega_1$, the first Brocard point. In this case

$$r_1 = \frac{b}{\sin\alpha} \sin^2 \omega, \quad r_2 = \frac{c}{\sin\beta} \sin^2 \omega, \quad r_1 = \frac{a}{\sin\gamma} \sin^2 \omega.$$

We obtain

$$\sin^4\omega \le \frac{\prod \sin^2\alpha}{4\sum \sin^2\alpha \sin^2\beta}.$$

Equivalently

$$\sin^4\omega \le \frac{1}{16R^2}\frac{1}{\left(\sum 1/a^2\right)}.$$

However using Cauchy-Schwarz inequality and **3.4.38**, we get

$$\sum \frac{1}{a^2} \ge \frac{1}{3}\left(\sum \frac{1}{a}\right)^2 \ge \frac{9}{4(R+r)^2}.$$

Hence, the inequality is

$$\sin^2\omega \le \frac{R+r}{6R}.$$

This is better than $\sin\omega \le 1/2$ in view of $2r \le R$. ■

3.6.11. Let P be a point in a triangle ABC. Prove that

$$\left(R_1 + R_2 + R_3\right)^2 \ge 4\sqrt{3}\,\left[ABC\right].$$

Proof: Draw AT parallel and equal to PB; BR parallel and equal to PC; and CS parallel and equal to PA. Then the area of the hexagon $ATBRCS$ is equal to $2\left[ABC\right]$, and perimeter of $ATBRCS$ is $2\left(R_1 + R_2 + R_3\right)$. But among all hexagons of fixed areas, a regular hexagon has the least perimeter. If a regular hexagon is inscribed in a circle of radius R, then its perimeter is $6R$ and its area is $3\sqrt{3}R^2/2$. Thus, we obtain

$$2\left(R_1 + R_2 + R_3\right) \ge 6R = 6\sqrt{\frac{2K}{3\sqrt{3}}},$$

where K is the area of the hexagon. Since we are minimising the perimeter of a hexagon of area $2\left[ABC\right]$, we obtain

$$2\left(R_1 + R_2 + R_3\right) \ge 6\sqrt{\frac{2\cdot 2\left[ABC\right]}{3\sqrt{3}}}.$$

This simplifies to

$$\left(R_1 + R_2 + R_3\right)^2 \ge 4\sqrt{3}\,\left[ABC\right].$$

■

3.6.12. Let P be an interior point of a triangle ABC and D, E, F be the feet of perpendiculars from P on to BC, CA, AB respectively. Show that

$$\left[DEF\right] \le \frac{1}{4}\left[ABC\right],$$

and equality holds if and only if P coincides with the circum-centre of ABC.

Proof:

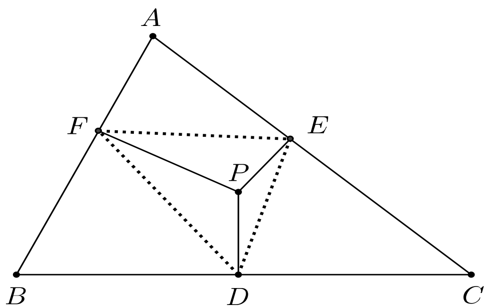

Fig. 3.8

Let $PD = r_1$, $PE = r_2$ and $PF = r_3$. Then

$$\begin{aligned}\left[DPE\right] &= \frac{1}{2} r_1 r_2 \sin \angle DPE \\ &= \frac{1}{2} r_1 r_2 \sin\left(\pi - \gamma\right) \\ &= \frac{1}{2} r_1 r_2 \sin \gamma \\ &= \frac{1}{4R} c r_1 r_2,\end{aligned}$$

and similar expressions for $\left[EPF\right]$, $\left[FPD\right]$ may be obtained. Adding these, one obtains

$$\begin{aligned}\left[DEF\right] &= \frac{1}{4R}\left(c r_1 r_2 + a r_2 r_3 + b r_3 r_1\right) \\ &= \frac{abc}{4R} \sum \frac{r_1 r_2}{ab} \\ &\le \frac{1}{4}\left[ABC\right],\end{aligned}$$

using (**3.6.10**).

Consequences: Using the concyclicity of A, F, P, E, it is easy to get $EF = AP\sin A$. Similarly, we get $FD = BP\sin B$ and $DE = CP\sin C$. If R', s' and r' denote respectively the circum-radius, semi-perimeter and in-radius of triangle DEF, then

$$\begin{aligned}
R' &= \frac{EF\cdot FD\cdot DE}{4[DEF]} = \frac{R'\prod R_1\sin\alpha}{\sum r_1r_2c}\\
s' &= \frac{R_1\sin\alpha + R_2\sin\beta + R_3\sin\gamma}{2}\\
r' &= \frac{[DEF]}{s'} = \frac{\sum r_1r_2c}{2R\sum R_1\sin\alpha}.
\end{aligned}$$

Now the inequality $s' \geq 27(r')^2$ gives

$$\frac{\left(\sum R_1\sin\alpha\right)^2}{4} \geq \frac{27\left(\sum r_1r_2c\right)^2}{4R^2\left(\sum R_1\sin\alpha\right)^2}.$$

This simplifies to

$$\left(\sum R_1\sin\alpha\right)^2 \geq \frac{3\sqrt{3}}{R}\sum r_1r_2c.$$

Using $a = 2R\sin\alpha$, etc., this may be written in the form

$$\left(\sum R_1\sin\alpha\right)^2 \geq 6\sqrt{3}\sum r_1r_2\sin\gamma.$$

However,

$$\begin{aligned}
(R_1\sin\alpha)^2 = EF^2 &= r_2^2 + r_3^2 - 2r_1r_2\cos(\pi-\alpha)\\
&= r_2^2 + r_3^2 + 2r_1r_2\cos\alpha,
\end{aligned}$$

and similar expressions for the other two. Thus, we deduce

$$\begin{aligned}
6\sqrt{3}\sum r_1r_2\sin\gamma &\leq \left(\sum R_1\sin\alpha\right)^2\\
&\leq 3\sum(R_1\sin\alpha)^2\\
&= 3\sum\left(r_2^2+r_3^2+2r_2r_3\cos\alpha\right)\\
&= 6\left(\sum r_1^2 + \sum r_2r_3\cos\alpha\right),
\end{aligned}$$

or

$$\sqrt{3}\sum r_1r_2\sin\gamma \leq \sum r_1^2 + \sum r_2r_3\cos\alpha.$$

Suppose $P = I$, the in-centre of ABC. Then $r_1 = r_2 = r_3 = r$ and hence

$$\sqrt{3}\,r^2\left(\sin\alpha+\sin\beta+\sin\gamma\right) \leq 3r^2 + r^2\left(\cos\alpha+\cos\beta+\cos\gamma\right),$$

or which is the same as

$$\sqrt{3}\Big(\sum \sin A\Big) \le 3 + \sum \cos A.$$

If we choose $P = O$, the circum-centre, then $r_1 = R\cos\alpha$, $r_2 = R\cos\beta$, $r_3 = R\cos\gamma$ and hence,

$$\sqrt{3}\sum \cos\alpha\cos\beta\sin\gamma \le \sum \cos^2\alpha + 3\prod \cos\alpha.$$

Using the standard identities $4\cos\alpha\cos\beta\sin\gamma = \sin 2\alpha + \sin 2\beta - \sin 2\gamma$, etc., and $\sum \sin^2\alpha = 2 + 2\prod\cos\alpha$, this reduces to

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma \le \frac{2}{\sqrt{3}}\Big(\sin^2\alpha + \sin^2\beta + \sin^2\gamma\Big).$$

■

3.6.13. (AMM, 1978) Let G denote the centroid of a triangle ABC and let $\theta = \angle GAB$, $\mu = \angle GBC$ and $\nu = \angle GCA$. Prove that

$$\sin\theta + \sin\mu + \sin\nu \le \frac{3}{2}.$$

Proof: Observe that

$$\sin\theta = \frac{\Delta}{cm_a}, \quad \sin\mu = \frac{\Delta}{am_b}, \quad \sin\nu = \frac{\Delta}{bm_c}.$$

Thus,

$$\sin\theta + \sin\mu + \sin\nu = \Delta\left\{\frac{1}{cm_a} + \frac{1}{am_b} + \frac{1}{bm_c}\right\}.$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\Big(\sin\theta + \sin\mu + \sin\nu\Big)^2 &\le \Delta^2\Big(\sum \frac{1}{a^2}\Big)\Big(\sum \frac{1}{m_a^2}\Big)\\ &= \Delta^2\frac{\Big(\sum a^2b^2\Big)\Big(\sum m_a^2m_b^2\Big)}{\big(abcm_am_bm_c\big)^2}.\end{aligned}$$

Using the standard identities $16\Delta^2 = 2\sum a^2b^2 - \sum a^4$, $4m_a^2 = 2b^2 + 2c^2 - a^2$, etc., the above inequality reduces to

$$\frac{4}{9}\big(\sin\theta + \sin\mu + \sin\nu\big)^2 \le \frac{P}{Q},$$

where

$$\begin{aligned}P &= \Big(2\sum a^2b^2 - \sum a^4\Big)\Big(\sum a^2b^2\Big)^2,\\ Q &= a^2b^2c^2\Big(2a^2 + 2b^2 - c^2\Big)\Big(2b^2 + 2c^2 - a^2\Big)\Big(2c^2 + 2a^2 - b^2\Big).\end{aligned}$$

Thus, it is sufficient to verify the inequality $P/Q \le 1$. For this, it is sufficient to prove that $Q = P + R$ for some non-negative quantity R. Consider $Q - P$ as a function of the variables $x = a^2$, $y = b^2$, $z = c^2$, say, $Q - P = f(x, y, z)$. Thus,

$$f(x,y,z) = xyz\big(2x+2y-z\big)\big(2y+2z-x\big)\big(2z+2x-y\big) \\ - \big(2xy+2yz+2zx-x^2-y^2-z^2\big)\big(y^2+2yz\big)^2.$$

It is easy to verify that $f(y, y, z) = 0$ and $f_x(y, y, z) = 0$. It follows that $(x - y)^2$ is a factor of $f(x, y, z)$. The symmetry of $f(x, y, z)$ in x, y, z shows that $(x - y)^2(y - z)^2(z - x)^2$ is a factor of $f(x, y, z)$. Looking at the degrees, it may be concluded that

$$f(x,y,z) = K(x-y)^2(y-z)^2(z-x)^2.$$

Now K may be seen equal to 1. Thus, it follows that $Q - P = \big[(x - y)(y - z)(z - x)\big]^2 = R \ge 0$ and hence $P \le Q$. Equality holds if and only if $x = y = z$ which is equivalent to $a = b = c$. Thus, equality holds if and only if ABC is equilateral.

A Generalisation

The above inequality may be written in the form

$$\frac{h_a}{m_b} + \frac{h_b}{m_c} + \frac{h_c}{m_a} \le 3.$$

However, a more general inequality holds. Let $\big(h_1, h_2, h_3\big)$ be any permutation of $\big(h_a, h_b, h_c\big)$. Then the inequality

$$\frac{h_1}{m_a} + \frac{h_2}{m_b} + \frac{h_3}{m_c} \le 3,$$

is true. Suppose $a \le b \le c$. Then it is easy to see that $h_a \ge h_b \ge h_c$ and $m_a \ge m_b \ge m_c$. Hence, the rearrangement inequality implies that

$$\frac{h_a}{m_a} + \frac{h_b}{m_b} + \frac{h_c}{m_c} \le \frac{h_1}{m_a} + \frac{h_2}{m_b} + \frac{h_3}{m_c} \le \frac{h_a}{m_c} + \frac{h_b}{m_b} + \frac{h_c}{m_a}.$$

Thus, it is sufficient to prove that

$$\frac{h_a}{m_c} + \frac{h_b}{m_b} + \frac{h_c}{m_a} \le 3.$$

Again the Cauchy-Schwarz inequality gives

$$\left(\frac{h_a}{m_c} + \frac{h_b}{m_b} + \frac{h_c}{m_a}\right)^2 \le \left(\sum h_a^2\right)\left(\sum \frac{1}{m_a^2}\right),$$

and the rest is as in the early part. ■

3.6.14. Prove the inequality

$$\sum \frac{r_2 + r_3}{r_2 + 2R_1 + r_3} \le 1 \le \frac{1}{3} \sum \frac{R_1}{r_2 + r_3}.$$

Find conditions on P and triangle ABC for equality to hold in the above inequality.

Proof: Let PD, PE, PF be the perpendiculars from P on to the sides BC, CA, AB respectively. Thus, $AP = R_1$, $BP = R_2$, $CP = R_3$, $PD = r_1$, $PE = r_2$ and $PF = r_3$. It is easy to see that $a \cdot R_1 \ge b \cdot r_2 + c \cdot r_3$, with equality if and only if AP is perpendicular to BC. If we reflect P in the bisector of $\angle BAC$, we also get $a \cdot R_1 \ge b \cdot r_3 + c \cdot r_2$. Adding these we get

$$R_1 \ge \frac{b+c}{2a}\big(r_2 + r_3\big).$$

Equality holds if and only if AP is is perpendicular to BC and bisects $\angle BAC$. This gives

$$R_1 + \frac{1}{2}\big(r_2 + r_3\big) \ge \frac{s}{a}\big(r_2 + r_3\big),$$

where s is the semi-perimeter of the triangle ABC. Dividing by $r_2 + r_3$, we obtain

$$\frac{R_1}{r_2 + r_3} + \frac{1}{2} \ge \frac{h_1}{2r},$$

where r is the in-radius and h_1 is the altitude from A on to BC. This simplifies to

$$\frac{r_2 + r_3}{r_2 + 2R_1 + r_3} \le \frac{r}{h_1}.$$

Using

$$\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} = \frac{1}{r}$$

we get the left hand side of the required inequality. We also observe that

$$\frac{R_1}{r_2 + r_3} \ge \frac{b+c}{2a}.$$

Hence, we get

$$\sum \frac{R_1}{r_2 + r_3} \ge \sum \left(\frac{b}{2a} + \frac{a}{2b}\right) \ge 3,$$

which gives the right hand side. Equality holds if and only if ABC is equilateral and P is its centre. ■

3.6.15. Prove that

$$\sum R_1^2 \sin^2 \alpha \le 3 \sum r_1^2,$$

and equality holds if and only if P is the symmedian point of ABC.

Proof: Consider the cyclic quadrilateral $AEPF$. Ptolemy's theorem gives

$$\begin{aligned} AP \cdot EF &= r_3 AE + r_2 AF \\ &= R_1 \sin\theta_2 \cdot R_1 \cos\theta_1 + R_1 \sin\theta_1 \cdot R_1 \cos\theta_2 \\ &= R_1^2 \sin\left(\theta_1 + \theta_2\right) = R_1^2 \sin\alpha, \end{aligned}$$

where $\angle PAB = \theta_2$ and $\angle CAP = \theta_1$.(See Fig. 3.9.)

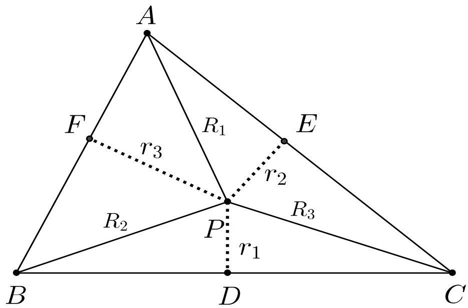

Fig. 3.9

Thus, $FE = R_1 \sin\alpha$. This gives

$$\sum R_1^2 \sin^2\alpha = EF^2 + FD^2 + DE^2.$$

Consider the triangle DEF and the point P inside it. We show that

$$DE^2 + EF^2 + FD^2 \leq 3\left(PD^2 + PE^2 + PF^2\right), \qquad (3.11)$$

valid for any triangle DEF and any point P inside it. Let K be the mid-point of EF. Join PK and DK. (See Fig. 3.10.) Appolonius' theorem applied to the triangle PEF gives

$$PK = \frac{1}{2}\sqrt{2PE^2 + 2PF^2 - EF^2}.$$

Similarly,

$$DK = \frac{1}{2}\sqrt{2DE^2 + 2DF^2 - EF^2}.$$

But $DK \leq PK + KD$, with equality if and only if P lies on DK. Thus,

$$\sqrt{2DE^2 + 2DF^2 - EF^2} \leq 2PD + \sqrt{2PE^2 + 2PF^2 - EF^2}.$$

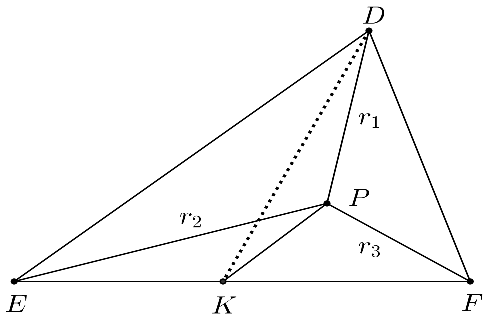

Fig. 3.10

Squaring and simplification gives

$$\begin{aligned} 2DE^2 + 2DF^2 &\le 4PD^2 + 2PE^2 + 2PF^2 \\ &\quad + PD\sqrt{2PE^2 + 2PF^2 - EF^2}. \end{aligned}$$

Summing over the cyclic permutation of D, E, F, this leads to

$$\begin{aligned} DE^2 + EF^2 + FD^2 &\le 2\big(PD^2 + PE^2 + PF^2\big) \\ &\quad + \sum PD\sqrt{2PE^2 + 2PF^2 - EF^2}. \end{aligned}$$

If $DE^2 + EF^2 + FD^2 \le 2\big(PD^2 + PE^2 + PF^2\big)$, then (3.11) is trivially valid. Otherwise, we have

$$\begin{aligned} 0 &\le DE^2 + EF^2 + FD^2 - 2\big(PD^2 + PE^2 + PF^2\big) \\ &\le \sum PD\sqrt{2PE^2 + 2PF^2 - EF^2} \\ &\le \left(\sum PD^2\right)^{1/2}\left(4\sum PD^2 - \sum DE^2\right)^{1/2}. \end{aligned}$$

Setting $\sum PD^2 = x^2$ and $\sum DE^2 = y^2$, the above relation is

$$0 \le y^2 - 2x^2 \le x\left(4x^2 - y^2\right)^{1/2}.$$

This is equivalent to $y^2\big(y^2 - 3x^2\big) \le 0$. It follows that $y^2 \le 3x^2$, which is precisely (3.11). Equality holds if and only if P coincides with the centroid of DEF. This is equivalent to

$$[PDE] = [PEF] = [PFD],$$

or

$$r_1 r_2 \sin\gamma = r_2 r_3 \sin\alpha = r_3 r_1 \sin\beta.$$

This shows that

$$r_1 : r_2 : r_3 = a : b : c.$$

Thus, equality holds if and only if P is the symmedian point. ■

Chapter 4

Applications involving inequalities

Some problems, though not direct inequality problems, use inequalities in their solutions. This is the case when the problems on maximisation and minimisation are considered. Inequalities are also useful in solving some Diophantine equations by way of estimating the bounds for solutions. They also help us to determine whether some polynomial equations have real roots. Here we consider several problems whose solutions use inequalities.

Example 4.1. (IMO, 1987) Let $x_1,x_2,x_3,\ldots,x_n$ be a sequence of n real numbers such that $\sum_{j=1}^{n} x_j^2 = 1$. Prove that for every $k \geq 2$, there are integers $a_1,a_2,a_3,\ldots,a_n$, not all zero, such that $|a_j| \leq k-1$ and

$$\left|a_1x_1 + a_2x_2 + \cdots + a_nx_n\right| \leq \frac{(k-1)\sqrt{n}}{k^n - 1}.$$

Solution: For each n-tuple $\left(a_1,a_2,a_3,\ldots,a_n\right)$ of non-negative integers with $0 \leq a_j \leq k-1$, associate the sum $\sum_{j=1}^{n} a_jx_j$. Observe that there are k^n such n-tuples. Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \left|\sum_{j=1}^{n} a_jx_j\right| &\leq \left(\sum_{j=1}^{n} a_j^2\right)^{1/2}\left(\sum_{j=1}^{n} x_j^2\right)^{1/2} \\ &\leq \left(k-1\right)\sqrt{n}. \end{aligned}$$

Now, we split the interval $\left[0,(k-1)\sqrt{n}\right]$ into $k^n - 1$ sub-intervals each of length $\left(k-1\right)\sqrt{n}/\left(k^n-1\right)$. By the pigeonhole principle, we can find two n-tuples $\left(b_1,b_2,b_3,\ldots,b_n\right)$ and $\left(c_1,c_2,c_3,\ldots,c_n\right)$ such that the sums $\sum_{j=1}^{n} b_jx_j$ and $\sum_{j=1}^{n} c_jx_j$ lie in the same sub-interval. Hence, it follows that

$$\left|\sum_{j=1}^{n} b_jx_j - \sum_{j=1}^{n} c_jx_j\right| \leq \frac{(k-1)\sqrt{n}}{\left(k^n-1\right)}.$$

Taking $a_j = b_j - c_j$, we see that $\left|a_j\right| \leq k-1$ for $1 \leq j \leq n$ and

$$\left|\sum_{j=1}^{n} a_jx_j\right| \leq \frac{(k-1)\sqrt{n}}{k^n-1}.$$

$\blacksquare$

Example 4.2. Find the maximum value of

$$x\left(1-y^2\right)\left(1-z^2\right) + y\left(1-z^2\right)\left(1-x^2\right) + z\left(1-x^2\right)\left(1-y^2\right),$$

subject to the conditions that x, y, z are non-negative and $xy + yz + zx = 1$.

Solution: Observe that

$$\begin{aligned}\sum_{\text{cyclic}} x\left(1-y^2\right)\left(1-z^2\right) &= \sum_{\text{cyclic}} x - \sum_{\text{cyclic}} \left(xy^2 + xz^2\right) + xyz \sum_{\text{cyclic}} yz \\ &= \sum_{\text{cyclic}} x - \sum_{\text{cyclic}} \left(x^2y + x^2z\right) + xyz \\ &= \sum_{\text{cyclic}} x - \sum_{\text{cyclic}} x\left(xy + xz\right) + xyz \\ &= \sum_{\text{cyclic}} x - \sum_{\text{cyclic}} x\left(1 - yz\right) + xyz \\ &= 4xyz.\end{aligned}$$

Now the AM-GM inequality gives

$$\left(xyz\right)^{2/3} \le \frac{1}{3}\left(xy + yz + zx\right) = \frac{1}{3}.$$

Thus, it follows that

$$\sum_{\text{cyclic}} x\left(1-y^2\right)\left(1-z^2\right) = 4xyz \le \frac{4}{3\sqrt{3}}.$$

Here equality holds if and only if $x = y = z = 1/\sqrt{3}$. Thus, the maximum value is $4/3\sqrt{3}$. ■

Example 4.3. (USSR, 1974) Consider a square grid S of 169 points which are uniformly arranged in 13 rows and 13 columns. Show that no matter what subset T consisting of 53 points is selected from these 169 points, some four points of T will always form the vertices of a rectangle R whose sides are parallel to the sides of S.

Solution: Let m_j, $1 \le j \le 13$, denote the number of elements of T in the j-th row. Then

$$m_1 + m_2 + \cdots + m_{13} = 53.$$

Now to get a rectangle R from the points of T, with sides parallel to those of S, we must get some two points from some j-th row and some two points from some k-th row such that the columns determined by the two points on the j-th row coincide with the columns determined by the two points on the k-th row.

Now the number of pairs of columns determined by m_j points on the j-th row is $\binom{m_j}{2}$. Hence, the total pairs of columns collectively determined by

points of T is $\sum_{j=1}^{13}\binom{m_j}{2}$. We observe that

$$\begin{aligned}\sum_{j=1}^{13}\binom{m_j}{2} &= \frac{1}{2}\sum_{j=1}^{13}(m_j^2 - m_j)\\ &\geq \frac{1}{2\times 13}\left(\sum_{j=1}^{13} m_j^2\right) - \frac{1}{2}\sum_{j=1}^{13} m_j\\ &= \frac{1}{26}(53^2) - \frac{1}{2}(53)\\ &> 2\times 53 - 27\\ &= 79 > 78 = \binom{13}{2}.\end{aligned}$$

We have used the Cauchy-Schwarz inequality in the second step above. The total number of pairs of columns in the grid S is $\binom{13}{2}$. Hence, we may apply the pigeonhole principle to conclude that there is a pair of columns determined by some two points on some j-th row which coincides with a pair of columns determined by some two points on some k-th row, $j \neq k$. We thus get a rectangle with sides parallel to the sides of S. ■

Example 4.4. Consider the set $S = \{a_1, a_2, \ldots, a_n\}$. Let $\mathcal{C} = \{P_1, P_2, \ldots, P_n\}$ be a collection of n distinct 2-element subsets of S such that whenever $P_j \cap P_k$ is not empty then $\{a_j, a_k\}$ is also one of the sets in the collection $\mathcal{C}$. Prove that each a_j appears exactly in two elements of $\mathcal{C}$.

Solution: For each j, let m_j denote the number of P_r's in which a_j lies. We have to show that $m_j = 2$ for each j, $1 \leq j \leq n$. Since P_j is a 2-element set, we have $|P_j| = 2$ for $1 \leq j \leq n$. Hence, we obtain

$$m_1 + m_2 + \cdots + m_n = 2n.$$

Suppose $a_j \in P_k \cap P_l$. Then the given condition implies that $\{a_k, a_l\}$ is in the collection $\{P_1, P_2, \ldots, P_n\}$. Thus, for each j, there are $\binom{m_j}{2}$ number of P_r's being contributed to the collection $\{P_1, P_2, \ldots, P_n\}$. Moreover all such sets are distinct. In fact if a set $\{a_k, a_l\}$ corresponds to a_s and a_t, then both a_s and a_t are in $P_k \cap P_l$ forcing $P_k = \{a_s, a_t\} = P_l$. It follows that

$$\sum_{j=1}^{n}\binom{m_j}{2} \leq n.$$

This simplifies to

$$\sum_{j=1}^{n}(m_j^2 - m_j) \leq 2n.$$

Since $\sum_{j=1}^{n} m_j = 2n$, we obtain

$$\sum_{j=1}^{n} m_j^2 \leq 4n.$$

However an application of the Cauchy-Schwarz inequality gives

$$4n^2 = \Big(\sum_{j=1}^{n} m_j\Big)^2 \leq n \sum_{j=1}^{n} m_j^2 \leq 4n^2.$$

This shows that equality holds in the Cauchy-Schwarz inequality. We conclude $m_j = \lambda$, for $1 \leq j \leq n$. Now using $\sum_{j=1}^{n} m_j = 2n$, we obtain $\lambda = 2$. This gives $m_j = 2$, for $1 \leq j \leq n$. ■

Example 4.5. Let ABC be a triangle in which $C = 90°$. Let m_a, m_b, m_c be its medians from the vertices A, B, C on to BC, CA, AB respectively. Find the maximum value of $\dfrac{m_a + m_b}{m_c}$.

Solution: Let us start with the standard expression:

$$4m_a^2 = 2b^2 + 2c^2 - a^2.$$

However, $c^2 = b^2 + a^2$ and hence $4m_a^2 = 4b^2 + a^2$. Similarly, $4m_b^2 = 4a^2 + b^2$; $4m_c^2 = 2b^2 + 2a^2 - c^2 = b^2 + a^2$. Thus, it follows that

$$\frac{m_a^2 + m_b^2}{m_c^2} = 5.$$

Now the Cauchy-Schwarz inequality gives

$$\left(\frac{m_a + m_b}{m_c}\right)^2 \leq \frac{2(m_a^2 + m_b^2)}{m_c^2} = 10.$$

We obtain

$$\frac{m_a + m_b}{m_c} \leq \sqrt{10}.$$

Equality holds if and only if $a = b$. Thus, the maximum value is $\sqrt{10}$. ■

Example 4.6. Let ABC be a triangle with circum-circle Γ. The medians AD, BE, CF, when extended, meet Γ in A_1, B_1, C_1 respectively. Show that the sum of the areas of the triangles BA_1C, CB_1A, AC_1B is at least equal to the area of ABC. When does equality hold?

Solution: Let h_a, h_b and h_c be the altitudes; m_a, m_b, m_c be the medians. Let A_1K, B_1L and C_1M be the perpendiculars from A_1, B_1, C_1 on to BC, CA, AB respectively. Then

$$\frac{[BA_1C]}{[ABC]} = \frac{A_1K}{h_a} = \frac{DA_1}{AD}.$$

But $DA_1 \cdot AD = BD \cdot DC = a^2/4$, since $BD = DC = a/2$. This gives $DA_1 = a^2/4AD$, and hence

$$\frac{[BA_1C]}{[ABC]} = \frac{a^2}{4AD^2} = \frac{a^2}{4m_a^2}.$$

Similarly,

$$\frac{[CB_1A]}{[ABC]} = \frac{b^2}{4m_b^2}, \quad \frac{[AC_1B]}{[ABC]} = \frac{c^2}{4m_c^2}.$$

Thus,

$$[BA_1C] + [CB_1A] + [AC_1B] = \left(\frac{a^2}{4m_a^2} + \frac{b^2}{4m_b^2} + \frac{c^2}{4m_c^2}\right)[ABC],$$

and it is sufficient to prove that

$$\frac{a^2}{4m_a^2} + \frac{b^2}{4m_b^2} + \frac{c^2}{4m_c^2} \geq 1.$$

Equivalently, we need to prove that

$$\frac{a^2}{2b^2+2c^2-a^2} + \frac{b^2}{2c^2+2a^2-b^2} + \frac{c^2}{2a^2+2b^2-c^2} \geq 1.$$

Using the Cauchy-Schwarz inequality, observe that

$$\begin{aligned} \left(a^2+b^2+c^2\right)^2 &= \left(\sum_{\text{cyclic}} \frac{a}{\sqrt{2b^2+2c^2-a^2}} \cdot a\sqrt{2b^2+2c^2-a^2}\right)^2 \\ &\leq \left(\sum_{\text{cyclic}} \frac{a^2}{2b^2+2c^2-a^2}\right)\left(\sum_{\text{cyclic}} a^2\left(2b^2+2c^2-a^2\right)\right). \end{aligned}$$

Thus, we obtain

$$\sum_{\text{cyclic}} \frac{a^2}{2b^2+2c^2-a^2} \geq \frac{\left(a^2+b^2+c^2\right)^2}{4\left(a^2b^2+b^2c^2+c^2a^2\right) - \left(a^4+b^4+c^4\right)}.$$

Thus, it is sufficient to prove that

$$\left(a^2+b^2+c^2\right)^2 \geq 4\left(a^2b^2+b^2c^2+c^2a^2\right) - \left(a^4+b^4+c^4\right).$$

This further reduces to

$$\sum_{\text{cyclic}} a^4 \geq \sum_{\text{cyclic}} a^2b^2,$$

which is a direct consequence of another application of the Cauchy-Schwarz inequality. Equality holds if and only if $a = b = c$, which corresponds to the case of an equilateral triangle. ■

Example 4.7. Let $n > 1$ be an integer, and let $a_1, a_2, \ldots, a_n$ be real numbers such that $(n-1)a_1^2 - 2na_2 < 0$. Prove that the roots of the equation

$$x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

cannot all be real.

Solution: Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation, and suppose all of them are real. Using Viette's formulae relating the roots and the coefficients of a polynomial equation, we get

$$\begin{aligned} \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 &= \big(\alpha_1 + \alpha_2 + \cdots + \alpha_n\big)^2 - 2\sum_{j<k} \alpha_j\alpha_k \\ &= a_1^2 - 2a_2. \end{aligned}$$

Since $\alpha_1, \alpha_2, \ldots, \alpha_n$ are real, the Cauchy-Schwarz inequality is applicable. This implies that

$$\begin{aligned} \big(\alpha_1 + \alpha_2 + \cdots + \alpha_n\big)^2 &\leq n\big(\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2\big) \\ &= n\big(a_1^2 - 2a_2\big). \end{aligned}$$

Thus,

$$a_1^2 \leq na_1^2 - 2na_2,$$

and hence $(n-1)a_1^2 - 2na_2 \geq 0$. This violates the given condition. Hence not all $\alpha_1, \alpha_2, \ldots, \alpha_n$ can be real. ■

Example 4.8. (Indian Team Selection, 1998) Let M be a positive integer and consider the set $S = \{n \in \mathbb{N} \mid M^2 \leq n < (M+1)^2\}$. Show that the products of the form ab where $a, b \in S$ are all distinct.

Solution: Suppose there are integers a, b, c, d in the given interval such that $ab = cd$. We may assume d is the largest among them so that $a < d$ and $b < d$. This implies that $c < a$ and $c < b$. Put $c = a - k$ and $d = a + l$ for some positive numbers k and l. Then we have

$$ab = (a-k)(a+l) = a^2 + a(l-k) - kl.$$

This shows that a divides kl. Since c, d lie in the set S, we observe that $a - k \geq M^2$ and $a + l \leq M^2 + 2M$. Thus, we have

$$l + k = (a + l) - (a - k) \leq \big(M^2 + 2M\big) - M^2 = 2M.$$

Using the AM-GM inequality, we now get

$$\sqrt{kl} \leq \frac{k + l}{2} \leq M.$$

This implies that $kl \leq M^2$. Since a divides kl, we get $a \leq M^2$. If $a = M^2$, then $c < a$ implies that c is not in the set S. Thus, $a < M^2$, a contradiction. Thus, if $ab = cd$, then $\{a, b\}$ and $\{c, d\}$ must be the same sets. ■

Example 4.9. Let $f : \mathbb{N} \to [\mathbf{1}, \infty)$ such that

(a) $f(2) = 2$;

(b) $f(mn) = f(m)f(n)$ for all m, n in $\mathbb{N}$;

(c) $f(m) < f(n)$ whenever $m < n$.

Prove that $f(n) = n$ for all natural numbers n.

Solution: Observe that $f(1) = f(1 \cdot 1) = f(1)^2$ and hence $f(1) = 1$. Now induction shows that $f(2^k) = 2^k$ for all integers $k \geq 1$. Let us take any $m \in \mathbb{N}$ and suppose $f(m) = l$. Then $f(m^n) = l^n$ for all $n \in \mathbb{N}$. If k is such that $2^k \leq m^n < 2^{k+1}$, then using (b) and (c) we obtain

$$2^k \leq l^n < 2^{k+1}.$$

Thus, we get the inequality

$$\frac{1}{2} < \left(\frac{m}{l}\right)^n < 2, \tag{4.1}$$

valid for all natural numbers n. If $m > l$, choose n such that $n > l/(m - l)$. Then Bernoulli's inequality gives

$$\left(\frac{m}{l}\right)^n = \left(1 + \frac{m - l}{l}\right)^n > 1 + n\left(\frac{m - l}{l}\right) > 2,$$

by our choice of n. This contradicts the right hand side of inequality (4.1). If $m < l$, we choose $n > m/(l - m)$. Again, Bernoulli's inequality is applicable and we get

$$\left(\frac{l}{m}\right)^n = \left(1 + \frac{l - m}{m}\right)^n > 1 + n\left(\frac{l - m}{m}\right) > 2,$$

by our choice of n. We thus obtain $\big(m/l\big)^n < 1/2$. But this contradicts the left hand side of inequality (4.1).

We conclude that $l = m$, thus forcing $f(m) = m$ for all natural numbers m. ■

Example 4.10. ([8]) Determine all real polynomials of degree n with each of its coefficients in the set $\{+1, -1\}$ and having only real zeros.

Solution: If $n = 1$ then $p(x) = (x-1)$, $(x+1)$, $-(x-1)$ and $-(x+1)$ give all the polynomials of degree 1. We may assume $n > 1$. We may also assume that the leading coefficient of $p(x)$ is 1 by changing the signs of all the coefficients, if necessary. Thus, $p(x)$ is of the form

$$p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_x + c_0,$$

where $\left|c_j\right| = 1$, for $0 \leq j \leq n-1$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ denote all the roots of $p(x) = 0$. We have $\alpha_1\alpha_2 \cdots \alpha_n = (-1)^n c_0$, so that $\alpha_j \neq 0$ for every j. This implies that

$$0 < \sum_{j=1}^{n} \alpha_j^2 = c_{n-1}^2 - 2c_{n-2} = 1 \pm 2.$$

Hence, $c_{n-2} = -1$ and $\sum_{j=1}^{n} \alpha_j^2 = 3$.

Consider the reciprocal polynomial of $p(x)$. It is also of the same form, i.e., having coefficients ± 1 and $1/\alpha_j$ are the zeroes of this reciprocal polynomial. The same analysis leads to the conclusion $\sum_{j=1}^{n}(1/\alpha_j^2) = 3$. These two lead to

$$\sum_{j=1}^{n}\left(\alpha_j^2 + \frac{1}{\alpha_j^2}\right) = 6.$$

However, the inequality

$$a + \frac{1}{a} \geq 2$$

holds good for any positive real a. This implies that

$$6 = \sum_{j=1}^{n}\left(\alpha_j^2 + \frac{1}{\alpha_j^2}\right) \geq 2n.$$

It follows that $n \leq 3$. We may verify that for $n = 3$, $\pm(x+1)^2(x-1)$ and $\pm(x+1)(x-1)^2$ are the required polynomials. In the case $n = 2$, the polynomials $\pm(x^2 - x - 1)$ and $\pm(x^2 + x - 1)$ are the required ones. ■

Example 4.11. ([8]) Suppose α, β, γ and δ are real numbers such that

$$\alpha + \beta + \gamma + \delta = \alpha^7 + \beta^7 + \gamma^7 + \delta^7 = 0.$$

Prove that $\alpha(\alpha+\beta)(\alpha+\gamma)(\alpha+\delta) = 0$.

Solution: Here we use a tricky factorisation:

$$\begin{aligned} 0 &= -\alpha^7-\beta^7-\gamma^7-\delta^7 \\ &= \big(\beta+\gamma+\delta\big)^7-\beta^7-\gamma^7-\delta^7 \\ &= 7\big(\beta+\gamma\big)\big(\gamma+\delta\big)\big(\beta+\delta\big)\Big\{\big(\beta^2+\gamma^2+\delta^2+\beta\gamma+\gamma\delta+\beta\delta\big)^2 \\ &\qquad\qquad + \beta\gamma\delta\big(\beta+\gamma+\delta\big)\Big\}. \end{aligned}$$

However we have

$$\begin{aligned} &4\Big\{\big(\beta^2+\gamma^2+\delta^2+\beta\gamma+\gamma\delta+\beta\delta\big)^2+\beta\gamma\delta\big(\beta+\gamma+\delta\big)\Big\} \\ &\qquad = \Big[\big(\beta+\gamma\big)^2+\big(\gamma+\delta\big)^2+\big(\beta+\delta\big)^2\Big]^2-4\alpha\beta\gamma\delta \\ &\qquad = \Big[\big(\alpha+\beta\big)^2+\big(\alpha+\gamma\big)^2+\big(\alpha+\delta\big)^2\Big]^2-4\alpha\beta\gamma\delta \\ &\qquad = \Big[3\alpha^2+2\alpha\big(\beta+\gamma+\delta\big)+\beta^2+\gamma^2+\delta^2\Big]^2-4\alpha\beta\gamma\delta. \end{aligned}$$

The last expression reduces to

$$\Big[\alpha^2+\beta^2+\gamma^2+\delta^2\Big]^2-4\alpha\beta\gamma\delta.$$

However we see that

$$\begin{aligned} \Big[\alpha^2+\beta^2+\gamma^2+\delta^2\Big]^2-4\alpha\beta\gamma\delta &\geq \Big[\big|\alpha\beta\big|+2\big|\gamma\delta\big|\Big]^2-4\alpha\beta\gamma\delta \\ &\geq \Big[4\sqrt{\big|\alpha\beta\gamma\delta\big|}\Big]^2-4\alpha\beta\gamma\delta \\ &\geq 12\big|\alpha\beta\gamma\delta\big| \geq 0. \end{aligned}$$

The AM-GM inequality has been used. We conclude that

$$4\Big\{\big(\beta^2+\gamma^2+\delta^2+\beta\gamma+\gamma\delta+\beta\delta\big)^2+\beta\gamma\delta\big(\beta+\gamma+\delta\big)\Big\}\geq 0$$

with equality if and only if $|\alpha|=|\beta|=|\gamma|=|\delta|=0$. Otherwise one of $\beta+\gamma$, $\beta+\delta$, $\gamma+\delta$ must be zero. It follows that

$$\alpha(\alpha+\beta)(\alpha+\gamma)(\alpha+\delta)=0.$$

■

Example 4.12. Find

$$\min_{|z|\leq 1}\Big\{\max\big\{\big|1+z\big|,\ \big|1+z^2\big|\big\}\Big\}.$$

Solution: Consider the functions f, g defined by

$$f(z) = \max\left\{\left|1+z\right|,\ \left|1+z^2\right|\right\}, g(z) = \max\left\{\left|1+z\right|^2,\ \left|1+z^2\right|^2\right\}.$$

Both f and g attain their minimum at the same point(Why?). Hence, it is sufficient to consider the latter and find the point where it attains its minimum. We have to find the least c for which there exists a z such that

$$\left|1+z\right|^2 \le c^2, \text{ and } \left|1+z^2\right|^2 \le c^2. \tag{4.2}$$

Since $\left|1+z\right| = \left|1+z^2\right| = 1$ for $z = 0$, we must have $c \le 1$.
Let us put $z = re^{i\theta}$. Then

$$\left|1+z\right|^2 = 1 + r^2 + 2r\cos\theta, \left|1+z^2\right|^2 = \left(1-r^2\right)^2 + 4r^2\cos^2\theta.$$

Using $\left|1+z\right|^2 \le c^2$, and $\left|1+z^2\right|^2 \le c^2$, we get

$$\cos^2\theta \ge \frac{\left(r^2+1-c^2\right)^2}{4r^2}, \quad \cos^2\theta \le \frac{c^2-\left(1-r^2\right)^2}{4r^2}.$$

Thus, there is a z such that $\left|1+z\right|^2 \le c^2$ and $\left|1+z^2\right|^2 \le c^2$ if and only if

$$\left(r^2+1-c^2\right)^2 \le c^2 - \left(1-r^2\right)^2.$$

This simplifies to

$$2r^4 - 2r^2c^2 + \left(c^4 - 3c^2 + 2\right) \le 0.$$

For getting a positive value for r^2, we need the discriminant of the above quadratic expression to be non-negative. This gives

$$c^4 - 6c^2 + 4 \le 0. \tag{4.3}$$

Thus, if (4.2) holds for some z, then (4.3) holds. The least number c satisfying (4.3) is

$$c = \sqrt{3-\sqrt{5}}.$$

Thus, for any z, we have

$$\max\left\{\left|1+z\right|,\ \left|1+z^2\right|\right\} \ge \sqrt{3-\sqrt{5}}.$$

Taking

$$z = re^{i\theta}, r = \sqrt{\frac{3-\sqrt{5}}{2}}, \theta = \frac{2\pi}{3},$$

we see that $\left|1+z\right| = \left|1+z^2\right| = \sqrt{3-\sqrt{5}}$. Hence, the required minimum is $\sqrt{3-\sqrt{5}}$. ■

Example 4.13. (Romania) Define a sequence $\langle x_n \rangle$ by

$$x_1 = 1, x_{n+1} = \frac{x_n}{n} + \frac{n}{x_n}, \text{ for } n \geq 1.$$

Show that $\langle x_n \rangle$ is increasing and $\left[x_n^2\right] = n$ for $n \geq 4$.

Solution: We observe that $x_1 = 1$, $x_2 = 2$, $x_3 = 2$. We show that for $n \geq 3$, the inequality

$$\sqrt{n} \leq x_n \leq \frac{n}{\sqrt{n-1}}, \tag{4.4}$$

holds good. Consider the function

$$f(x) = \frac{x}{n} + \frac{n}{x},$$

defined for all $x > 0$. Then it is easy to see that $f(x)$ is a decreasing function for $x \leq n$. Assuming (4.4) for n, we have

$$x_{n+1} = f\left(x_n\right) \leq f\left(\sqrt{n}\right) = \frac{n+1}{\sqrt{n}},$$

$$x_{n+1} = f\left(x_n\right) \geq f\left(\frac{n}{\sqrt{n-1}}\right) = \frac{n}{\sqrt{n-1}} > \sqrt{n+1}.$$

Thus, (4.4) is true for all n by induction. We now have

$$\begin{aligned} x_{n+1} - x_n &= x_n\left(\frac{1}{n} - 1\right) \\ &= \frac{(n-1)}{n x_n}\left(\frac{n^2}{n-1} - x_n^2\right) \geq 0. \end{aligned}$$

Thus, $x_n \leq x_{n+1}$ for all n. Now using $x_n \leq n/\sqrt{n-1}$, we get

$$x_{n+1} = f\left(x_n\right) \geq f\left(\frac{n}{\sqrt{n-1}}\right) = \frac{n}{\sqrt{n-1}}.$$

We thus obtain

$$\frac{n-1}{\sqrt{n-2}} \leq x_n \leq \frac{n}{\sqrt{n-1}}.$$

This gives

$$x_{n+1} = f\left(x_n\right) \leq f\left(\frac{n-1}{\sqrt{n-2}}\right) = \frac{n^2(n-2) + (n-1)^2}{n(n-1)\sqrt{n-2}}.$$

We show that

$$\frac{n^2(n-2) + (n-1)^2}{n(n-1)\sqrt{n-2}} < \sqrt{n+2}.$$

This is equivalent to the statement $2n^3 - 6n^2 + 4n - 1 > 0$, which is true for $n \geq 4$. Thus, $x_n < \sqrt{n+1}$ for $n \geq 4$. We obtain

$$\sqrt{n} < x_n < \sqrt{n+1},$$

for $n \geq 4$. Hence, $\left[x_n^2\right] = n$ for $n \geq 4$. ■

Example 4.14. ([8]) Find all real numbers a, b, c such that

$$(1-a)^2 + (a-b)^2 + (b-c)^2 + c^2 = \frac{1}{4}.$$

Solution: Observe that

$$\begin{aligned} 1 &= \left|(1-a) + (a-b) + (b-c) + c\right|^2 \\ &\leq 4\Big((1-a)^2 + (a-b)^2 + (b-c)^2 + c^2\Big) \\ &= 1. \end{aligned}$$

Thus, equality holds in the Cauchy-Schwarz inequality. It follows that

$$1 - a = a - b = b - c = c = \frac{1}{4}.$$

Hence, $a = 3/4$, $b = 1/2$, $c = 1/4$. ■

Example 4.15. ([8]) Show that if all the roots of $P(x) = ax^4 - bx^3 + cx^2 - x + 1 = 0$ are positive, then $c \geq 80a + b$.

Solution: Suppose $a \neq 0$. Let α, β, γ and δ be the roots of $P(x) = 0$. These are positive reals. Taking

$$u = \frac{1}{\alpha} + \frac{1}{\beta}, \quad v = \frac{1}{\gamma} + \frac{1}{\delta},$$

the following relations are easily obtained:

$$u + v = 1, \quad \alpha\beta u = \alpha + \beta, \quad \gamma\delta v = \gamma + \delta.$$

We have to show that

$$\frac{c}{a} - \frac{b}{a} \geq 80.$$

However, using the Viette's relations between the roots and coefficients,

$$\begin{aligned} \frac{c}{a} - \frac{b}{a} &= \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta - (\alpha + \beta + \gamma + \delta) \\ &= (\alpha + \beta)(\gamma + \delta) - (\alpha + \beta) - (\gamma + \delta) + \alpha\beta + \gamma\delta \\ &= \alpha\beta\gamma\delta uv + \alpha\beta(1 - u) + \gamma\delta(1 - v) \\ &= \alpha\beta\gamma\delta uv + \alpha\beta v + \gamma\delta u. \end{aligned}$$

But, observe that

$$\sqrt{\alpha\beta} \geq \frac{2}{\alpha^{-1}+\beta^{-1}}.$$

This gives $\alpha\beta \geq 4u^{-2}$. Similarly $\gamma\delta \geq 4v^{-2}$. Thus,

$$\begin{aligned}
\frac{c}{a}-\frac{b}{a} &\geq \frac{16}{uv}+\frac{4}{u^2v^2}\left(u^3+v^3\right) \\
&= \frac{16}{uv}+\frac{4}{u^2v^2}\left(u^2-uv++v^2\right) \\
&= 4\left\{\frac{3}{uv}+\frac{1}{u^2}+\frac{1}{v^2}\right\} \\
&= 4\left\{\frac{1}{uv}+\left(\frac{1}{u}+\frac{1}{v}\right)^2\right\} \\
&= 4\left\{\frac{1}{uv}+\frac{1}{u^2v^2}\right\} \\
&= \left(\frac{2}{uv}+1\right)^2-1.
\end{aligned}$$

But $u+v=1$ implies that $uv \leq 1/4$. Thus,

$$\frac{c}{a}-\frac{b}{a} \geq 9^2-1=80.$$

Equality holds if and only if $\alpha=\beta=\gamma=\delta=4$.

Suppose $a=0$ and $b \neq 0$. Then $P(x)$ reduces to a cubic polynomial: $P(x)=-bx^3+cx^2-x-1$. Let p,q,r be its positive roots. Then

$$\frac{c}{b}=p+q+r, \text{ and } \frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1.$$

Hence,

$$\frac{c}{b}=\left(p+q+r\right)\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right) \geq 3,$$

by the AM-GM inequality. In particular $c/b > 1$ and this proves the result in this case.

Finally, suppose $a=b=0$. Then $P(x)=cx^2-x-1$ and hence $c>0$; otherwise $P(x)=0$ has no positive root. This completes the proof. ∎

Example 4.16. (CRUX, 1995) Find the maximum value of

$$\frac{a}{c}+\frac{b}{d}+\frac{c}{a}+\frac{d}{b},$$

where a,b,c,d vary over all distinct real numbers such that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}=4, \quad ac=bd.$$

Solution: Introducing the new variables u, v by $u = a/b$ and $v = b/c$, we see that

$$\frac{c}{d} = \frac{1}{u}, \frac{d}{a} = \frac{1}{v}, \frac{a}{c} = uv, \frac{b}{d} = \frac{v}{u}.$$

Thus, the maximum of $uv + \frac{v}{u} + \frac{1}{uv} + \frac{u}{v}$ has to be found under the constraint $u + v + (1/u) + (1/v) = 4$. Now for any nonzero real p, we have $p + (1/p) \geq 2$ if $p > 0$ and $p + (1/p) \leq 2$ if $p < 0$. The condition $u + v + (1/u) + (1/v) = 4$ shows that u and v cannot both be negative. If u and v are both positive then $u + (1/u) = 2 = v + (1/v)$ which forces that $u = v = 1$. But then $a = b = c$ contradicting the distinctness of a, b, c. Thus, one of u, v is positive and the other negative. We may assume that $u > 0$ and $v < 0$. In this case $v + \frac{1}{v} \leq -2$ and hence

$$u + \frac{1}{u} = 4 - \left(v + \frac{1}{v}\right) \geq 6.$$

We finally get

$$uv + \frac{v}{u} + \frac{1}{uv} + \frac{v}{u} = \left(u + \frac{1}{u}\right)\left(v + \frac{1}{v}\right) \leq -12.$$

This maximum is attained when

$$a = 3 + 2\sqrt{2}, b = 1, c = -1, d = -3 + 2\sqrt{2}.$$

■

Example 4.17. Solve the system of equations for real values of x, y, z:

$$\begin{aligned} 3\left(x^2 + y^2 + z^2\right) &= 1, \\ x^2y^2 + y^2z^2 + z^2x^2 &= xyz(x + y + z)^3. \end{aligned}$$

Solution: The second relation shows that $xyz(x + y + z)^3 \geq 0$ and hence $xyz(x+y+z) \geq 0$. If $xyz(x+y+z) = 0$, then the second relation implies that $xy = yz = zx = 0$. Hence, two of x, y, z are zero. If $x = y = 0$, then $z^2 = 1/3$ giving $z = \pm 1/3$. We get six solutions:

$$(x, y, z) = \left(0, 0, \pm 1/\sqrt{3}\right), \left(0, \pm 1/\sqrt{3}, 0\right), \left(\pm 1/\sqrt{3}, 0, 0\right).$$

Suppose $xyz(x+y+z) > 0$. Using the Cauchy-Schwarz inequality, we see that

$$\left(xy + yz + zx\right)^2 \leq 3\left(x^2y^2 + y^2z^2 + z^2x^2\right).$$

This gives

$$xyz(x + y + z) \leq x^2y^2 + y^2z^2 + z^2x^2 = xyz(x + y + z)^3.$$

Since $xyz(x+y+z) > 0$, it follows that $(x+y+z)^2 \geq 1$. But then

$$1 \leq (x+y+z)^2 \leq 3\big(x^2+y^2+z^2\big) = 1.$$

Hence, equality is forced every where. We obtain $(x+y+z)^2 = 3\big(x^2+y^2+z^2\big)$ or equivalently $x^2+y^2+z^2 = xy+yz+zx$. This leads to

$$(x-y)^2 + (y-z)^2 + (z-x)^2 = 0,$$

which gives $x = y = z$. We obtain from the first equation $x = y = z = 1/3$ or $x = y = z = -1/3$, giving two more solutions. ■

Example 4.18. Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a real polynomial of degree n. Suppose n is even and

(i) $a_0 > 0$, $a_n > 0$;

(ii) $a_1^2 + a_2^2 + \cdots + a_{n-1}^2 \leq \dfrac{4}{n-1} \min\left\{a_0^2, a_n^2\right\}$.

Prove that $P(x) \geq 0$ for all real values of x.

Solution: Suppose x is a real number. The Cauchy-Schwarz inequality gives

$$\Big|a_{n-1}x^{n-1} + \cdots + a_1 x\Big| \leq \sqrt{a_{n-1}^2 + \cdots + a_1^2}\,\sqrt{x^2 + x^4 + \cdots + x^{2n-2}},$$

for any real x. Thus, for $x \in \mathbb{R}$,

$$\begin{aligned} P(x) &= \big(a_n x^n + a_0\big) + \big(a_{n-1}x^{n-1} + \cdots + a_1 x\big) \\ &\geq \big(a_n x^n + a_0\big) - \Big|a_{n-1}x^{n-1} + \cdots + a_1 x\Big| \\ &\geq \min\left\{a_n, a_0\right\}\big(1 + x^n\big) - \sqrt{a_{n-1}^2 + \cdots + a_1^2}\,\sqrt{x^2 + x^4 + \cdots + x^{2n-2}}. \end{aligned}$$

Using the condition (ii), it follows that

$$\begin{aligned} P(x) &\geq \min\left\{a_n, a_0\right\}\big(1 + x^n\big) - \frac{2\min\{a_0, a_n\}}{\sqrt{n-1}}\sqrt{x^2 + x^4 + \cdots + x^{2n-2}} \\ &= \min\left\{a_n, a_0\right\}\left[1 + x^n - \frac{2\sqrt{x^2 + x^4 + \cdots + x^{2n-2}}}{\sqrt{n-1}}\right]. \end{aligned}$$

Thus, it is sufficient to prove that

$$1 + x^n - \frac{2\sqrt{x^2 + x^4 + \cdots + x^{2n-2}}}{\sqrt{n-1}} \geq 0,$$

for all real x. Equivalently, one has to prove that

$$\big(n-1\big)\big(x^n + 1\big)^2 \geq 4\sum_{j=1}^{n-1} x^{2j}.$$

Since n is even, $n = 2m$ for some positive integer m. Introducing $x^2 = y$, we need to show that

$$(2m-1)\left(y^{2m} + 2y^m + 1\right) \geq 4 \sum_{j=1}^{2m-1} y^j.$$

We use the majorisation technique. The function $f(t) = t^k$ is a convex function for all $k \geq 1$. Hence, we need to show that

$$\Big(\underbrace{2m-1, \ldots, 2m-1}_{4}, \underbrace{2m-2, \ldots, 2m-2}_{4}, \ldots, \underbrace{1, 1, 1, 1}_{4}\Big)$$
$$\prec \Big(\underbrace{2m, 2m, \ldots, 2m}_{2m-1}, \underbrace{m, m, \ldots, m}_{2(2m-1)}, \underbrace{0, 0, \ldots, 0}_{2m-1}\Big)$$

Note that after $4(m-1)$ terms in the first vector, the remaining terms are not more than m. Hence, it is sufficient to prove that

$$4\big(2m - 1 + 2m - 2 + \cdots + 2m - (m-1)\big) < 2m(2m-1) + m(2m-3).$$

But the left-side is $6m^2 - 12m$ and the right-side is $6m^2 - 5m$ and hence the inequality holds. This completes the proof. ■

Example 4.19. (Nordic Contest-1992) Find all real numbers $x > 1$, $y > 1$, $z > 1$ such that

$$x + y + z + \frac{3}{x-1} + \frac{3}{y-1} + \frac{3}{z-1} = 2\left(\sqrt{x+2} + \sqrt{y+2} + \sqrt{z+2}\,\right).$$

Solution: For $t > 1$, the AM-GM inequality gives

$$t - 1 + \frac{t+2}{t-1} \geq 2\sqrt{t+2}.$$

Equality holds if and only if $(t-1) = (t+2)/(t-1)$, which is equivalent to $t = (3 + \sqrt{13})/2$. Thus,

$$\begin{aligned}
& x + y + z + \frac{3}{x-1} + \frac{3}{y-1} + \frac{3}{z-1} \\
&\qquad = \quad x - 1 + y - 1 + z - 1 + \frac{x+2}{x-1} + \frac{y+2}{y-1} + \frac{z+2}{z-1} \\
&\qquad \geq \quad 2\left(\sqrt{x+2} + \sqrt{y+2} + \sqrt{z+2}\,\right) \\
&\qquad = \quad x + y + z + \frac{3}{x-1} + \frac{3}{y-1} + \frac{3}{z-1}
\end{aligned}$$

Hence, equality holds and this corresponds to

$$x = y = z = \frac{3+\sqrt{13}}{2}.$$

$\blacksquare$

Example 4.20. (Bulgaria) Find the largest real number a such that

$$\frac{x}{\sqrt{y^2+z^2}} + \frac{y}{\sqrt{z^2+x^2}} + \frac{z}{\sqrt{x^2+y^2}} \geq a,$$

for all positive real numbers x, y, z.

Solution: Note that for any real α,

$$\frac{2\alpha}{1+\alpha^2} \leq 1.$$

If $\alpha > 0$, we also get

$$\frac{2\alpha^2}{1+\alpha^2} \leq \alpha.$$

Thus, it follows that

$$\frac{x}{\sqrt{y^2+z^2}} \geq \frac{2x^2/(y^2+z^2)}{1+\left(x^2/(y^2+z^2)\right)} = \frac{2x^2}{x^2+y^2+z^2}.$$

This implies that

$$\frac{x}{\sqrt{y^2+z^2}} + \frac{y}{\sqrt{z^2+x^2}} + \frac{z}{\sqrt{x^2+y^2}} \geq \sum_{\text{cyclic}} \frac{2x^2}{x^2+y^2+z^2} = 2$$

We show that $a = 2$ is the largest possible. In other words, for each $\epsilon > 0$, there are positive x, y, z such that

$$\sum_{\text{cyclic}} \frac{x}{\sqrt{y^2+z^2}} < 2 + \epsilon.$$

For this choose $x = y = 1$ and $z = \epsilon\sqrt{2}$. Then

$$\sum_{\text{cyclic}} \frac{x}{\sqrt{y^2+z^2}} = \frac{2}{\sqrt{1+2\epsilon^2}} + \epsilon < 2 + \epsilon.$$

$\blacksquare$

Example 4.21. (Romania, 1993) Prove that if x, y, z are positive integers such that $x^2 + y^2 + z^2 = 1993$, then $x + y + z$ cannot be the square of an integer.

Solution: Suppose, if possible, $x+y+z=k^2$ for some integer k. Then

$$k^2 = x+y+z \le \sqrt{3}\sqrt{x^2+y^2+z^2} = \sqrt{3}\sqrt{1993} < 70.$$

Thus, $k^2 \in \{1,4,9,16,25,36,49,64\}$. On the other hand $x^2 \equiv x \pmod 2$ for any integer x. Thus, $x^2+y^2+z^2 \equiv x+y+z \pmod 2$ and hence

$$k^2 = x+y+z \equiv x^2+y^2+z^2 \equiv 1993 \equiv 1 \pmod 2.$$

This shows that k^2 is odd and hence $k^2 \in \{1,9,25,49\}$. Now observe that

$$(x+y+z)^2 > x^2+y^2+z^2 = 1993, \text{ and } \; 25^2 = 625 < 1993.$$

Hence, $x+y+z=49$ is the only possibility. But then $x+y+z=49$ and $x^2+y^2+z^2=1993$ give

$$(49-x)^2 = (y+z)^2 > y^2+z^2 = 1993-x^2.$$

Simplification leads to $x^2-49x+204>0$. It follows that

$$\text{either } x < \frac{49-\sqrt{1505}}{2} \qquad \text{or} \qquad x > \frac{49+\sqrt{1505}}{2}.$$

However

$$x > \frac{49+\sqrt{1505}}{2} \Longrightarrow x > \frac{49+39}{2} = 44,$$

and hence $x \ge 45$. But then $x^2 \ge 45^2 = 2025 > 1993$. Similarly,

$$x < \frac{49-\sqrt{1505}}{2} \Longrightarrow x \le 4.$$

Symmetry shows that $y \le 4$ and $z \le 4$. It follows that

$$x+y+z \le 12 < 49.$$

Thus, $x+y+z$ cannot be a square. ■

Example 4.22. ([7]) Find all real numbers a, b, c such that

$$\left|ax^2+bx+c\right| \le 1, \text{ for all } |x| \le 1,$$

and $\frac{8}{3}a^2 + 2b^2$ is maximum.

Solution: It is sufficient to maximise $4a^2 + 3b^2$, since

$$4a^2 + 3b^2 = \frac{3}{2}\left(\frac{8}{3}a^2 + 2b^2\right).$$

Consider the polynomial $P(x) = ax^2 + bx + c$. Since $|P(0)| \le 1$ and $|P(1)| \le 1$, we have

$$2 \ge |P(1) - P(0)| = |a + b + c - c| = |a + b|.$$

Similarly using $|P(-1)| \le 1$, we also obtain

$$2 \ge |P(-1) - P(0)| = |a - b + c - c| = |a - b|.$$

Thus,

$$4a^2 + 3b^2 = 2(a + b)^2 + 2(a - b)^2 - b^2 \le 16 - b^2 \le 16.$$

Here equality holds only if $b = 0$. Thus, $4a^2 + 3b^2$ is maximum only if $b = 0$. In this case

$$|a + b| = |a - b| = |a| = 2.$$

Thus,

$$|P(1) - P(0)| = |a + c - c| = |a| = 2.$$

Hence,

$$2 = |P(1) - P(0)| \le |P(1)| + |P(0)| \le 2.$$

We conclude that $|P(1)| = |P(0)| = 1$ and hence $|c| = 1$, $|a + c| = 1$. It follows that either $c = 1, a = -2, b = 0$ or $c = -1, a = 2, b = 0$.

Now the required maximum is

$$\frac{8}{3}a^2 + 2b^2 = \frac{8}{3} \times 4 = \frac{32}{3}.$$

■

Example 4.23. ([7]) What is the maximum value of $a^2 + b^2$, given that the equation

$$x^4 + ax^3 + bx^2 + ax + 1 = 0$$

has real roots.

Solution: Consider the polynomial $P(x) = x^4 + ax^3 + bx^2 + ax + 1$. Observe that $P(x) = x^4 P(1/x)$. Hence, $P(x)$ is a reciprocal polynomial and hence the substitution $y = x + 1/x$ is meaningful. This leads to a quadratic equation

$$y^2 + ay + b - 2 = 0.$$

If x is real, we observe that y is also real and hence

$$(2 - y^2)^2 = (ay + b)^2 \le (a^2 + b^2)(1 + y^2).$$

This gives

$$a^2 + b^2 \geq \frac{(2-y^2)^2}{1+y^2}.$$

We also note that $y = x + 1/x \leq -2$ or $y \geq 2$. Hence, $y^2 \geq 4$. Consider the function

$$f(t) = \frac{(2-t)^2}{1+t},$$

for $t \geq 4$. This is a monotonically increasing function for $t \geq 4$, as may be verified using its derivatives. Its minimum is attained at $t = 4$;

$$a^2 + b^2 \geq f(4) = \frac{4}{5}.$$

If we take $y^2 = 4$, we have $x + 1/x = 2$ and hence $x = 1$ is a real solution. This corresponds to $2a + b = -2$ and we can take

$$a = -\frac{4}{5}, \quad b = -\frac{2}{5}.$$

Then $a^2 + b^2 = 4/5$. Thus, the minimum value of $a^2 + b^2$ is $4/5$. ■

Example 4.24. Suppose a, b, c are real numbers such that

$$|a| \leq |b - c|, \quad |b| \leq |c - a|, \text{and} \quad |c| \leq |a - b|$$

Prove that one of these numbers is the sum of the remaining two.

Solution: We get from the first relation, $a^2 \leq (b-c)^2$, or which is the same as

$$(b - c - a)(b - c + a) \geq 0.$$

Similarly

$$(c - a - b)(c - a + b) \geq 0, \text{ and } (a - b + c)(a - b - c) \geq 0.$$

Multiplying these (which is permissible since all are non-negative), we get

$$(a + b - c)^2(b + c - a)^2(c + a - b)^2 \leq 0.$$

It follows that at least one of $(a+b-c)$, $(b+c-a)$, $(c+a-b)$ is equal to zero. ■

Example 4.25. Let ABC be a triangle and P be an interior point of ABC. Let the lines AP, BP, CP meet BC, CA, AB respectively in D, E, F. Find all positions of the point P for which the area of triangle DEF is maximal.

Solution: It is sufficient to locate D, E, F such that AD, BE, CF are concurrent and $[DEF]$ is maximal, where $[DEF]$ denotes the area of triangle DEF. Let us put $BD = a - x$, $DC = x$, $CE = b - y$, $EA = y$, $AF = c - z$ and $FB = z$. Then by Ceva's theorem,

$$\frac{a-x}{x} \cdot \frac{b-y}{y} \cdot \frac{c-z}{z} = 1.$$

If we also introduce $\alpha = x/a$, $\beta = y/b$ and $\gamma = z/c$, then it takes the form

$$(1-\alpha)(1-\beta)(1-\gamma) = \alpha\beta\gamma.$$

Now observe that $[BDF] = \gamma(1-\alpha)[ABC]$ and similar expressions for other corner triangles. Thus,

$$\begin{aligned} [DEF] &= [ABC] - [BDF] - [CED] - [AFE] \\ &= [ABC]\{1 - \gamma(1-\alpha) - \alpha(1-\beta) - \beta(1-\gamma)\}. \end{aligned}$$

This shows that we have to minimize $\gamma(1-\alpha) + \alpha(1-\beta) + \beta(1-\gamma)$ in order to maximise $[DEF]$. However

$$\gamma(1-\alpha) + \alpha(1-\beta) + \beta(1-\gamma) = \alpha + \beta + \gamma - (\alpha\beta + \beta\gamma + \gamma\alpha).$$

Now using $(1-\alpha)(1-\beta)(1-\gamma) = \alpha\beta\gamma$, it is easy to get

$$\alpha + \beta + \gamma - (\alpha\beta + \beta\gamma + \gamma\alpha) = 1 - 2\alpha\beta\gamma.$$

Thus, we have to minimise $1 - 2\alpha\beta\gamma$ or which is same as maximising $\alpha\beta\gamma$. But

$$\begin{aligned} \alpha\beta\gamma &= \sqrt{\alpha\beta\gamma(1-\alpha)(1-\beta)(1-\gamma)} \\ &= \sqrt{\alpha(1-\alpha)\beta(1-\beta)\gamma(1-\gamma)} \\ &\leq \left(\frac{\alpha + 1 - \alpha + \beta + 1 - \beta + \gamma + 1 - \gamma}{3}\right)^3 \\ &= \frac{1}{8}. \end{aligned}$$

Equality holds if and only if $\alpha = \beta = \gamma$. This is possible if and only if D, E, F are the mid-points of BC, CA, AB respectively. We conclude that $P = G$, the centroid of ABC. ■

Example 4.26. (Short-list, IMO-1993) Let a, b, c, d be non-negative real numbers such that $a + b + c + d = 1$. Prove that

$$abc + bcd + cda + dab \leq \frac{1}{27} + \frac{176}{27}abcd.$$

Solution: Consider the expression

$$\begin{aligned}
F\big(a,b,c,d\big) &= abc+bcd+cda+dab-\frac{176}{27}abcd \\
&= bc(a+d)+ad\left(b+c-\frac{176}{27}bc\right).
\end{aligned}$$

The function $F\big(a,b,c,d\big)$ is symmetric in a,b,c,d. There are two possibilities:

(i) If $b+c-\dfrac{176}{27}bc \le 0$, then the AM-GM inequality implies that

$$F\big(a,b,c,d\big) \le bc(a+d) \le \left(\frac{b+c+a+d}{3}\right)^3 = \frac{1}{27}.$$

Hence, we may assume that this never occurs in any subsequent steps.

(ii) Suppose $b+c-\dfrac{176}{27}bc > 0$. Then again the AM-GM inequality implies that

$$\begin{aligned}
F\big(a,b,c,d\big) &\le bc(a+d)+\left(\frac{a+d}{2}\right)^2\left(b+c-\frac{176}{27}bc\right) \\
&= F\left(\frac{a+d}{2},b,c,\frac{a+d}{2}\right).
\end{aligned}$$

Now we iterate this under the assumption that we are in the second case in each stage of the iteration and we also exploit the symmetry of F:

$$\begin{aligned}
F\big(a,b,c,d\big) &\le F\left(\frac{a+d}{2},b,c,\frac{a+d}{2}\right) \\
&= F\left(b,\frac{a+d}{2},\frac{a+d}{2},c\right) \\
&\le F\left(\frac{b+c}{2},\frac{a+d}{2},\frac{a+d}{2},\frac{b+c}{2}\right) \\
&= F\left(\frac{a+d}{2},\frac{b+c}{2},\frac{a+d}{2},\frac{b+c}{2}\right) \\
&\le F\left(\frac{1}{4},\frac{b+c}{2},\frac{a+d}{2},\frac{1}{4}\right) \\
&= F\left(\frac{b+c}{2},\frac{1}{4},\frac{1}{4},\frac{a+d}{2}\right) \\
&\le F\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right) = \frac{1}{27}
\end{aligned}$$

This completes the solution. ■

Example 4.27. Determine the minimum value of $\sum_{j=1}^{n} a_j$ where the minimum is taken over all sequences $\langle a_1, a_2, a_3, \ldots, a_n \rangle$, $(n \geq 2)$, with non-negative terms such that $\displaystyle\sum_{j=1}^{n} a_j a_{j+1} = 1$, where $a_{n+1} = a_1$.

Solution: Let

$$M_n = \min\left\{ \sum_{j=1}^{n} a_j \,\middle|\, a_j \geq 0, \text{ for } 1 \leq j \leq n, \text{ and } \sum_{j=1}^{n} a_j a_{j+1} = 1 \right\}.$$

We show that $M_2 = \sqrt{2}$, $M_3 = \sqrt{3}$ and $M_n = 2$ for $n \geq 4$. Put $\alpha = \sum a_j$ and $\beta = \sum a_j a_{j+1}$. If $n = 2$, then $\alpha = a_1 + a_2$, $\beta = 2a_1 a_2 = 1$, so that

$$\alpha^2 - 2 = \alpha^2 - 2\beta = (a_1 - a_2)^2 \geq 0,$$

with equality if and only if $a_1 = a_2 = 1/\sqrt{2}$. Thus, $M_2 = \sqrt{2}$. If $n = 3$, then

$$\begin{aligned}
\alpha^2 - 3 &= \alpha^2 - 3\beta \\
&= (a_1 + a_2 + a_3)^2 - 3(a_1 a_2 + a_2 a_3 + a_3 a_1) \\
&= a_1^2 + a_2^2 + a_3^2 - a_1 a_2 - a_2 a_3 - a_3 a_1 \\
&= \frac{1}{2}\left\{ (a_1 - a_2)^2 + (a_2 - a_3)^2 + (a_3 - a_1)^2 \right\} \\
&\geq 0.
\end{aligned}$$

Here again, equality holds if and only if $a_1 = a_2 = a_3 = 1/\sqrt{3}$. Thus, $M_3 = \sqrt{3}$.

Consider the case $n = 4$. In this case,

$$\begin{aligned}
\alpha^2 - 4 &= \alpha^2 - 4\beta \\
&= \left\{ (a_1 + a_3) + (a_2 + a_4) \right\}^2 - 4(a_1 + a_3)(a_2 + a_4) \\
&= \left\{ (a_1 + a_3) - (a_2 + a_4) \right\}^2 \geq 0.
\end{aligned}$$

Equality holds if and only if $a_1 + a_3 = a_2 + a_4 = 1/2$. Thus, $M_4 = 2$.

Suppose $n \geq 5$. We may assume that a_1 is the least number by using the cyclical nature of the condition $\sum a_j a_{j+1} = 1$. Let

$$\gamma = \left(a_1 + a_2 + \cdots + a_n\right)^2 - \left(a_1 - a_2 + a_3 - \cdots + (-1)^{n+1} a_n\right)^2.$$

If n is even, say $n = 2k$, then

$$\begin{aligned}
\gamma &= 4\left(a_1 + a_3 + \cdots + a_{2k-1}\right)\left(a_2 + a_4 + \cdots + a_{2k}\right) \\
&\geq 4\left(a_1 a_2 + a_2 a_3 + \cdots + a_{2k-1} a_{2k} + a_{2k} a_1\right) \\
&= 4.
\end{aligned}$$

Suppose n is odd, say $n = 2k - 1$; then

$$\begin{aligned}
\gamma &= 4\big(a_1 + a_3 + \cdots + a_{2k-1}\big)\big(a_2 + a_4 + \cdots + a_{2k-2}\big) \\
&\geq 4\big(a_1a_2 + a_2a_3 + \cdots + a_{2k-2}a_{2k-1} + a_{2k-1}a_2\big) \\
&\geq 4\big(a_1a_2 + a_2a_3 + \cdots + a_{2k-1}a_1\big) \\
&= 4,
\end{aligned}$$

using $a_2 \geq a_1$. Thus, $\gamma \geq 4$. This implies that

$$\big(a_1 + a_2 + \cdots + a_n\big)^2 \geq 4. \tag{4.5}$$

We show that equality holds in (4.5) if and only if there is a k such that $1 \leq k \leq n$, $a_k = 1$, $a_{k-1} + a_{k+1} = 1$ and $a_j = 0$ for all other j. (Here $a_0 = a_n$, $a_{n+1} = a_1$.) Clearly this is sufficient. Conversely, suppose equality holds in (4.5). Then

$$\begin{aligned}
&\big(a_1 + a_2 + \cdots + a_n\big)^2 - \big(a_1 - a_2 + a_3 - \cdots + (-1)^{n+1}a_n\big)^2 \\
&\qquad\qquad = 4\big(a_1a_2 + a_2 + a_3 + \cdots + a_{n-1}a_n + a_na_1\big).
\end{aligned}$$

Thus, for j, k with $j + 2k + 1 \leq n$, it must be the case that $a_ja_{j+2k+1} = 0$. In particular, $a_1a_4 = 0$. This gives $a_1 = 0$, since $a_1 \leq a_4$. Further we have

$$a_1 + a_2 + \cdots + a_n = 2, \text{ and } a_1 - a_2 + a_3 - \cdots \pm a_n = 0.$$

Thus, it follows that

$$\begin{aligned}
a_1 + a_3 + \cdots &= 1, \\
a_2 + a_4 + \cdots &= 1.
\end{aligned}$$

If $a_2 \neq 0$, then the condition $a_ja_{j+2k+1} = 0$ implies that $a_5 = a_7 = \cdots = 0$, and $a_3 = 1$. Further it also follows that $a_6 = a_8 = \cdots = 0$. Hence, $a_2 + a_4 = 1$. Suppose $a_2 = 0$. Consider a_3. Using a similar argument, it is easy to show that there is some l such that $a_l = 1$ and $a_{l-1} + a_{l+1} = 1$ and the remaining are all equal to zero. ■

Example 4.28. (Taiwan, 2002) Determine all positive integers n and integers $a_1, a_2, a_3, \ldots, a_n$ such that $a_j \geq 0$, for $1 \leq j \leq n$, and

$$\sum_{j=1}^{n} a_j^2 = 1 + \frac{4}{4n+1}\left(\sum_{j=1}^{n} a_j\right)^2.$$

Solution: Let us put

$$A = \sum_{j=1}^{n} a_j.$$

Suppose $a_j \leq 1$, for all j. Then $a_j = 0$ or 1, for each j. Let α be the number of those j's for which $a_j = 1$. Then $n - \alpha$ is the number of those j's for which $a_j = 0$. The given equation reduces to

$$\alpha = 1 + \frac{4}{4n+1}\alpha^2. \tag{4.6}$$

This shows that $\alpha > 1$. Moreover,

$$4n + 1 = 4(\alpha + 1) + \frac{4}{\alpha - 1}.$$

Hence, $\alpha - 1$ divides 4, so that $\alpha - 1 = 1, 2$ or 4. The first two cases may be disposed of using (4.6). The third choice gives $n = 6$. In this case $(0, 1, 1, 1, 1, 1)$ and its permutations give all the solutions.

We may assume $a_j \geq 2$ for some j. In this case

$$A + 1 < \sum a_j^2 = 1 + \frac{4}{4n+1}A^2.$$

Thus, $A > n + (1/4)$. Since A is an integer, it follows that $A \geq n + 1$. Using the Cauchy-Schwarz inequality, one obtains,

$$\frac{A^2}{n} \leq \left(\sum_{j=1}^{n} a_j^2\right) = 1 + \frac{4}{4n+1}A^2.$$

This implies that $A^2 \leq 4n^2 + n$. Again the nature of A shows that $A \leq 2n$. Thus, $n + 1 \leq A \leq 2n$. We hence obtain

$$1 < 1 + \frac{1}{n} \leq \frac{A}{n} \leq 2.$$

But we also have

$$\sum_{j=1}^{n}\left(a_j - \frac{A}{n}\right)^2 = 1 - \frac{A^2}{n(4n+1)} < 1,$$

which gives

$$-1 < a_j - \frac{A}{n} < 1,$$

for $1 \leq j \leq n$. Thus,

$$0 < -1 + \frac{A}{n} < a_j < 1 + \frac{A}{n} < 3,$$

for $1 \leq j \leq n$. This shows that $a_j = 1$ or 2 for all j. Let b denote number of those j's for which $a_j = 2$. Then there are $n - b$ values of j for which $a_j = 1$. In this case (4.6) takes the form

$$4b + n - b = 1 + \frac{4}{4n+1}(2b + n - b)^2.$$

This simplifies to $n = b + \dfrac{1}{4b-3}$. Since n and b are integers the only possibility is $b = 1$ and $n = 2$.

We conclude that $n = 2$ or $n = 6$. In these cases, a_j's can be computed using the bounds we have derived:

$n = 2$: $(a_1, a_2) = (1, 2), (2, 1)$;

$n = 6$: the 6-tuples are $(0, 1, 1, 1, 1, 1)$ and permutations of this. ■

Example 4.29. (IMO, 1992) Consider a finite set V in three-dimensional space consisting of points with integer coordinates. Let S_x, S_y, S_z denote the sets consisting of the orthogonal projections of the points of S on to yz-plane, zx-plane, xy-planes respectively. Prove that

$$\left|V\right|^2 \le \left|S_x\right|^2 \left|S_y\right|^2 \left|S_z\right|^2,$$

where $|X|$ denotes the number of elements of a finite set X.

Solution: Let us introduce

$$\begin{aligned} S(x) &= \{(y,z) \,:\, (x,y,z) \in V\}; \\ S_y(x) &= \{z \,:\, (x,z) \in S_y\}; \\ S_z(x) &= \{y \,:\, (x,y) \in S_z\}. \end{aligned}$$

Observe that $S(x) \subset S_x$ and $S(x) \in S_y(x) \times S_z(x)$. Thus

$$\begin{aligned} |V| = \sum_x |S(x)| &\le \sum_x \sqrt{|S_x||S_y(x)|S_z(x)|} \\ &= \sqrt{|S_x|} \sum_x \sqrt{|S_y(x)|S_z(x)|}. \end{aligned}$$

But Cauchy-Schwarz inequality gives

$$\sum_x \sqrt{|S_y(x)|S_z(x)|} \le \sqrt{\sum_x |S_y(x)|} \sqrt{\sum_x |S_z(x)|} = \sqrt{|S_y||S_z|}.$$

Combining the two inequalities, it follows that

$$|V| \le \sqrt{|S_x||S_y||S_z|}.$$

■

Example 4.30. For $k \ge 2$, define $f : \mathbb{N} \to \mathbb{N}$ by

$$f(n) = \left[\left(n + n^{1/k}\right)^{1/k}\right] + n,$$

whee $[x]$ denotes the integer part of a real number x. Determine the range of f.

Solution: We show that the range of f is precisely the set of all those natural numbers which are not k-th power of some natural number. Suppose $a \in \mathbb{N}$ is not a k-th power. Then there is a natural number m such that $m^k < a < (m+1)^k$. Using the binomial theorem, it is easy to derive that

$$(x+1)^k \geq x^k + x + 1,$$

for all $x \geq 0$. Taking $x = m - 1$, it follows that $(m-1)^k \leq m^k - m$. Taking $n = a - m$, we also get $m^k - m < a - m = n < a$. Thus,

$$\begin{aligned} m^k \leq a - 1 &= (a-m) + (m-1) \\ &< n + n^{1/k} \\ &< (a-m) + (m+1) \\ &= a + 1 \\ &\leq (m+1)^k. \end{aligned}$$

The definition of f shows that $f(n) = m + n = a$, i.e., $f(a-m) = a$. This shows that if $a \in \mathbb{N}$ is not a k-th power, then a is in the range of f. Observe that f is strictly increasing. Since $f(1) = 2$, it follows that 1 is not in the range of f. Now consider any k-th power, say, m^k. We have proved that if $a \in \mathbb{N}$ is not a k-th power, then

$$a = f\Big(a - \big[a^{1/k}\big]\Big).$$

Thus, we obtain

$$f\Big(m^k - m\Big) = f\Big(m^k - 1 - (m-1)\Big) = m^k - 1,$$

and

$$f\Big(m^k + 1 - m\Big) = m^k + 1.$$

It follows that m^k is not in the range of f. Thus, the range of f consists of precisely all those natural numbers which are not the k-th power.

■

Example 4.31. (IMO, 1992) Find all integers a, b, c with $1 < a < b < c$ such that $(a-1)(b-1)(c-1)$ is a divisor of $abc - 1$.

Solution: Putting $a - 1 = x, b - 1 = y$ and $c - 1 = z$, the problem is to find integers $1 \leq x < y < z$ such that xyz divides $(x+1)(y+1)(z+1) - 1$. This is equivalent to find integers $1 \leq x < y < z$ for which

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx} = p$$

is also an integer. We observe that $x \geq 1$, $y \geq 2$ and $z \geq 3$, and hence

$$p \leq 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = \frac{17}{6} < 3.$$

Thus, $p = 1$ or $p = 2$.

Case 1. $p = 1$.
Here we have

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx} = 1.$$

Obviously $x \geq 2$. If $x \geq 3$ then $y \geq 4$ and $z \geq 5$ and we get an estimate

$$p \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{12} + \frac{1}{15} + \frac{1}{20} = \frac{59}{60} < 1$$

which is not tenable. We conclude that $x = 2$ and hence

$$\frac{1}{2} + \frac{1}{y} + \frac{1}{z} + \frac{1}{2y} + \frac{1}{yz} + \frac{1}{2z} = 1.$$

This simplifies to $(y-3)(z-3) = 11$. Keeping in mind that $y < z$, we see that the only solution is $(y, z) = (4, 14)$. This leads to the triple $(a, b, c) = (3, 5, 15)$.

Case 2. $p = 2$.
Here again, we see that $x \geq 2$ forces $p < 2$, and hence $x = 1$. Using this information in the expression for p, we obtain $(y-2)(z-2) = 5$. Since $y < z$, the only solution is $(y, z) = (3, 7)$ and this gives $(a, b, c) = (2, 4, 8)$.

We obtain two solutions to the problem: $(a, b, c) = (3, 5, 15)$, $(2, 4, 8)$. ■

Example 4.32. Find all integer sided triangles with the property that the area of each triangle is numerically equal to its perimeter.

Solution: Let a, b, c denote the three sides of such a triangle. If $s = (a+b+c)/2$ is its semi perimeter, Heron's formula gives

$$\Delta^2 = s(s-a)(s-b)(s-c).$$

The given condition translates to

$$4s = (s-a)(s-b)(s-c).$$

Taking $s - a = x, s - b = y, s - c = z$, we can write an equation for x, y, z:

$$\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx} = \frac{1}{4}. \tag{4.7}$$

We can rule out isosceles triangles. If, for example, $a = b$, we get $x = y$ and the equation reduces to

$$\frac{1}{x^2} + \frac{2}{xz} = \frac{1}{4}.$$

This can be written as a quadratic equation in x: $zx^2 - 8x - 4z = 0$. The discriminant of this equation is $D = 16(4 + z^2)$. The equation has an integer

root only if D is the square of an integer. This can happen only if $z = 0$ which is impossible.

Thus, we may assume that $1 \le x < y < z$. If $x \ge 3$, we have $y \ge 4$ and $z \ge 5$. Going back to the equation (4.7) with this information, we see that

$$\frac{1}{4} = \frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx} \le \frac{1}{12} + \frac{1}{20} + \frac{1}{15} = \frac{1}{5}.$$

This absurd implication shows that $x \le 2$. Thus, $x = 1$ or $x = 2$.

Case 1. $x = 1$.
Here the equation (4.7) takes the form

$$\frac{1}{y} + \frac{1}{yz} + \frac{1}{z} = \frac{1}{4}.$$

This can be written as $(y-4)(z-4) = 20$. In this case, we get $(y, z) = (5, 24)$, $(6, 14)$, $(8, 9)$. We obtain the triangles (with $c < b < a$) $(6, 25, 29)$, $(7, 15, 20)$, and $(9, 10, 17)$.

Case 2. $x = 2$
By a similar analysis, the only triangles when $x = 2$ are $(5, 12, 13)$ and $(6, 8, 10)$.

■

Example 4.33. Show that the only solutions of the equation

$$u^3 - v^3 = uv + 1$$

in integers are $(u, v) = (1, 0)$ and $(0, -1)$.

Solution: We show that there is no solution (u, v) such that $uv \neq 0$. Suppose $uv > 0$ then $u^3 - v^3 > 0$ and hence $u - v \ge 1$. We get

$$\begin{aligned} uv + 1 = u^3 - v^3 &= (u-v)(u^2 + uv + v^2) \\ &= (u-v)\{(u-v)^2 + 3uv\} \\ &> 3uv \end{aligned}$$

This shows that $2uv < 1$ which is impossible because $uv \ge 1$.

Suppose $uv < 0$. If $uv = -1$, we have $u^3 - v^3 = uv + 1 = 0$ so that $u = v$. But then $u^2 = uv = -1$, which is impossible. Hence, we may assume that $uv < -1$. We have

$$\begin{aligned} |uv + 1| &= |u - v||u^2 + uv + v^2| \\ &= |u - v||(u+v)^2 - uv|. \end{aligned}$$

If $u + v = 0$, we see that $u = -v$ and hence we get $2u^3 = -u^2 + 1$ which has no integer solution. Hence, we may assume that $u + v \neq 0$ and thus $(u+v)^2 \ge 1$. Using the condition $uv < 0$, we get

$$1 - uv \le |uv + 1| = -(uv + 1)$$

because $uv < -1$. This leads to the absurd conclusion that $1 \leq -1$.

We conclude that $uv = 0$. If $u = 0$, we get $v = -1$. If $v = 0$, we get $u = 1$. Thus, the only solutions are $(u, v) = (1, 0), (0, -1)$.

■

Example 4.34. (Romania, 2003) Two unit squares with parallel sides overlap in a rectangular region of area $1/8$ square units. Find the minimum and maximum possible distances between the centres of these two squares.

Solution: Let $ABCD$ be the rectangle obtained as the intersection of two unit squares. Let X and Y denote the centres of the two squares. Let $AB = a$, $BC = b$ so that $ab = 1/8$ sq. units, where $a, b \in [0, 1]$. Draw a line from X parallel to AB and a line from Y parallel to BC. Let them intersect in Z. (See Fig. 4.1)

Then $XZ = 1 - a$ and $YZ = 1 - b$ and

$$\begin{aligned} XY^2 = XZ^2 + YZ^2 &= (1-a)^2 + (1-b)^2 \\ &= a^2 + b^2 - 2(a+b) + 2 \\ &= a^2 + 2ab + b^2 - 2(a+b) - \frac{1}{4} + 2 \\ &= (a+b)^2 - 2(a+b) + \frac{7}{4} \\ &= (a+b-1)^2 + \frac{3}{4} \\ &\geq \frac{3}{4}. \end{aligned}$$

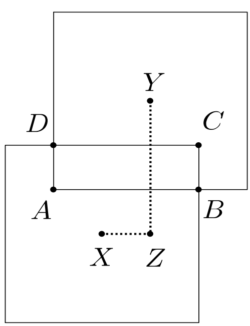

Fig. 4.1

Thus, $XY \geq \sqrt{3}/2$ and hence the minimal distance is $\sqrt{3}/2$. This is achieved

when $a+b=1$ and $ab=1/8$. This corresponds to

$$a=\frac{2+\sqrt{2}}{4}, b=\frac{2-\sqrt{2}}{4} \text{ or } a=\frac{2-\sqrt{2}}{4}, b=\frac{2+\sqrt{2}}{4}.$$

Observe

$$0 \leq (1-a)(1-b) = 1-a-b+ab = \frac{9}{8}-(a+b).$$

This shows that $a+b \leq 9/8$. On the other hand

$$a+b \geq 2\sqrt{ab} = \frac{1}{\sqrt{2}}.$$

Thus,

$$\frac{1}{\sqrt{2}}-1 \leq a+b-1 \leq \frac{1}{8}.$$

However,

$$\left(\frac{1}{8}\right)^2 \leq \left(\frac{1}{\sqrt{2}}-1\right)^2,$$

as may be easily verified. This implies that

$$\begin{aligned} XY^2 &= (a+b-1)^2+\frac{3}{4} \\ &\leq \left(\frac{1}{\sqrt{2}}-1\right)^2+\frac{3}{4} \\ &= \frac{9}{4}-\sqrt{2} \\ &= \left(2-\frac{1}{\sqrt{2}}\right)^2. \end{aligned}$$

Thus, it follows that $XY \leq 2-\left(1/\sqrt{2}\right)$. Equality holds only if $a=b=1/2\sqrt{2}$. We hence have

$$\min XY = \sqrt{3}/2, \text{ and } \max XY = 2-\frac{1}{\sqrt{2}}.$$

■

Chapter 5

Problems on inequalities

1. (Romania,1996) Let $a_1, a_2, \ldots, a_n, a_{n+1}$ be $n+1$ positive real numbers such that $a_1 + a_2 + \cdots + a_n = a_{n+1}$. Prove that

$$\sum_{j=1}^{n} \sqrt{a_j\left(a_{n+1} - a_j\right)} \leq \sqrt{\sum_{j=1}^{n} a_{n+1}\left(a_{n+1} - a_j\right)}.$$

2. (Czech and Slovak, 1999) If a, b, c are positive real numbers, prove that

$$\frac{a}{b+2c} + \frac{b}{c+2a} + \frac{c}{a+2b} \geq 1.$$

3. (Balkan Olympiads, 2001) Let a, b, c be positive real numbers such that $abc \leq a+b+c$. Prove that

$$a^2 + b^2 + c^2 \geq \sqrt{3}(abc).$$

4. (Balkan Olympiads, 2002) For any positive real numbers a, b, c, prove that

$$\frac{2}{b(a+b)} + \frac{2}{c(b+c)} + \frac{2}{a(c+a)} \geq \frac{27}{(a+b+c)^2}.$$

5. (Ireland, 2003) Let a, b, c be three sides of a triangle such that $a+b+c=2$. Prove that

$$1 \leq ab + bc + ca - abc \leq 1 + \frac{1}{27}.$$

6. (Ukraine, 2004) If a, b, c be positive real numbers such that $a+b+c=1$, prove that

$$\sqrt{ab+c} + \sqrt{bc+a} + \sqrt{ca+b} \geq 1 + \sqrt{ab} + \sqrt{bc} + \sqrt{ca}.$$

7. (Romania, 2003) Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that

$$\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+cd}{1+c} + \frac{1+da}{1+d} \geq 4.$$

8. If a, b, c, d are positive real numbers, prove that

$$\sqrt{\frac{a^2+b^2+c^2+d^2}{4}} \geq \sqrt[3]{\frac{abc+abd+acd+bcd}{4}}.$$

9. (Hungary, 1990) Let a, b, c be the sides of a triangle such that $a+b+c=2$. Prove that

$$a^2+b^2+c^2+2abc<2.$$

10. If a, b, c are positive real numbers such that $a^2+b^2+c^2=1$, prove that

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+a+b+c \geq 4\sqrt{3}.$$

11. (Austria, 2001) Find all triplets (a, b, c) of positive real numbers which satisfy the system of equations:

$$\begin{aligned} a+b+c &= 6, \\ \frac{1}{a}+\frac{1}{b}+\frac{1}{c} &= 2-\frac{4}{abc}. \end{aligned}$$

12. (Bosnia and Herzegovina, 2002) Let a, b, c be real numbers such that $a^2+b^2+c^2=1$. Prove that

$$\frac{a^2}{1+2bc}+\frac{b^2}{1+2ca}+\frac{c^2}{1+2ab} \geq \frac{3}{5}.$$

13. (Ukraine, 2001) Let a, b, c and α, β, γ be positive real numbers such that $\alpha+\beta+\gamma=1$. Prove that

$$\alpha a+\beta b+\gamma c+2\sqrt{(\alpha\beta+\beta\gamma+\gamma\alpha)(ab+bc+ca)} \leq a+b+c.$$

14. (Estonia, 1996-97) Prove that for all real numbers a, b, the inequality

$$a^2+b^2+1>a\sqrt{b^2+1}+b\sqrt{a^2+1}$$

holds.

15. (Poland, 1994-95) For a fixed positive integer n, compute the minimum value of the sum

$$x_1+\frac{x_2^2}{2}+\frac{x_3^3}{3}+\cdots+\frac{x_n^n}{n},$$

where $x_1, x_2, x_3, \ldots, x_n$ are positive real numbers such that

$$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\cdots+\frac{1}{x_n}=n.$$

16. Let a, b, c, d be positive real numbers such that $a+b+c+d \leq 1$. Prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a} \leq \frac{1}{64abcd}.$$

17. Let a, b, c be positive real numbers, all less than 1, such that $a+b+c=2$. Prove that

$$abc \geq 8(1-a)(1-b)(1-c).$$

18. (USAMO, 2003) Let a, b, c be three positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \leq 8.$$

19. Three positive real numbers a, b, c are such that $(1+a)(1+b)(1+c) = 8$. Prove that $abc \leq 1$.

20. (IMO, 1983) If a, b, c are the sides of a triangle, prove that

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

21. Let $a_1, a_2, \ldots, a_n$ be $n(\geq 2)$ real numbers whose sum is 1. Prove that

$$\sum_{j=1}^{n} \frac{a_j}{2-a_j} \geq \frac{n}{2n-1}.$$

22. Let $a_1, a_2, \ldots, a_n$ be n positive real numbers whose sum is 1. Prove that

$$\sum_{j=1}^{n} \frac{a_j^2}{a_j + a_{j+1}} \geq \frac{1}{2}.$$

(Here $a_{n+1} = a_1$.)

23. Let a, b, c, d be four positive real numbers. Prove that

$$\frac{1}{a} + \frac{4}{b} + \frac{9}{c} + \frac{16}{d} \geq \frac{100}{a+b+c+d}.$$

24. Let $a_1, a_2, \ldots, a_n$ be $n(> 2)$ positive real numbers such that $a_1 + a_2 + \cdots + a_n = 1$ and $a_j < 1/2$ for each j, $1 \leq j \leq n$. Prove that

$$\sum_{j=1}^{n} \frac{a_j^2}{1-2a_j} \geq \frac{1}{n-2}.$$

25. (Romania, 1999) Let $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$ be $2n$ positive real numbers such that $x_1 + x_2 + \cdots + x_n \geq x_1y_1 + x_2y_2 + \cdots + x_ny_n$, where $n \geq 2$ is an integer. Prove that

$$x_1 + x_2 + \cdots + x_n \leq \frac{x_1}{y_1} + \frac{x_2}{y_2} + \cdots + \frac{x_n}{y_n}.$$

26. (Proposed for IMO-2001) If $x_1, x_2, \ldots, x_n$ are n positive real numbers, prove that

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+x_2^2+\cdots+x_n^2} < \sqrt{n}.$$

27. (CRUX, 2001) If a, b, c are positive real numbers, prove that

$$3\left(a^2b+b^2c+c^2a\right)\left(ab^2+bc^2+ca^2\right) \geq abc(a+b+c)^3.$$

28. (Czech and Slovak, 2003) Let $P(x) = ax^2+bx+c$ be a quadratic polynomial with non-negative coefficients and let α be a positive real number. Prove that

$$P(\alpha)P(1/\alpha) \geq P(1)^2.$$

29. If a, b, c, d, e are positive reals, prove the inequality

$$\sum \frac{a}{b+c} \geq \frac{5}{2},$$

where the sum is taken cyclically over a, b, c, d, e.

30. If a, b, c are the sides of an acute-angled triangle, prove that

$$\sum_{\text{cyclic}} \sqrt{a^2+b^2-c^2}\sqrt{a^2-b^2+c^2} \leq ab+bc+ca.$$

31. (Iran, 2006) Let a, b, c be non-negative real numbers such that

$$\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} = 2.$$

Prove that $ab+bc+ca \leq 3/2$.

32. (Thailand, 2006) Suppose a, b, c are positive real numbers. Prove that

$$3\left(a+b+c\right) \geq 8\left(abc\right)^{1/3} + \left(\frac{a^3+b^3+c^3}{3}\right)^{1/3}.$$

When does equality hold?

33. Let c_1,c_2,c_3,...,c_n be n real numbers such that either $0 \leq c_j \leq 1$ for all j or $c_j \geq 1$ for all j, $1 \leq j \leq n$. Prove that the inequality

$$\prod_{j=1}^{n} (1-p+pc_j) \leq 1-p+p\prod_{j=1}^{n} c_j$$

holds, for any real p with $0 \leq p \leq 1$.

34. (Taiwan, 2002) Let x_1, x_2, x_3, x_4 be real numbers in the interval $(0, 1/2]$. Prove that

$$\frac{x_1x_2x_3x_4}{(1-x_1)(1-x_2)(1-x_3)(1-x_4)} \leq \frac{x_1^4+x_2^4+x_3^4+x_4^4}{(1-x_1)^4+(1-x_2)^4+(1-x_3)^4+(1-x_4)^4}.$$

35. Let $x_1, x_2, x_3, \ldots, x_n$ be n real numbers such that $0 < x_j \leq 1/2$. Prove that

$$\left(\prod_{j=1}^n x_j \Big/ \left(\sum_{j=1}^n x_j\right)^n\right) \leq \left(\prod_{j=1}^n (1-x_j) \Big/ \left(\sum_{j=1}^n (1-x_j)\right)^n\right).$$

36. (CRUX) Consider a sequence $\langle a_n \rangle$ of real numbers satisfying $a_{j+k} \leq a_j + a_k$. Prove that

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \geq a_n,$$

for all n.

37. For positive real numbers x, y, z, prove the inequality

$$\sum \frac{x}{x+\sqrt{(x+y)(x+z)}} \leq 1,$$

where the sum is taken cyclically over x, y, z.

38. (INMO, 2002) Let x, y be non-negative real numbers such that $x + y = 2$. Prove the inequality

$$x^3y^3\left(x^3+y^3\right) \leq 2.$$

39. (INMO, 1998) A convex quadrilateral $ABCD$ is inscribed in a unit circle. Suppose its sides satisfy the inequality $AB \cdot BC \cdot CD \cdot DA \geq 4$. Prove that the quadrilateral is a square.

40. Let $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ be two sets of reals such that $0 < h \leq a_j \leq H$ and $0 < m \leq b_j \leq M$ for some reals h, H, m, M. Prove that

$$1 \leq \frac{(\sum {a_j}^2)(\sum {b_j}^2)}{(\sum a_jb_j)^2} \leq \frac{1}{4}\left(\sqrt{\frac{HM}{hm}} + \sqrt{\frac{hm}{HM}}\right)^2.$$

41. Let $f : [0, a] \to \mathbb{R}$ be a convex function. Consider n points $x_1, x_2, x_3, \ldots, x_n$ in $[0, a]$ such that $\sum_{j=1}^n x_j$ is also in $[0, a]$. Prove that

$$\sum_{j=1}^n f(x_j) \leq f\left(\sum_{j=1}^n x_j\right) + (n-1)f(0).$$

42. For any natural number n, prove

$$\binom{2n}{n}\sqrt{3n} < 4^n.$$

43. Let a, b, c be positive real numbers and let x be a non-negative real number. Prove that

$$a^{x+2} + b^{x+2} + c^{x+2} \geq a^x bc + ab^x c + abc^x.$$

44. Let $(a_1, a_2, \ldots, a_n)$, $(b_1, b_2, \ldots, b_n)$, and $(c_1, c_2, \ldots, c_n)$ be three sequences of positive real numbers. Prove that

$$\sum_{j=1}^{n} a_j b_j c_j \leq \left(\sum_{j=1}^{n} a_j^3\right)^{1/3} \left(\sum_{j=1}^{n} b_j^3\right)^{1/3} \left(\sum_{j=1}^{n} c_j^3\right)^{1/3}.$$

45. ([9]) Prove for any three real numbers a, b, c, the inequality

$$3\left(a^2 - a - 1\right)\left(b^2 - b - 1\right)\left(c^2 - c + 1\right) \geq \left(abc\right)^2 - abc + 1.$$

46. In a triangle ABC, show that

$$\frac{1}{\sin A} + \frac{1}{\sin B} \geq \frac{8}{3 + 2\cos C}.$$

Find the conditions for equality.

47. Consider a real polynomial of the form

$$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + 1,$$

where $a_j \geq 0$ for $1 \leq j \leq (n-1)$. Suppose $P(x) = 0$ has n real roots. Prove that $P(2) \geq 3^n$.

48. Let $a_1{<}a_2{<}a_3{<}\ldots{<}a_n$ be n positive integers. Prove that

$$\left(a_1{+}a_2{+}a_3{+}\cdots{+}a_n\right)^2 \leq a_1^3 + a_2^3 + a_3^3 + \cdots + a_n^3.$$

49. (Poland, 1996) Consider a sequence $a_1, a_2, a_3, \ldots, a_n$ of positive real numbers which add up to 1, where $n \geq 2$ is an integer. Prove that for any positive real numbers $x_1, x_2, x_3, \ldots, x_n$ with $\sum_j^n x_j = 1$, the inequality

$$2\sum_{j<k} x_j x_k \leq \frac{n-2}{n-1} + \sum_{j=1}^{n} \frac{a_j x_j^2}{1 - a_j},$$

holds.

50. (Iran, 1998) Let x_1, x_2, x_3, x_4 be four positive real numbers such that $x_1x_2x_3x_4 = 1$. Prove that

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 \geq \min\left\{x_1 + x_2 + x_3 + x_4, \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right\}.$$

51. (Romania) Let $\{x\}$ denote the fractional part of x; i.e., $\{x\} = x - [x]$. Prove for any positive integer n,

$$\sum_{j=1}^{n} \{\sqrt{j}\} \leq \frac{n^2 - 1}{2}.$$

52. If a, b, c are positive real numbers, prove that

$$\frac{a^2}{(a+b)(a+c)} + \frac{b^2}{(b+c)(b+a)} + \frac{c^2}{(c+a)(c+b)} \geq \frac{3}{4}.$$

53. Suppose a, b, c are the sides of a triangle. Prove that

$$a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \leq 3abc.$$

54. Let x, y, z be positive real numbers such that $xyz \geq xy + yz + zx$. Prove that

$$xyz \geq 3(x + y + z).$$

55. Let $b_1, b_2, b_3, \ldots, b_n$ be n non-negative real numbers and let b denote the sum of these numbers. Prove that

$$\sum_{j=1}^{n-1} b_j b_{j+1} \leq \frac{b^2}{4}.$$

56. (AMM) Let a, b, c, d be complex numbers such that $ac \neq 0$. Prove that

$$\frac{\max\{|ac|, |ad + bc|, |bd|\}}{\max\{|a|, |b|\}\{|c|, |d|\}} \geq \frac{-1 + \sqrt{5}}{2}.$$

57. Let x, y, z be three real numbers in the interval $[0, 1]$ such that $xyz = (1-x)(1-y)(1-z)$. Find the least possible value of $x(1-z) + y(1-x) + z(1-y)$.

58. Let $x_1, x_2, x_3, \ldots, x_n$ be non-negative real numbers such that

$$\sum_{j=1}^{n} \frac{1}{1 + x_j} \leq 1.$$

Prove that $x_1 x_2 x_3 \cdots x_n \geq (n-1)^n$.

59. (Bulgaria, 1998) For positive real numbers a, b, c, prove the inequality

$$3\left(a + \sqrt{ab} + \sqrt[3]{abc}\right) \le 4(a+b+c).$$

60. ([1]) Show that for any two natural numbers m, n, the inequality

$$\frac{1}{m+n+1} - \frac{1}{(m+1)(n+1)} \le \frac{4}{45}$$

holds.

61. If a, b are two positive real numbers, prove that

$$a^b + b^a > 1.$$

62. Let a, b be positive real numbers such that $a+b=1$ and let p be a positive real. Prove that

$$\left(a + \frac{1}{a}\right)^p + \left(b + \frac{1}{b}\right)^p \ge \frac{5^p}{2^{p-1}}.$$

63. (IMO, 2000) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \le 1.$$

64. (Proposed for IMO-2001) Let x, y, z be real numbers in the interval $[-1, 2]$ such that $x+y+z=0$. Prove that

$$\sqrt{\frac{(2-x)(2-y)}{(2+x)(2+y)}} + \sqrt{\frac{(2-y)(2-z)}{(2+y)(2+z)}} + \sqrt{\frac{(2-z)(2-x)}{(2+z)(2+x)}} \ge 3.$$

65. (IMO, 1978) Let $\langle a_n \rangle$ be a sequence of distinct positive integers. Prove that

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \ge \sum_{k=1}^{n} \frac{1}{k},$$

for every positive integer n.

66. (IMO, 1984) Let x, y, z be non-negative real numbers such that $x+y+z=1$. Prove that

$$0 \le xy + yz + zx - 2xyz \le \frac{7}{27}.$$

67. Let $x_1, x_2, \ldots, x_n$ be n positive real numbers. Prove that

$$\sum_{j=1}^{n} \frac{x_j^3}{x_j^2 + x_j x_{j+1} + x_{j+1}^2} \ge \frac{1}{3} \sum_{j=1}^{n} x_j \ ,$$

where $x_{n+1} = x_1$.

68. Suppose x, y, z are non-negative real numbers. Prove the inequality

$$x(x-z)^2 + y(y-z)^2 \geq (x-z)(y-z)(x+y-z).$$

Find conditions for equality.

69. (Indian Team Selection, 2002) Prove that for any positive reals a, b, c, the inequality,

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{c+a}{c+b} + \frac{a+b}{a+c} + \frac{b+c}{b+a}$$

holds.

70. If a, b are real numbers, prove that

$$a^2 + ab + b^2 \geq 3(a+b-1).$$

71. ([7]) Define a sequence $\langle x_n \rangle$ by

$$x_1 = 2, \quad x_{n+1} = \frac{x_n^4 + 9}{10x_n}.$$

Prove that $\frac{4}{5} < x_n \leq \frac{5}{4}$ for all $n > 1$.

72. ([7]) Let a, b, c be positive real numbers such that $a^2 - ab + b^2 = c^2$. Prove that $(a-c)(b-c) \leq 0$.

73. ([7]) Let a, b, c be positive real numbers. Prove that

$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{a^2 + ac + c^2}.$$

74. ([7]) For all real numbers a, show that

$$(a^3 - a + 2)^2 \geq 4a^2(a^2+1)(a-2)$$

holds.

75. (Thailand, 2004) Let a, b, c be distinct real numbers. Prove that

$$\left(\frac{2a-b}{a-b}\right)^2 + \left(\frac{2b-c}{b-c}\right)^2 + \left(\frac{2c-a}{c-a}\right)^2 \geq 5.$$

76. Let α, β, x_1,x_2,x_3,...,x_n be positive reals such that $\alpha + \beta = 1$, and $x_1 + x_2 + x_3 + \cdots + x_n = 1$. Prove that

$$\sum_{j=1}^{n} \frac{x_j^{2m+1}}{\alpha x_j + \beta x_{j+1}} \geq \frac{1}{n^{2m-1}},$$

for every positive integer m, where $x_{n+1} = x_1$.

77. Given positive reals a, b, c, d, prove that

$$\sqrt{(a+c)^2 + (b+d)^2} \leq \sqrt{a^2+b^2} + \sqrt{c^2+d^2}$$
$$\leq \sqrt{(a+c)^2 + (b+d)^2} + \frac{2|ad-bc|}{\sqrt{(a+c)^2+(b+d)^2}}.$$

78. (APMO, 1996) In a triangle ABC, prove that

$$\sqrt{s-a} + \sqrt{s-b} + \sqrt{s-c} \leq \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{2}}.$$

79. Let ABC be an acute-angled triangle with altitudes AD, BE, CF and ortho-centre H. Prove that

$$\frac{HD}{HA} + \frac{HE}{HB} + \frac{HF}{HC} \geq \frac{3}{2}.$$

80. (IMO, 1981) For any point P inside a triangle ABC, let r_1, r_2, r_3 denote the distances of P from the lines BC, CA, AB respectively. Find all points P for which $a/r_1 + b/r_2 + c/r_3$ is minimal.

81. (IMO, 1996) Let $ABCDEF$ be a convex hexagon in which AB, BC, CD are respectively parallel to DE, EF, FA. Let R_A, R_B, R_C be the circum-radii of the triangles FAB, BCD, DEF respectively, and let p denote the perimeter of the hexagon. Prove that

$$R_A + R_B + R_C \geq \frac{p}{2}.$$

82. (AMM) Let h_a, m_b, w_c denote respectively the altitude from A to BC, the median from B to CA and the internal angle bisector of angle C. Prove that

$$h_a + m_b + w_c \leq \frac{\sqrt{3}}{2}(a+b+c).$$

83. (Bulgaria, 1997) Let ABC be a triangle with centroid G. Prove that

$$\sin\angle ABG + \sin\angle ACG \leq \frac{2}{\sqrt{3}}.$$

84. (CRUX) Let ABC be a triangle with in-radius r. Let Γ_1, Γ_2, Γ_3 be three circles inscribed inside ABC such that each touches other circles and also two of the sides. (Such a configuration is called *Malafatti* circles.) Let O_1, O_2, O_3

be respectively the centres of the circles Γ_1, Γ_2, Γ_3. If r' denotes the in-radius of $O_1O_2O_3$, prove that

$$\frac{r}{r'} \geq 1 + \sqrt{3}.$$

Find conditions for equality.

85. Let Ω be a Brocard point of a triangle ABC. Let $A\Omega$, $B\Omega$,$C\Omega$ extended meet the circum-circle of ABC in K, L, M respectively. Prove that

$$\frac{A\Omega}{\Omega K} + \frac{B\Omega}{\Omega L} + \frac{C\Omega}{\Omega M} \geq 3.$$

86. (Romania, 2006) Let P be a point inside a triangle ABC. Let r_A, r_B, r_C denote the in-radii of triangles PBC, PCA, PAB respectively. Prove that

$$\frac{a}{r_A} + \frac{b}{r_B} + \frac{c}{r_C} \geq 6\left(2 + \sqrt{3}\right).$$

87. (Bulgaria, 1995) Suppose $A_1A_2 \cdots A_7$, $B_1B_2 \cdots B_7$, $C_1C_2 \cdots C_7$ are three regular heptagons which are such that $A_1A_2 = B_1B_3 = C_1C_4$. If Δ_1, Δ_2, Δ_3 denote respectively their areas, prove that

$$\frac{1}{2} < \frac{\Delta_2 + \Delta_3}{\Delta_1} < 2 - \sqrt{2}.$$

88. (Germany, 1995) Let ABC be a triangle, and let D, E be points on BC, CA such that the in-centre of ABC lies on DE. Prove that $[ABC] \geq 2r^2$.

89. (Ireland, 1995) Let A, X, D be points on a line with X between A and D. Let B be a point such that $\angle ABX \geq 120°$ and let C be a point between B and X. Prove that $2AD \geq \sqrt{3}\left(AB + BC + CD\right)$.

90. (South Korea, 1995) Let the internal bisectors of the angles A, B, C of a triangle ABC meet the sides BC, CA, AB in D, E, F and the circum-circle in L, M, N respectively. Prove that

$$\frac{AD}{DL} + \frac{BE}{EM} + \frac{CF}{FN} \geq 9.$$

91. (IMO, 1995) Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$, $DE = EF = FA$, and $\angle BCD = \angle EFA = \pi/3$. Suppose G and H are two interior points of the hexagon such that $\angle AGB = \angle DHE = 2\pi/3$. Prove that

$$AG + GB + GH + DH + HE \geq CF.$$

92. (Balkan Olympiads, 1996) Let O and G be respectively the circum-centre and centroid of a triangle ABC. If R and r are its circum-radius and in-radius, prove that $OG \leq \sqrt{R(R - 2r)}$.

93. (St. Petersburg Olympiad, 1996) Let M be the point of intersection of two diagonals of a cyclic quadrilateral. Let N be the point of intersection of two lines joining the midpoints of opposite pair of sides. If O is the centre of the circumscribing circle, prove that $OM \geq ON$.

94. (APMO, 1997) Let ABC be a triangle with internal angle bisectors AD, BE, CF. Suppose AD, BE, CF when extended meet the circum-circle again in K, L, M respectively. If $l_a = AD/AK$, $l_b = BE/BL$ and $l_c = CF/CM$, prove that

$$\frac{l_a}{\sin^2 A} + \frac{l_b}{\sin^2 B} + \frac{l_c}{\sin^2 C} \geq 3.$$

95. (Armenia, 1999) Let O be the circum-circle of a triangle ABC. Suppose AO, BO, CO when extended meet the circum-circles of triangles BOC, COA, AOB in K, L, M respectively. Prove that

$$\frac{AK}{OK} + \frac{BL}{OL} + \frac{CM}{OM} \geq \frac{9}{2}.$$

96. (Romania, 2003) Show that in any triangle ABC

$$\frac{1}{m_a m_b} + \frac{1}{m_b m_c} + \frac{1}{m_c m_a} \leq \frac{\sqrt{3}}{[ABC]}.$$

97. (Romania, 2003) In any triangle ABC, prove that

$$w_a + w_b + w_c \leq \sqrt{3}s.$$

98. (Ireland, 1998) Let ABC be a triangle with points D, E, F respectively on the sides BC, CA, AB. Let the lines AD, BE, CF, when produced meet the circum-circle respectively in K, L, M. Prove that

$$\frac{AD}{DK} + \frac{BE}{EL} + \frac{CF}{FM} \geq 9.$$

99. Show that in a triangle ABC,

$$\max\left\{am_a, bm_b, cm_c\right\} \leq sR.$$

100. Let $ABCD$ be a convex quadrilateral of area 1 unit. Prove that

$$AB + BC + CD + DA + AC + BD \geq 4 + \sqrt{8}.$$

101. Let $ABCD$ be a square inscribed in circle. If M is a point on the arc AB (arc not containing C and D), prove that

$$MC \cdot MD \geq (3 + 2\sqrt{2})MA \cdot MB.$$

102. (Estonia, 1996) Let a, b, c be the sides of a triangle ABC with in-radius r. Prove that

$$a \sin A + b \sin B + c \sin C \geq 9r.$$

103. Suppose ABC is an acute-angled triangle with area Δ and in-radius r. Prove that

$$\left(\sqrt{\cot A} + \sqrt{\cot B} + \sqrt{\cot C}\right)^2 \leq \frac{\Delta}{r^2}.$$

104. (US Team Selection, 2000) Let ABC be a triangle having the circum-radius R. Let P be an interior point of ABC. Prove that

$$\frac{AP}{BC^2} + \frac{BP}{CA^2} + \frac{CP}{AB^2} \geq \frac{1}{R}.$$

105. With every natural number n, associate a real number a_n by

$$a_n = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k},$$

where $\{p_1, p_2, \ldots, p_k\}$ is the set of all prime divisors of n. Show that for any natural number $N \geq 2$,

$$\sum_{n=2}^{N} a_2 a_3 \cdots a_n < 1.$$

106. (IMO, 1999) Let n be a fixed integer, with $n \geq 2$.

(a) Determine the least constant C such that the inequality

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_{1 \leq i \leq n} x_i \right)^4$$

holds for all real numbers $x_1, \ldots, x_n \geq 0$.

(b) For this constant C, determine when equality holds.

107. Let a, b, c, d be real numbers such that

$$\left(a^2 + b^2 - 1\right)\left(c^2 + d^2 - 1\right) > \left(ac + bd - 1\right)^2.$$

Prove that $a^2 + b^2 - 1 > 0$ and $c^2 + d^2 - 1 > 0$.

108. ([9]) Let $x_1, x_2, x_3, \ldots, x_{100}$ be 100 positive integers such that

$$\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_{100}}} = 20.$$

Prove that at least two of x_j's are equal.

109. Let $f(x)$ be a polynomial with integer coefficients and of degree $n > 1$. Suppose $f(x) = 0$ has n real roots in the interval $(0, 1)$, not all equal. If a is the leading coefficient of $f(x)$, prove that

$$|a| \geq 2^n + 1.$$

110. Show that the equation

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{w} + \frac{w}{x} = m,$$

has no solutions in positive reals for $m = 2, 3$.

111. Solve the system of equations

$$x = \frac{4z^2}{1+4z^2}, \quad y = \frac{4x^2}{1+4x^2}, \quad z = \frac{4x^2}{1+4x^2},$$

for real numbers x, y, z.

112. Suppose a, b are nonzero real numbers and that all the roots of the real polynomial

$$ax^n - ax^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 - n^2bx + b = 0$$

are real and positive. Prove that all the roots are in fact equal.

113. Find all triples (a, b, c) of positive integers such that the product of any two leaves the remainder 1 when divided by the third number.

114. (CRUX, 2000) Show that a triangle is equilateral if and only if

$$a\cos(\beta - \gamma) + b\cos(\gamma - \alpha) + c\cos(\alpha - \beta) = \frac{a^4 + b^4 + c^4}{abc},$$

where a, b, c are the sides and α, β, γ are the angles opposite to the sides a, b, c respectively.

115. ([7]) Find all positive solutions of the system

$$x_1 + \frac{1}{x_2} = 4, x_2 + \frac{1}{x_3} = 1, \ldots, x_{1999} + \frac{1}{x_{2000}} = 4, x_{2000} + \frac{1}{x_1} = 1.$$

116. Find all positive solutions of the system

$$\begin{aligned} x + y + z &= 1, \\ x^3 + y^3 + z^3 + xyz &= x^4 + y^4 + z^4 + 1. \end{aligned}$$

117. (Bulgaria, 2001) Let a, b be positive integers such that each equation

$$(a+b-x)^2 = a-b, \quad (ab+1-x)^2 = ab-1$$

has two distinct real roots. Suppose the bigger of these roots are the same. Show that the smaller roots are also the same.

118. (Short-list, IMO-1989) Suppose the polynomial

$$P(x) = x^n + nx^{n-1} + a_2x^{n-2} + \cdots + a_n$$

has real roots $\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n$. If

$$\alpha_1^{16} + \alpha_2^{16} + \cdots + \alpha_n^{16} = n.$$

Find $\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n$.

119. (IMO, 1972) Find all the solutions of the following system of inequalities:

$$\begin{array}{rrcl}
(i) & \left(x_1^2 - x_3x_5\right)\left(x_2^2 - x_3x_5\right) & \leq & 0, \\
(ii) & \left(x_2^2 - x_4x_1\right)\left(x_3^2 - x_4x_1\right) & \leq & 0, \\
(iii) & \left(x_3^2 - x_5x_2\right)\left(x_4^2 - x_5x_2\right) & \leq & 0, \\
(iv) & \left(x_4^2 - x_1x_3\right)\left(x_5^2 - x_1x_3\right) & \leq & 0, \\
(v) & \left(x_5^2 - x_2x_4\right)\left(x_1^2 - x_2x_4\right) & \leq & 0.
\end{array}$$

120. (Short-list, IMO-1993) Solve the following system of equations, when a is a real number such that $|a| > 1$:

$$\begin{array}{rcl}
x_1^2 & = & ax_2 + 1, \\
x_2^2 & = & ax_3 + 1, \\
\vdots & \vdots & \vdots \\
x_{999}^2 & = & ax_{1000} + 1, \\
x_{1000}^2 & = & ax_1 + 1.
\end{array}$$

121. (Indian Team Selection,1999) Let $a_1,a_2,a_3,\ldots,a_n$ be n positive integers such that $\sum_{j=1}^n a_j = \prod_{j=1}^n a_j$. Let V_n denote this common value. Show that $V_n \geq n + s$, where s is the least positive integer such that $2^s - s \geq n$.

122. Let $z_1,z_2,z_3,\ldots,z_n$ be n complex numbers such that $\sum_{j=1}^n |z_j| = 1$. Prove that there exists a subset S of the set $\left\{z_1,z_2,z_3,\ldots,z_n\right\}$ such that

$$\left|\sum_{z\in S} z\right| \geq \frac{1}{4}.$$

123. ([1]) Let $(a_1,a_2,a_3,\ldots,a_n)$ and $(b_1,b_2,b_3,\ldots,b_n)$ be two sequences of real numbers which are not proportional. Let $(x_1,x_2,x_3,\ldots,x_n)$ be another sequence of real numbers such that

$$\sum_{j=1}^{n} a_j x_j = 0, \quad \sum_{j=1}^{n} b_j x_j = 1.$$

Prove that

$$\sum_{j=1}^{n} x_j^2 \geq \frac{\sum_{j=1}^{n} a_j^2}{\left(\sum_{j=1}^{n} a_j^2\right)\left(\sum_{j=1}^{n} b_j^2\right) - \left(\sum_{j=1}^{n} a_j b_j\right)^2} \ .$$

When does equality hold?

124. ([1]) Let $(a_1,a_2,a_3,\ldots,a_n)$ and $(b_1,b_2,b_3,\ldots,b_n)$ be two sequences of real numbers such that

$$b_1^2 - b_2^2 - \cdots - b_n^2 > 0 \text{ or } a_1^2 - a_2^2 - \cdots - a_n^2 > 0.$$

Prove that

$$\left(a_1^2 - a_2^2 - \cdots - a_n^2\right)\left(b_1^2 - b_2^2 - \cdots - b_n^2\right) \leq \left(a_1 b_1 - a_2 b_2 - \cdots - a_n b_n\right)^2,$$

and show that equality holds if and only if $a_j = \lambda b_j$, $1 \leq j \leq n$, for some real constant λ.

125. ([1]) Let $x_1,x_2,x_3,\ldots,x_n$ be n positive real numbers. Prove that

$$\sum_{j=1}^{n} \frac{x_j}{2x_j + x_{j+1} + \cdots + x_{j+n-2}} \leq n,$$

where $x_{n+k} = x_k$.

126. ([1]) Let $x_1,x_2,x_3,\ldots,x_n$ be $n \geq 2$ positive real numbers and k be a fixed integer such that $1 \leq k \leq n$. Show that

$$\sum_{\text{cyclic}} \frac{x_1 + 2x_2 + \cdots + 2x_{k-1} + x_k}{x_k + x_{k+1} + \cdots + x_n} \geq \frac{2n(k-1)}{n-k+1}.$$

127. ([1]) Let z and ξ be two complex numbers such that $|z| \leq r$, $|\xi| \leq r$ and $z \neq \xi$. Show that for any natural number n, the inequality

$$\left|\frac{z^n - \xi^n}{z - \xi}\right| \leq \frac{1}{2} n(n-1) r^{n-2} |z - \xi|$$

holds.

128. For any three vectors, $\boldsymbol{x} = \big(x_1,x_2,x_3,\ldots,x_n\big)$, $\boldsymbol{y} = \big(y_1,y_2,y_3,\ldots,y_n\big)$, and $\boldsymbol{z} = \big(z_1,z_2,z_3,\ldots,z_n\big)$ in $\mathbb{R}^n$, prove that

$$||\boldsymbol{x}|| + ||\boldsymbol{y}|| + ||\boldsymbol{z}|| - ||\boldsymbol{x}+\boldsymbol{y}|| - ||\boldsymbol{y}+\boldsymbol{z}|| - ||\boldsymbol{z}+\boldsymbol{x}|| + ||\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}|| \geq 0.$$

129. Let $A_1A_2A_3\cdots A_{n+1}$ be a polygon with centre O, in which $A_1 = A_{n+1}$ is fixed and the remaining A_j's vary on the circle. Show that the area and the perimeter of the polygon are the largest when the polygon is regular.

130. ([1]) A sequence $\langle a_n\rangle$ is said to be convex if $a_n - 2a_{n+1} + a_{n+2} \geq 0$ for all $n \geq 1$. Let $a_1,a_2,a_3,\ldots,a_{2n+1}$ be a convex sequence. Show that

$$\frac{a_1 + a_3 + a_5 + \cdots + a_{2n+1}}{n+1} \geq \frac{a_2 + a_4 + a_6 + \cdots + a_{2n}}{n},$$

and equality holds if and only if $a_1,a_2,a_3,\ldots,a_{2n+1}$ is an arithmetic progression.

131. Suppose $a_1,a_2,a_3,\ldots,a_n$ are n positive real numbers. For each k, define

$$x_k = a_{k+1} + a_{k+2} + \cdots + a_{k+n-1} - (n-2)a_k,$$

where $a_j = a_{j-n}$ for $j > n$. Suppose $x_k \geq 0$ for $1 \leq k \leq n$. Prove that

$$\prod_{k=1}^{n} a_k \geq \prod_{k=1}^{n} x_k.$$

Show that for $n = 3$ the inequality is still true without the non-negativity of x_k's, but for $n > 3$ these conditions are essential.

132. Let a, c be positive reals and b be a complex number such that

$$f(z) = a|z|^2 + 2\,\mathrm{Re}(bz) + c \geq 0,$$

for all complex numbers z, where $\mathrm{Re}(z)$ denotes the real part of z. Prove that

$$|b|^2 \leq ac,$$

and

$$f(z) \leq \big(a+c\big)\big(1+|z|^2\big).$$

Show that $|b|^2 = ac$ only if $f(z) = 0$ for some $z \in \mathbb{C}$.

133. (IMO, 2003) Suppose $x_1 \leq x_2 \leq x_3 \leq \ldots \leq x_n$ be n real numbers. Show that

$$\left(\sum_{j=1}^{n}\sum_{k=1}^{n}\big|x_j - x_k\big|\right)^2 \leq \frac{2\big(n^2-1\big)}{3}\sum_{j=1}^{n}\sum_{k=1}^{n}\big(x_j - x_k\big)^2.$$

Prove also that equality holds if and only if the sequence $\langle x_j\rangle$ is in arithmetic progression.

134. (Short-list, IMO-2002) Suppose $\langle a_n \rangle$ is an infinite sequence of real numbers with the properties:

(i) there is some real constant c such that $0 \le a_n \le c$, for all $n \ge 1$;

(ii) $\left|a_j - a_k\right| \ge \dfrac{1}{j+k}$ for all j, k with $j \ne k$.

Prove that $c \ge 1$.

135. Let ABC be a right-angle triangle with medians m_a, m_b, m_c. Let $A'B'C'$ denote the triangle whose sides are m_a, m_b, m_c. If R and R' denote respectively the circum-radii of ABC and $A'B'C'$, prove that $R' \ge \dfrac{5}{6}R$.

136. Let ABC be an equilateral triangle and D, E, F be arbitrary points on the segments BC, CA, AB respectively. Prove that

$$[DEF]\left\{\frac{1}{[BDF]} + \frac{1}{[CED]} + \frac{1}{[AFE]}\right\} \ge 3.$$

(Here $[XYZ]$ denotes the area of the triangle XYZ.)

137. Let the diagonals of a convex quadrilateral $ABCD$ meet in P. Prove that

$$\sqrt{[APB]} + \sqrt{[CPD]} \le \sqrt{[ABCD]},$$

where as usual square-bracket denotes the area.

138. Let AD be the median from A on to BC of a triangle ABC; let r, r_1, r_2 denote the in-radii of triangles ABC, ABD, ADC respectively. Prove that

$$\frac{1}{r_1} + \frac{1}{r_2} \ge 2\left(\frac{1}{r} + \frac{2}{BC}\right).$$

139. (Japan, 2005) Let a, b, c be positive reals such that $a + b + c = 1$. Prove that

$$a\big(1 + b - c\big)^{1/3} + b\big(1 + c - a\big)^{1/3} + c\big(1 + a - b\big)^{1/3} \le 1.$$

140. Show that in a triangle ABC,

$$\left(a^2m_a^2 + b^2m_b^2 + c^2m_c^2\right)\left(a^2 + b^2 + c^2\right) \ge 16m_a^2m_b^2m_c^2,$$

where m_a, m_b, m_c denote the medians on to the sides BC, CA, AB from A, B, C respectively.

141. Let $x_1, x_2, x_3, \ldots, x_n$ be n positive reals which add up to 1. Find the minimum value of

$$\sum_{j=1}^{n} \frac{x_j}{1 + \sum_{k \neq j} x_k} \quad .$$

142. (IMO, 1974) Find all possible values of

$$\frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d},$$

when a, b, c, d vary over positive reals.

143. Let $\langle F_n \rangle$ be the Fibonacci sequence defined by

$$F_1 = F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n, \text{ for } n \geq 1.$$

Prove that

$$\sum_{j=1}^{n} \frac{F_j}{2^j} < 2,$$

for all $n \geq 1$.

144. ([7]) Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with real coefficients such that $|P(0)| = P(1)$. Suppose all the roots of $P(x) = 0$ are real and lie in the interval $(0, 1)$. Prove that the product of the roots does not exceed $1/2^n$.

145. If x, y are real numbers such that

$$2x + y + \sqrt{8x^2 + 4xy + 32y^2} = 3 + 3\sqrt{2},$$

prove that $x^2 y \leq 1$.

146. (South Africa) If α, β, γ are the angles of a triangle whose circum-radius is R and in-radius r, prove that

$$\cos^2\left(\frac{\alpha - \beta}{2}\right) \geq \frac{2r}{R}.$$

147. (IMO, 1991) Let I be the in-centre of a triangle ABC. Suppose the internal bisectors of angles A, B, C meet the opposite sides at A', B' and C'. Prove that

$$\frac{1}{4} < \frac{AI \cdot BI \cdot CI}{AA' \cdot BB' \cdot CC'} \leq \frac{8}{27}.$$

148. (Proposed for IMO-1991) Determine the maximum value of

$$\sum_{j<k} x_j x_k (x_j + x_k),$$

over all n-tuples $(x_1,x_2,x_3,\ldots,x_n)$ of reals such that $x_j \geq 0$ for $1 \leq j \leq n$.

149. If α, β, γ are the angles of a triangle, prove that

$$3 \sum_{\text{cyclic}} \cos\alpha \geq 2 \sum_{\text{cyclic}} \sin\alpha \sin\beta.$$

150. (Indian Team Selection, 1994) Let $x_1,x_2,x_3,\ldots,x_N$ be positive real numbers. Prove that

$$\sum_{j=1}^{N} \left(x_1 x_2 \cdots x_j\right)^{1/j} < 3\left(\sum_{j=1}^{N} x_j\right).$$

151. (Indian Team Selection, 2000) Let $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n$ be n real numbers with the property $\sum_{j=1}^{n} a_j = 0$. Prove that

$$n a_1 a_n \sum_{j=1}^{n} a_j^2 \leq 0.$$

152. (Indian Team Selection, 1997) Suppose a, b, c are positive real numbers. Prove that

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{1+abc}.$$

153. Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 = 2$. Prove that

$$x + y + z \leq 2 + xyz.$$

Find conditions under which equality holds.

154. (Wolschaum County Problem book) Let $0 \leq x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n$ be such that $\sum_{j=1}^{n} x_j = 1$, where $n \geq 2$ is an integer. If $x_n \leq 2/3$, prove that there exists a k such that $1 \leq k \leq n$ and

$$\frac{1}{3} \leq \sum_{j=1}^{k} x_j \leq \frac{2}{3}.$$

155. (Vietnam, 1996) Let x, y, z be non-negative real numbers such that $xy + yz + zx + xyz = 4$. Prove that

$$x + y + z \geq xy + yz + zx.$$

156. (Canada) Let x, y, z be non-negative real numbers such that $x+y+z = 1$. Prove that

$$x^2y + y^2z + z^2x \leq \frac{4}{27}.$$

157. Let x, y, z be real numbers and let p, q, r be real numbers in the interval $(0, 1/2)$ such that $p + q + r = 1$. Prove that

$$pqr(x + y + z)^2 \geq xyr(1 - 2r) + yzp(1 - 2p) + zxq(1 - 2q).$$

When does equality hold?

158. (Romania) Let $x_1, x_2, x_3, \ldots, x_n$ be n real numbers in the interval $[0, 1]$. Prove that

$$\left(\sum_{j=1}^{n} x_j\right) - \left(\sum_{j=1}^{n} x_j x_{j+1}\right) \leq \left[\frac{n}{2}\right],$$

where $x_{n+1} = x_1$.

159. (IMO, 2005) Suppose x, y, z are positive real numbers such that $xyz \geq 1$. Prove that the inequality

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \geq 0$$

holds.

160. (Indian Team Selection, 1992) Consider two sequences of positive real numbers, $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq b_3 \leq \cdots \leq b_n$, such that

$$\sum_{j=1}^{n} a_j \geq \sum_{j=1}^{n} b_j.$$

Suppose there exists a k, $1 \leq k \leq n$, such that $b_j \leq a_j$ for $1 \leq j \leq k$ and $b_j \geq a_j$ for $j > k$. Prove that

$$\prod_{j=1}^{n} a_j \geq \prod_{j=1}^{n} b_j.$$

161. (Bulgaria, 1997) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \leq \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c}.$$

162. (PUTNAM) Let $n \geq 4$ and let $a_1, a_2, a_3, \ldots, a_n$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n \geq n, \quad a_1^2 + a_2^2 + \cdots + a_n^2 \geq n^2.$$

Prove that

$$\max\left\{a_1, a_2, a_3, \ldots, a_n\right\} \geq 2.$$

163. Let $x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_{n+1}$ be $n+1$ positive integers. Prove that

$$\sum_{j=1}^{n+1} \frac{\sqrt{x_{j+1} - x_j}}{x_{j+1}} < \sum_{j=1}^{n^2} \frac{1}{j}.$$

164. Let a, b, c be three positive real numbers which satisfy $abc = 1$ and $a^3 > 36$. Prove that

$$\frac{2}{3}a^2 < a^2 + b^2 + c^2 - ab - bc - ca.$$

165. ([1]) Let $z_1, z_2, z_3, \ldots, z_n$ be n complex numbers and consider n positive real numbers $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ which have the property that $\sum 1/\lambda_j = 1$. Prove that

$$\left|\sum_{j=1}^{n} z_j\right|^2 \leq \sum_{j=1}^{n} \lambda_j \left|z_j\right|^2.$$

166. ([1]) Let a, b, c be three distinct real numbers. Prove that

$$3\min\left\{a, b, c\right\} < \sum a - \left(\sum a^2 - \sum ab\right)^{1/2} < \sum a + \left(\sum a^2 - \sum ab\right)^{1/2} < 3\max\left\{a, b, c\right\},$$

where the sum is cyclically over a, b, c.

167. Suppose a, b, c are real numbers such that $a^3 + b^3 + c^3 = 0$. Prove that

$$\left(\sum a^2\right)^3 \leq \left(\sum (b-c)^2\right)\left(\sum a^4\right),$$

where the sum is cyclically over a, b, c.

168. Show that for all complex numbers z with real part of $z > 1$, the following inequality holds:

$$|z^{n+1} - 1| > |z^n||z - 1|, \text{ for all } n \geq 1.$$

169. Suppose a, b, c are positive real numbers and let

$$x = a + b - c, \quad y = b + c - a, \quad z = c + a - b.$$

Prove that

$$abc(xy + yz + zx) \geq xyz(ab + bc + ca).$$

170. Let a, b, c be positive real numbers. Prove that

$$\sum_{\text{cyclic}} \frac{a^3}{b^2 - bc + c^2} \geq \frac{3\sum_{\text{cyclic}} ab}{\sum_{\text{cyclic}} a}.$$

171. (CRUX, 2004) Let $a_1, a_2, \ldots, a_n < 1$ be non-negative real numbers satisfying

$$a = \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{\sqrt{3}}{3}.$$

Prove that

$$\frac{a_1}{1 - a_1^2} + \frac{a_2}{1 - a_2^2} + \cdots + \frac{a_n}{1 - a_n^2} \geq \frac{na}{1 - a^2}.$$

172. (USAMO, 2001) Suppose a, b, c are non-negative real numbers such that $a^2 + b^2 + c^2 + abc = 4$. Prove that

$$0 \leq ab + bc + ca - abc \leq 2.$$

173. (CRUX, 2003) Suppose a, b, c are complex numbers such that $|a| = |b| = |c|$. Prove that

$$\left|\frac{ab}{a^2 - b^2}\right| + \left|\frac{bc}{b^2 - c^2}\right| + \left|\frac{ca}{c^2 - a^2}\right| \geq \sqrt{3}.$$

174. (CRUX, 2001) Suppose x, y, z are non-negative real numbers such that $x^2 + y^2 + z^2 = 1$. Prove that

(a) $1 \leq \sum_{\text{cyclic}} \frac{x}{1 - yz} \leq \frac{3\sqrt{3}}{2}$;

(b) $1 \leq \sum_{\text{cyclic}} \frac{x}{1 + yz} \leq \sqrt{2}$.

175. (CRUX, 2004) Let x, y, z be non-negative real numbers satisfying $x + y + z = 1$. Prove that

$$xy^2 + yz^2 + zx^2 \geq xy + yz + zx - \frac{2}{9}.$$

176. (CRUX, 1990) Let a, b, c, d be four positive real numbers such that $a + b + c + d = 2$. Prove that

$$\sum_{\text{cyclic}} \frac{a^2}{\left(a^2+1\right)^2} \leq \frac{16}{25}.$$

177. (IMO, 2001) Prove that

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1$$

for all positive real numbers a, b and c.

178. Prove that in a triangle ABC, the inequality

$$\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} \leq \frac{9R^2}{4[ABC]},$$

holds.

179. (CRUX, 2002) Prove that in a triangle with angles α, β, γ, the inequality

$$\sum_{\text{cyclic}} \sin\alpha \leq \sqrt{\frac{15}{4} + \sum \cos(\alpha - \beta)}$$

holds.

180. (KÖMAL) If x, y are real numbers such that $x^3 + y^4 \leq x^2 + y^3$, prove that $x^3 + y^3 \leq 2$.

181. (CRUX, 2003) Let a, b, c be three positive real numbers. Prove that

$$\sum \frac{ab}{c(c+a)} \geq \sum \frac{a}{c+a},$$

where the sum is taken cyclically over a, b, c.

182. Prove that for any real x, and real numbers a, b,

$$\left(\sin x + a\cos x\right)\left(\sin x + b\cos x\right) \leq 1 + \left(\frac{a+b}{2}\right)^2.$$

183. (PUTNAM, 1988) Let x, y be two real numbers, where y is non-negative and $y(y+1) \leq (x+1)^2$. Prove that $y(y-1) \leq x^2$.

184. Let x, y, z be positive real numbers. Prove that

$$\left(\frac{xy+yz+zx}{3}\right)^{1/2} \leq \left(\frac{(x+y)(y+z)(z+x)}{8}\right)^{1/3}.$$

185. (Romania, 1997) Let a, b, c be positive real numbers such that $abc = 1$. Show that

$$\sum \frac{a^9 + b^9}{a^6 + a^3b^3 + b^6} \geq 2,$$

where the sum is cyclical.

186. Let a, b, c be the sides of a triangle and set $x = 2(s-a)$, $y = 2(s-b)$, $z = 2(s-c)$, where s denotes the semi-perimeter. Prove that

$$abc(ab + bc + ca) \geq xyz(xy + yz + zx).$$

187. (AMM) Let $a_1, a_2, a_3, \ldots, a_n$ $(n > 2)$ be positive real numbers and let s denote their sum. Let $0 < \beta \leq 1$ be a real number. Prove that

$$\sum_{k=1}^{n} \left(\frac{s - a_k}{a_k}\right)^{\beta} \geq (n-1)^{2\beta} \sum_{k=1}^{n} \left(\frac{a_k}{s - a_k}\right)^{\beta}.$$

When does equality hold?

188. A point D on the segment BC of a triangle ABC is such that the in-radii of ABD and ACD are equal, say r_1. Similarly define r_2 and r_3. Prove that

(i) $2r_1 + 2\dfrac{\sqrt{s(s-a)}}{a} r = h_a$.

(ii) $2\big(r_1 + r_2 + r_3\big) + s \geq h_a + h_b + h_c$.

189. (Romania) For $n \geq 4$, let $a_1, a_2, a_3, \ldots, a_n$ be n positive real numbers such that $\sum_{j=1}^{n} a_j^2 = 1$. Show that

$$\frac{a_1}{a_2^2 + 1} + \frac{a_2}{a_3^2 + 1} + \cdots + \frac{a_n}{a_1^2 + 1} \geq \frac{4}{5}\Big(a_1\sqrt{a_1} + a_2\sqrt{a_2} + \cdots + a_n\sqrt{a_n}\Big)^2.$$

190. (Belarus) Does there exist an infinite sequence $\langle x_n \rangle$ of positive real numbers such that

$$x_{n+2} = \sqrt{x_{n+1}} - \sqrt{x_n},$$

for all $n \geq 2$.

191. (Belarus) Let $a_1, a_2, a_3, \ldots, a_n$ be n positive real numbers and consider a permutation $b_1, b_2, b_3, \ldots, b_n$ of it. Prove that

$$\sum_{j=1}^{n} \frac{a_j^2}{b_j} \geq \sum_{j=1}^{n} a_j.$$

192. Let $a_1, a_2, a_3, \ldots, a_n$ and $b_1, b_2, b_3, \ldots, b_n$ be two sequences of positive real numbers such that $\sum_{j=1}^{n} a_j = \sum_{j=1}^{n} b_j = 1$. Prove that

$$\sum_{j=1}^{n} \frac{a_j^2}{a_j + b_j} \geq \frac{1}{2}.$$

193. Let x, y, z be positive real numbers. Prove that

$$\frac{y^2 - x^2}{z + x} + \frac{z^2 - y^2}{x + y} + \frac{x^2 - z^2}{y + z} \geq 0.$$

194. (Japan, 2003) Find the greatest real value of k such that for every triple (a, b, c) of positive real numbers, the inequality

$$\left(a^2 - bc\right)^2 > k\left(b^2 - ca\right)\left(c^2 - ab\right)$$

holds.

195. (Romania, 2003) Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that

$$\frac{1 + ab}{1 + a} + \frac{1 + bc}{1 + b} + \frac{1 + cd}{1 + c} + \frac{1 + da}{1 + d} \geq 4.$$

196. (UK, 2004) Let a, b, c be the sides of a triangle such that $a + b + c = 1$, and let $n \geq 2$ be a natural number. Prove that

$$\left(a^n + b^n\right)^{1/n} + \left(b^n + c^n\right)^{1/n} + \left(c^n + a^n\right)^{1/n} < 1 + \frac{2^{1/n}}{2}.$$

197. Let a, b, c, d be positive real numbers. Prove that

$$\sum_{\text{cyclic}} \frac{a}{b + 2c + d} \geq 1.$$

198. (Balkan Olympiads, 2005) Let a, b, c be positive real numbers such that $(a + b)(b + c)(c + a) = 1$. Prove that

$$ab + bc + ca \leq \frac{3}{4}.$$

199. (Iran, 2005) Let ABC be a right-angled triangle with $A = 90°$. Let AD be the bisector of angle A, and I_a be the ex-centre opposite to A. Prove that

$$\frac{AD}{DI_a} \leq \sqrt{2} - 1.$$

200. Let x, y, z be non-negative real numbers such that $x + y + z = 1$. Prove that

$$x^2 + y^2 + z^2 + 18xyz \leq 1.$$

201. Let ABC be a triangle with circum-circle Γ, and G be its centroid. Extend AG, BG, CG to meet Γ in D, E, F respectively. Prove that

$$AG + BG + CG \leq GD + GE + GF.$$

202. Prove with usual notation that in a triangle ABC, the inequality

$$\left(a + b + c\right)\left(h_a + h_b + h_c\right) \geq 18\Delta.$$

203. (Short-list, IMO-2004) Let a, b, c be three positive real numbers such that $ab + bc + ca = 1$. Prove that

$$\left(\frac{1}{a} + 6b\right)^{1/3} + \left(\frac{1}{b} + 6c\right)^{1/3} + \left(\frac{1}{c} + 6a\right)^{1/3} \leq \frac{1}{abc}.$$

204. (IMO, 1974) Let ABC be a triangle. Show that there exists a point D on AB such that $CD^2 = AD \cdot BD$ if and only if $\sqrt{\sin A \sin B} \leq \sin(C/2)$.

205. (Short-list, IMO-2004) Let $a_1, a_2, a_3, \ldots, a_n$ be $n > 1$ positive real numbers. For each k, $1 \leq k \leq n$, let $A_k = (a_1 + a_2 + \cdots + a_k)/k$. Let $g_n = (a_1 a_2 \cdots a_n)^{1/n}$ and $G_n = (A_1 A_2 \cdots A_n)^{1/n}$. Prove that

$$n\left(\frac{G_n}{A_n}\right)^{1/n} + \frac{g_n}{G_n} \leq n + 1.$$

Find the cases of equality.

206. Let x, y, z be real numbers in the interval $[0, 1]$. Prove that

$$3\left(x^2y^2 + y^2z^2 + z^2x^2\right) - 2xyz\left(x + y + z\right) \leq 3.$$

207. (UK, 1999) Let x, y, z be non-negative real numbers such that $x+y+z = 1$. Prove that

$$7(xy + yz + zx) \leq 2 + 9xyz.$$

208. (South Africa, 2003-04) Let x, y, z be real numbers in the interval $[0, 1]$. Prove that

$$\frac{x}{yz + 1} + \frac{y}{zx + 1} + \frac{z}{xy + 1} \leq 2.$$

209. (Greece, 2003) Let a, b, c, d be positive reals such that $a^3 + b^3 + 3ab = c + d = 1$. Prove that

$$\left(a+\frac{1}{a}\right)^3 + \left(b+\frac{1}{b}\right)^3 + \left(c+\frac{1}{c}\right)^3 + \left(d+\frac{1}{d}\right)^3 \geq 40.$$

210. Let x, y, z be positive real numbers such that $x+y+z = xyz$. Prove that

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} + \frac{1}{\sqrt{1+z^2}} \leq \frac{3}{2}.$$

211. Let P be an interior point of a triangle ABC whose sides are a, b, c. Let $R_1 = PA$, $R_2 = PB$, and $R_3 = PC$. Prove that

$$\left(R_1R_2 + R_2R_3 + R_3R_1\right)\left(R_1 + R_2 + R_3\right) \geq a^2R_1 + b^2R_2 + c^2R_3.$$

When does equality hold?

212. (IMO, 2004) Let $t_1, t_2, t_3, \ldots, t_n$ be positive real numbers such that

$$n^2 + 1 > \left(t_1 + t_2 + \cdots + t_n\right)\left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n}\right),$$

where $n \geq 3$ is an integer. Show that for each triple (j, k, l) with $1 \leq j < k < l \leq n$, there is a triangle with sides t_j, t_k, t_l.

213. (Moldova, 2004) Let x, y, z be non-negative real numbers. Prove that

$$x^3 + y^3 + z^3 \geq x^2\sqrt{yz} + y^2\sqrt{zx} + z^2\sqrt{xy}.$$

214. (Romania, 2004) Find all positive real numbers such that

$$4\left(ab + bc + ca\right) - 1 \geq a^2 + b^2 + c^2 \geq 3\left(a^3 + b^3 + c^3\right).$$

215. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a}{(a+1)(b+1)} + \frac{b}{(b+1)(c+1)} + \frac{c}{(c+1)(a+1)} \geq \frac{3}{4}.$$

216. (CRUX, 2000) Suppose a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 1$. Prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq 3 + \frac{2\left(a^3 + b^3 + c^3\right)}{abc}.$$

217. (Taiwan, 1999) Let ABC be a triangle with circum-centre O and circum-radius R. Suppose the line AO, when extended, meets the circum-circle of OBC in D; similarly define E and F. Prove that

$$OD \cdot OE \cdot OF \geq 8R^3.$$

218. (Proposed for IMO-1998) Let x, y, z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

219. (Nordic Contest, 2003-04) Let D, E, F be respectively the points of contact of the in-circle of a triangle ABC with its sides BC, CA, AB. Prove that

$$\frac{BC}{FD} + \frac{CA}{DE} + \frac{AB}{EF} \geq 6.$$

220. (Short-list, IMO-1990) Let a, b, c, d be non-negative real numbers such that $ab + bc + cd + da = 1$. Show that

$$\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \geq \frac{1}{3}.$$

221. (Bulgaria, 1974) Find all real λ for which the inequality

$$x_1^2 + x_2^2 + x_3^2 \geq \lambda(x_1x_2 + x_2x_3),$$

holds for all real numbers x_1, x_2, x_3.

222. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

223. (Indian Team Selection, 2007) Let a, b, c be non-negative reals such that

$$a + b \leq 1 + c, \quad b + c \leq 1 + a, \quad c + a \leq 1 + b.$$

Prove that

$$a^2 + b^2 + c^2 \leq 2abc + 1.$$

224. (Indian Team Selection, 1997) If a, b, c are non-negative real numbers such that $a + b + c = 1$ then show that

$$\frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} \geq \frac{9}{10}.$$

225. (Indian Team Selection, 2006) Let ABC be a triangle with sides a, b, c, circum-radius R and in-radius r. Prove that

$$\frac{R}{2r} \geq \left(\frac{64a^2b^2c^2}{\bigl(4a^2 - (b-c)^2\bigr)\bigl(4b^2 - (c-a)^2\bigr)\bigl(4c^2 - (a-b)^2\bigr)} \right)^2.$$

226. (INMO, 2004) Let R denote the circum-radius of a triangle ABC; a, b, c its sides BC, CA, AB; and r_a, r_b, r_c its ex-radii opposite A, B, C. If $2R \leq r_a$, prove that

(i) $a > b$ and $a > c$;

(ii) $2R > r_b$ and $2R > r_c$.

227. (USSR, 1974) Given a square grid S containing 49 points in 7 rows and 7 columns, a subset T consisting of k points is selected. What is the maximum value of k such that no four points of T determine a rectangle R having sides parallel to the sides of S?

228. (Proposed for IMO-1977) Let a, b, c, d be real numbers such that $0 \leq a \leq b \leq c \leq d$. Prove that

$$a^b b^c c^d d^a \geq b^a c^b d^c a^d.$$

229. (Ukraine, 2001) In a triangle ABC, let AA_1, CC_1 be the bisectors of the angles A, C respectively. Let M be point on the segment AC, and I be the in-centre of ABC. Draw a line through M, parallel to AA_1 and let it meet CC_1 in N and BC in Q. Similarly, let the line through M parallel to CC_1 meet AA_1 in H and AB in P. Let d_1, d_1, d_3 respectively denote the distances of H, I, N from the line PQ. Prove that

$$\frac{d_1}{d_2} + \frac{d_2}{d_3} + \frac{d_3}{d_1} \geq \frac{2ab}{a^2+bc} + \frac{2bc}{b^2+ca} + \frac{2ca}{c^2+ab}.$$

230. (CRMO, 1996) Let ABC be a triangle and h_a be the altitude through A. Prove that

$$(b+c)^2 \geq a^2 + 4h_a^2.$$

(As usual a, b, c denote the sides BC, CA, AB respectively.)

231. (CRMO, 2003) Let a, b, c be three positive real numbers such that $a+b+c=1$. Prove that among the three numbers $a-ab$, $b-bc$, $c-ca$ there is one which is at most $1/4$ and there is one which is at least $2/9$.

232. (CRMO, 2004) Let x and y be positive real numbers such that $y^3 + y \leq x - x^3$. Prove that

(a) $y < x < 1$; and

(b) $x^2 + y^2 < 1$.

233. (CRMO, 2005) Let a, b, c be three positive real numbers such that $a + b + c = 1$. Let

$$\lambda = \min\left\{a^3 + a^2bc,\ b^3 + ab^2c,\ c^3 + abc^2\right\}.$$

Prove that the roots of the equation $x^2 + x + 4\lambda = 0$ are real.

234. (CRMO, 2006) If a, b, c are three positive real numbers, prove that

$$\frac{a^2+1}{b+c} + \frac{b^2+1}{c+a} + \frac{c^2+1}{a+b} \geq 3.$$

235. (Hungary, 1990) If d is the largest among the positive numbers a, b, c, d, prove that

$$a(d-b) + b(d-c) + c(d-a) \leq d^2.$$

236. (INMO, 2007) If x, y, z are positive real numbers, prove that

$$(x+y+z)^2 \left(yz+zx+xy\right)^2 \leq 3\left(y^2+yz+z^2\right)\left(z^2+zx+x^2\right)\left(x^2+xy+y^2\right).$$

237. (USAMO, 1974) Suppose a, b, c are positive real numbers. Prove that

$$a^a b^b c^c \geq (abc)^{(a+b+c)/3}.$$

238. (Bulgaria, 1988) Find all real p and q for which the equation

$$x^4 - \frac{8p^2}{q}x^3 + 4qx^3 - 3px + p^2 = 0$$

has four positive roots.

239. (Russia, 2005) Let a_1, a_2, a_3 be real numbers, each greater than 1. Let $S = a_1 + a_2 + a_3$ and suppose $S < a_j^2/(a_j - 1)$ for $j = 1, 2, 3$. Prove that

$$\frac{1}{a_1+a_2} + \frac{1}{a_2+a_3} + \frac{1}{a_3+a_1} > 1.$$

240. (Macedonia, 2010) Let a, b, c be positive real numbers such that $ab+bc+ca = 1/3$. Prove that

$$\frac{a}{a^2-bc+1} + \frac{b}{b^2-ca+1} + \frac{c}{c^2-ab+1} \geq \frac{1}{a+b+c}.$$

241. (Balkan Olympiads, 2010) Suppose a, b, c are positive real numbers. Prove that

$$\frac{a^2b(b-c)}{a+b} + \frac{b^2c(c-a)}{b+c} + \frac{c^2a(a-b)}{c+a} \geq 0.$$

242. (Macedonia, 2010) Let $a_1, a_2, a_3, \ldots, a_n$ be $n > 2$ positive real numbers such that $a_1 + a_2 + a_3 + \cdots + a_n = 1$. Prove that

$$\sum_{j=1}^{n} \frac{a_1 a_2 \cdots a_{j-1} a_{j+1} \cdots a_n}{a_j + n - 2} \leq \frac{1}{(n-1)^2}.$$

243. (Thailand, 2014) Determine the largest value of k such that the inequality

$$\left(k+\frac{a}{b}\right)\left(k+\frac{b}{c}\right)\left(k+\frac{c}{ba}\right) \geq \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)$$

holds for all positive real numbers a, b, c.

244. (Macedonia, 2010) Let $x_1, x_2, x_3, \ldots, x_n$ be $n \geq 3$ positive real numbers. Prove that

$$\frac{x_1x_3}{x_1x_3+x_2x_4}+\frac{x_2x_4}{x_2x_4+x_3x_5}+\cdots+\frac{x_{n-1}x_1}{x_{n-1}x_1+x_nx_2}+\frac{x_nx_2}{x_nx_2+x_1x_3} \leq n-1.$$

245. Let $a_1, a_2, a_3, \ldots, a_{2017}$ be positive real numbers. Prove that

$$\sum_{j=1}^{2017} \frac{a_j}{a_{j+1}+a_{j+2}+\cdots+a_{j+1008}} \geq \frac{2017}{1008},$$

where the indices are taken modulo 2017.

246. (Russia, 2014) Let a, b, c be three positive real numbers such that $ab+bc+ca=1$. Prove that

$$\sqrt{a+\frac{1}{a}}+\sqrt{b+\frac{1}{b}}+\sqrt{c+\frac{1}{c}} \geq 2(\sqrt{a}+\sqrt{b}+\sqrt{c}).$$

247. (Macedonia, 2010) Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{a^3+2}{b+2}+\frac{b^3+2}{c+2}+\frac{c^3+2}{a+2} \geq 3.$$

248. (Russia, 2014) Let a, b, c, d be real numbers such that $a^2+b^2+c^2+d^2=4$. Prove that $(2+a)(2+b) \geq cd$.

249. (Belarus, 2014) Find all real λ such that

$$\frac{a+b}{2} \geq \lambda\sqrt{ab}+(1-\lambda)\sqrt{\frac{a^2+b^2}{2}}$$

holds for all positive real numbers a, b.

250. (Russia, 2014) Let a, b, c, d be real numbers having absolute value greater than 1 and such that $abc+abd+acd+bcd+a+b+c+d=0$. Prove that

$$\frac{1}{a-1}+\frac{1}{b-1}+\frac{1}{c-1}+\frac{1}{d-1} > 0.$$

251. (Belarus, 2014) Show that

$$\frac{1}{x+y+1}-\frac{1}{(x+1)(y+1)}<\frac{1}{11},$$

for all positive real numbers x, y.

252. (Kazaksthan, 2008) Let a, b, c be three positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{b(a+b)}+\frac{1}{c(b+c)}+\frac{1}{a(c+a)} \geq \frac{3}{2}.$$

253. (Belarus, 2014) Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{a^2}{(b+c)^3}+\frac{b^2}{(c+a)^3}+\frac{c^2}{(a+b)^3} \geq \frac{9}{8}$$

254. (Belarus, 2014) Suppose a, b, c are positive reals such that $ab+bc+ca \geq a+b+c$. Prove that

$$(a+b+c)(ab+bc+ca)+3abc \geq 4(ab+bc+ca).$$

255. (Kazaksthan, 2008) Let a, b, c, d be four real numbers such that $a+b+c+d=0$. Prove that

$$(ab+ac+ad+bc+bd+cd)^2+12 \geq 6(abc+abd+acd+bcd).$$

256. (Vietnam, 2014) Consider the expression

$$P=\frac{x^3y^4z^3}{(x^4+y^4)(xy+z^2)^3}+\frac{y^3z^4x^3}{(y^4+z^4)(yz+x^2)^3}+\frac{z^3x^4y^3}{(z^4+x^4)(zx+y^2)^3}.$$

Find the maximum value of P when x, y, z vary over the set of all positive real numbers.

257. (Ukraine, 2014) Let a, b, c be the sides of an acute-angled triangle. Prove that

$$\sqrt{a^2+b^2-c^2}+\sqrt{b^2+c^2-a^2}+\sqrt{c^2+a^2-b^2} \leq \sqrt{3(ab+bc+ca)}.$$

258. (Ukraine, 2014) Let $x_1, x_2, x_3, \ldots, x_n$ be Positive real numbers such that $x_1x_2x_3\cdots x_n=1$. Let $S=x_1^3+x_2^3+x_3^3+\cdots+x_n^3$. Prove that

$$\frac{x_1}{S-x_1^3+x_1^2}+\frac{x_2}{S-x_2^3+x_2^2}+\frac{x_3}{S-x_3^3+x_3^2}+\cdots+\frac{x_n}{S-x_n^3+x_n^2} \leq 1.$$

259. (Ireland, 2014) Let $a_1, a_2, a_3, \ldots, a_n$ be $n(>1)$ positive real numbers whose sum is 1. Define $b_k = \frac{a_k^2}{\sum_{j=1}^n a_j^2}$, $1 \leq k \leq n$. Prove that

$$\sum_{k=1}^{n} \frac{a_k}{1-a_k} \leq \sum_{k=1}^{n} \frac{b_k}{1-b_k}.$$

260. (Bulgaria, 2014) Suppose a, b, c, d are positive real numbers. Prove that

$$\sum_{\text{cyclic}} \frac{a^4}{a^3 + a^2 b + ab^2 + b^3} \geq \frac{a+b+c+d}{4}.$$

261. (Turkey, 2014) Let a, b, c be non-negative real numbers satisfying $a^2 + b^2 + c^2 = 1$. Prove that

$$\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a} \geq 5abc + 2.$$

262. (Turkey, 2014) Let x, y, z be positive real numbers such that $x^2+y^2+z^2 \leq x + y + z$. Prove that

$$\frac{x^2+3}{x^3+1} + \frac{y^2+3}{y^3+1} + \frac{z^2+3}{z^3+1} \geq 6.$$

263. (Belarus, 2010) For any three positive real numbers a, b, c, prove that

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} \geq \frac{3a+2b-c}{4}.$$

264. (APMO, 2003) Let a, b, c be the sides of a triangle with perimeter equal to 1. Prove that

$$\sqrt{a^2+b^2} + \sqrt{b^2+c^2} + \sqrt{c^2+a^2} < 1 + \frac{\sqrt{2}}{2}.$$

265. (Tajikisthan, 2014) Let a, b, c be the sides of a triangle. Prove that

$$\sqrt{a^2+ab+b^2} + \sqrt{b^2+bc+c^2} + \sqrt{c^2+ca+a^2} \\ \leq \sqrt{5(a^2+b^2+c^2) + 4(ab+bc+ca)}.$$

266. (Tajikisthan, 2014) Suppose a, b, c are non-negative real numbers such that $a^3 + b^3 + c^3 + abc = 4$. Prove that

$$a^3 b + b^3 c + c^3 a \leq 3.$$

267. (JBMO, 2014) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(a + \frac{1}{b}\right)^2 + \left(b + \frac{1}{c}\right)^2 + \left(c + \frac{1}{a}\right)^2 \geq 3(a+b+c+1).$$

268. (Tajikisthan, 2014) Let a, b, c be positive real numbers with $abc = 1$. Prove that

$$\frac{a}{c(a+1)} + \frac{b}{a(b+1)} + \frac{c}{b(c+1)} \geq \frac{3}{2}.$$

269. (Estonia, 2014) Let a, b, c be positive real number such that $abc = 1$. Prove that

$$\frac{1}{1+a^{2014}} + \frac{1}{1+b^{2014}} + \frac{1}{1+c^{2014}} > 1.$$

270. (CRUX, 2000) For positive real numbers a, b, c, prove the inequality

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left(\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}\right) \geq \frac{9}{1+abc}.$$

271. ([10]) Let x, y, z be positive real numbers such that $x+y+z = 3$. Prove that

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \geq xy + yz + zx.$$

272. ([10]) Let a, b, c be positive real numbers. Prove that

$$\frac{9abc}{2(a+b+c)} \leq \frac{ab^2}{a+b} + \frac{bc^2}{b+c} + \frac{ca^2}{c+a} \leq \frac{a^2+b^2+c^2}{2}.$$

273. ([10]) For positive real numbers a, b, c, prove that

$$\frac{abc}{(1+a)(a+b)(b+c)(c+16)} \leq \frac{1}{81}.$$

274. ([10]) Let a, b, c, d be positive real numbers such that a + b + c + d = 4. Prove that

$$\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} + \frac{1}{d^2+1} \geq 2.$$

275. Let a, b, c be the sides of a triangle. Prove that inequality

$$64(s-a)(s-b)(s-c) \leq (a+b)(b+c)(c+a),$$

where $s = (a+b+c)/2$ is the semi-perimeter of the triangle.

276. (Estonia, 2014) Let a, b, c be positive real numbers. Prove that

$$\frac{1+ab}{c} + \frac{1+bc}{a} + \frac{1+ca}{b} \geq \sqrt{a^2+2} + \sqrt{b^2+2} + \sqrt{c^2+2}.$$

277. (Estonia, 2014) Let I Be the incentre of a triangle ABC. Let R_A, R_B, R_C be respectively the circum-radii of triangles BCI, CAI, ABI. If R is the circum-radius of $\triangle ABC$, prove that

$$R_A + R_B + R_C \leq 3R.$$

278. (Estonia, 2014) Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{a^2}{b^3+c^4+1}+\frac{b^2}{c^3+a^4+1}+\frac{c^2}{a^3+b^4+1}>\frac{1}{5}.$$

279. (Balkan Olympiads, 2014) Let x, y, z be three positive real numbers such that $xy+yz+zx=3xyz$. Prove that

$$x^2y+y^2z+z^2x \geq 2(x+y+z)-3.$$

280. (Janous inequality) Let a, b, c and x, y, z be two sets of positive real numbers. Prove that

$$\frac{x}{y+z}(b+c)+\frac{y}{z+x}(c+a)+\frac{z}{x+y}(a+b) \geq \sqrt{3(ab+bc+ca)}.$$

281. Let x, y, z be positive real numbers such that $xy+yz+zx=1$. Prove that

$$\frac{x}{x^2+1}+\frac{y}{y^2+1}+\frac{z}{z^2+1} \leq \frac{3\sqrt{3}}{4}.$$

282. Suppose x, y, z are positive real numbers such that $x+y+z=1$. Prove that

$$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right) \geq 64.$$

283. Let x, y, z be positive real numbers such that $x+y+z=1$. Prove that

$$\frac{1}{1-xy}+\frac{1}{1-yz}+\frac{1}{1-zx} \leq \frac{27}{8}.$$

284. Let x, y, z be positive real numbers such that $x+y+z=1$. Show that

$$\frac{z-xy}{x^2+xy+y^2}+\frac{x-yz}{y^2+yz+z^2}+\frac{y-zx}{z^2+zx+x^2} \geq 2.$$

285. ([10]) Let a, b, c be positive real numbers define

$$u=a+b+c, \quad \frac{u^2-v^2}{3}=ab+bc+ca, \quad w=abc,$$

where $v \geq 0$. Then

$$\frac{(u+v)^2(u-2v)}{27} \leq r \leq \frac{(u-v)^2(u+2v)}{27}.$$

286. ([10]) Let a, b, c be positive real numbers. Prove that

$$a^4+b^4+c^4 \geq abc(a+b+c).$$

287. Let a, b, c be real numbers such that $a^2 + b^2 + c^2 = 9$. Prove that

$$2(a + b + c) - abc \leq 10.$$

288. (Serbia, 2008) Let a, b, c be positive real numbers such that $a+b+c = 1$. Prove that

$$a^2 + b^2 + c^2 + 3abc \geq \frac{9}{4}.$$

289. (China, 2004) Determine the maximum value of λ such that

$$a + b + c \geq \lambda$$

for all positive reals a, b, c with $a\sqrt{bc} + b\sqrt{c} + c\sqrt{a} \geq 1$.

290. (China, 2004) If a, b, c are real numbers such that $a + b + c = 1$, prove that

$$10(a^3 + b^3 + c^3) - 9(a^5 + b^5 + c^5) \geq 1.$$

291. (IMO, 2004) Suppose $a_1, a_2, a_3, \ldots, a_n$ are n positive real numbers such that

$$\Big(a_1 + a_2 + a_3 + \cdots + a_n\Big)\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n}\right) < n^2 + 1.$$

Show that for any three distinct numbers j, k, l the numbers a_j, a_k, a_l form the sides of a triangle.

292. (Colombia, 2013) Let a, b, c be positive real numbers. Prove that

$$24abc \leq \left|a^3 + b^3 + c^3 - (a + b + c)^3\right| \leq \frac{8}{9}(a + b + c)^3.$$

Show further that equality holds in both the inequalities if and only if $a = b = c$.

293. (Czech and Slovak, 2013) Two circles Γ_1, Γ_2 with respective centre S_1, S_2 and radii r_1, r_2 are externally tangent to each other and lie in a square $ABCD$ of side a units so that Γ_1 touches DC, DA while Γ_2 touches CD, CB. Prove that the area of at least one of the triangles AS_1S_2 and BS_1S_2 is no more than $\frac{3}{16}a^2$ units.

294. (Czech and Slovak, 2013) Find all $\lambda > 0$ such that the inequality

$$\sqrt{a^2 + \lambda b^2} + \sqrt{b^2 + \lambda a^2} \geq a + b + (\lambda - 1)\sqrt{ab}$$

holds for all positive real numbers a and b.

295. (Netherlands, 2013) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$a + b + c \geq \sqrt{\frac{1}{3}(a+2)(b+2)(c+2)}.$$

296. (Belarus, 2013) Let $x_1, x_2, x_3, \ldots, x_n$ be $n \geq 3$ positive real numbers such that $x_1 x_2 x_3 \cdots x_n = 1$. Prove that

$$\sum_{j=1}^{n} \frac{x_j^8}{x_{j+1}(x_j^4 + x_{j+1}^4)} \geq \frac{n}{2},$$

where $x_{n+1} = x_1$.

297. (Belarus, 2013) Let a, b, c be positive real number such that $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = 1$. Prove that

$$\frac{a^2 + b^2 + c^2 + ab + bc + ca - 3}{5} \geq \frac{a}{b} + \frac{b}{c} + \frac{c}{a}.$$

298. (Turkey, 2012) Show that for all positive real numbers x, y, z, the inequality

$$\frac{x(2x-y)}{y(2z+x)} + \frac{y(2y-z)}{z(2x+y)} + \frac{z(2z-x)}{x(2y+z)} \geq 1.$$

299. (Turkey, 2012) Suppose

$$\frac{z(xz + yz + y)}{xy_y^2 + z^2 + 1} \leq K,$$

for all real numbers $x, y, z \in (-2, 2)$ with $x^2 + y^2 + z^2 + xyz = 4$. Find the smallest value of K.

300. (Turkey, 2012) Suppose a, b, c are positive real numbers such that $a^3 + b^3 + c^3 = a^4 + b^4 + c^4$. Prove that

$$\frac{a}{a^2 + b^3 + c^3} + \frac{b}{b^2 + c^3 + a^3} + \frac{c}{c^2 + c^3 + a^3} \geq 1.$$

301. (Turkey, 2013) Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{a^4 + 5b^4}{a(a+2b)} + \frac{b^4 + 5c^4}{b(b+2c)} + \frac{c^4 + 5a^4}{c(c+2a)} \geq 1 - (ab + bc + ca).$$

302. (USAMO, 1998) Let $a_0, a_1, a_2, \ldots, a_n$ be real numbers in the interval $\left(0, \frac{\pi}{2}\right)$ such that

$$\tan\left(a_0 - \frac{\pi}{4}\right)\tan\left(a_1 - \frac{\pi}{4}\right)\tan\left(a_2 - \frac{\pi}{4}\right)\cdots\tan\left(a_n - \frac{\pi}{4}\right) \geq n - 1.$$

Prove that

$$\tan(a_0)\tan(a_1)\tan(a_2)\cdots\tan(a_n) \geq n^{n+1}.$$

303. (Iran, 1996) Let x, y, z be positive real numbers. Prove that

$$(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right) \geq \frac{9}{4}.$$

304. Suppose a, b, c are positive real numbers such that $abc = 1$. Prove that

$$\sum_{\text{cyclic}} \frac{a^2+bc}{a^2(b+c)} \geq ab+bc+ca.$$

305. Let a, b, c be non-negative real numbers. Prove that

$$4\left(a^3+b^3+c^3\right)+15abc \geq (a+b+c)^3.$$

306. Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{1}{a^4+b+c}+\frac{1}{b^4+c+a}+\frac{1}{c^4+a+b} \leq \frac{3}{a+b+c}.$$

307. Let a, b, c be positive reals. Prove that

$$a^4(b+c)+b^4(c+a)+c^4(a+b) \leq \frac{1}{12}(a+b+c)^5.$$

308. Suppose a, b, c are positive reals such that $ab+bc+ca=1$. Prove that

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}-\frac{1}{a+b+c} \geq 2.$$

309. Let a, b, c be positive real numbers such that $ab+bc+ca=1$. Prove that

$$\frac{1+a^2b^2}{(a+b)^2}+\frac{1+b^2c^2}{(b+c)^2}+\frac{1+c^2a^2}{(c+a)^2} \geq \frac{5}{2}.$$

310. [10] Let a, b, c, d be non-negative real numbers. Prove that

$$a^4+b^4+c^4+d^4+2abcd \geq a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2.$$

311. (Kvant, 1988) Let a, b, c be positive real numbers. Prove that

$$3+a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3\left(\frac{(a+1)(b+1)(c+1)}{1+abc}\right).$$

312. (Indian Team Selection, 2017) Let a, b, c be distinct positive real numbers such that $abc = 1$. Prove that

$$\sum_{\text{cyclic}} \frac{a^6}{(a-b)(a-c)} > 15.$$

313. (Indian Team Selection, 2010) Let a, b, c be real numbers such that $a^2 + b^2 + c^2 = 1$. Prove that

$$a + b + c \leq 2abc + \sqrt{2}.$$

314. (Japan, 1997) Let a, b, c be positive real numbers. Prove that

$$\frac{(b+c-a)^2}{a^2+(b+c)^2} + \frac{(c+a-b)^2}{b^2+(c+a)^2} + \frac{(a+b-c)^2}{c^2+(a+b)^2} \geq \frac{3}{5}.$$

315. (Ukraine, 2005) Let a, b, c be positive real numbers such that $a+b+c = 1$. Prove that

$$\sqrt{\frac{1}{a}-1}\sqrt{\frac{1}{b}-1} + \sqrt{\frac{1}{b}-1}\sqrt{\frac{1}{c}-1} + \sqrt{\frac{1}{c}-1}\sqrt{\frac{1}{a}-1} \geq 6.$$

Chapter 6

Solutions to problems

1. Let $a_1, a_2, \ldots, a_n, a_{n+1}$ be $n+1$ positive real numbers such that $a_1 + a_2 + \cdots + a_n = a_{n+1}$. Prove that

$$\sum_{j=1}^{n} \sqrt{a_j\left(a_{n+1} - a_j\right)} \leq \sqrt{\sum_{j=1}^{n} a_{n+1}\left(a_{n+1} - a_j\right)}.$$

Solution: We observe that

$$\begin{aligned} \sum_{j=1}^{n} a_{n+1}\left(a_{n+1} - a_j\right) &= n a_{n+1}^2 - \left(\sum_{j=1}^{n} a_j\right) a_{n+1} \\ &= (n-1) a_{n+1}^2. \end{aligned}$$

Thus, it is sufficient to prove that

$$\sum_{j=1}^{n} \sqrt{a_j(a_{n+1} - a_j)} \leq \sqrt{(n-1)}\, a_{n+1}.$$

However, this follows from the Cauchy-Schwarz inequality:

$$\begin{aligned} \sum_{j=1}^{n} \sqrt{a_j\left(a_{n+1} - a_j\right)} &= \sum_{j=1}^{n} \sqrt{a_j} \sqrt{\left(a_{n+1} - a_j\right)} \\ &\leq \sqrt{\sum_{j=1}^{n} a_j} \sqrt{\sum_{j=1}^{n} \left(a_{n+1} - a_j\right)} \\ &= \sqrt{a_{n+1}} \sqrt{(n-1) a_{n+1}} \\ &= \sqrt{(n-1)}\, a_{n+1}. \end{aligned}$$

■

2. If a, b, c are positive real numbers, prove that

$$\frac{a}{b+2c} + \frac{b}{c+2a} + \frac{c}{a+2b} \geq 1.$$

Solution: The inequality may be written in an equivalent form:

$$\frac{a^2}{ab+2ca}+\frac{b^2}{bc+2ab}+\frac{c^2}{ca+2bc}\geq 1.$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}(a+b+c)^2 &= \left(\sum_{\text{cyclic}}\frac{a}{\sqrt{ab+2ca}}\sqrt{ab+2ca}\right)^2\\ &\leq \left(\sum_{\text{cyclic}}\frac{a^2}{ab+2ca}\right)\left(\sum_{\text{cyclic}}(ab+2ca)\right).\end{aligned}$$

It follows that

$$\sum_{\text{cyclic}}\frac{a^2}{ab+2ca}\geq\frac{(a+b+c)^2}{3(ab+bc+ca)}.$$

Thus, it is sufficient to prove that

$$(a+b+c)^2\geq 3(ab+bc+ca).$$

Equivalently, we need to prove that $a^2+b^2+c^2\geq ab+bc+ca$, which is clear from the Cauchy-Schwarz inequality. (Or one can use $(a-b)^2+(b-c)^2+(c-a)^2\geq 0$.)

■

3. Let a,b,c be positive real numbers such that $abc\leq a+b+c$. Prove that

$$a^2+b^2+c^2\geq\sqrt{3}(abc).$$

Solution: We observe that

$$(a+b+c)^2-3(a^2+b^2+c^2)=2(ab+bc+ca-a^2-b^2-c^2)\leq 0.$$

Thus, we get

$$3(a^2+b^2+c^2)\geq(a+b+c)^2\geq(abc)^2.$$

On the other hand, we also have

$$a^2+b^2+c^2\geq 3(abc)^{2/3},$$

as a consequence of the AM-GM inequality. This gives

$$(a^2+b^2+c^2)^3\geq 27(abc)^2.$$

Multiplying these inequalities, we get

$$9(abc)^4\leq(a^2+b^2+c^2)^4.$$

Taking the fourth-root, we get the desired inequality. ■

4. For any positive real numbers a, b, c, prove that

$$\frac{2}{b(a+b)} + \frac{2}{c(b+c)} + \frac{2}{a(c+a)} \ge \frac{27}{(a+b+c)^2}.$$

Solution: Taking $A = \sqrt[3]{abc}$ and $B = \sqrt[3]{(a+b)(b+c)(c+a)}$, we have

$$\frac{2}{b(a+b)} + \frac{2}{c(b+c)} + \frac{2}{a(c+a)} \ge \frac{6}{AB},$$

by the AM-GM inequality. However, the AM-GM inequality also gives

$$A \le \frac{a+b+c}{3}, \quad B \le \frac{2(a+b+c)}{3}.$$

Thus

$$\frac{6}{AB} \ge \frac{27}{(a+b+c)^2}.$$

■

5. Let a, b, c be three sides of a triangle such that $a+b+c = 2$. Prove that

$$1 \le ab + bc + ca - abc \le 1 + \frac{1}{27}.$$

Solution: We have

$$\begin{aligned}(1-a)(1-b)(1-c) &= 1 - (a+b+c) + (ab+bc+ca) - abc \\ &= -1 + (ab+bc+ca) - abc.\end{aligned}$$

Thus it is sufficient to prove that

$$0 \le (1-a)(1-b)(1-c) \le \frac{1}{27}.$$

But, this is a consequence of the AM-GM inequality: observe $a < b+c = 2-a$ so that $1-a > 0$, and similarly $1-b > 0$, $1-c > 0$;

$$(1-a)(1-b)(1-c) \le \left(\frac{1-a+1-b+1-c}{3}\right)^3 = \frac{1}{27}.$$

■

6. If a, b, c be positive real numbers such that $a+b+c = 1$, prove that

$$\sqrt{ab+c} + \sqrt{bc+a} + \sqrt{ca+b} \ge 1 + \sqrt{ab} + \sqrt{bc} + \sqrt{ca}.$$

Solution: First we observe that $\sqrt{ab+c} \geq \sqrt{ab}+c$. In fact, this is equivalent to $c \geq c^2 + 2c\sqrt{ab}$, which is equivalent to $1-c \geq 2\sqrt{ab}$, since $c > 0$. Using $1-c = a+b$, we get $a+b \geq 2\sqrt{ab}$. Similarly, we derive $\sqrt{bc}+a \geq \sqrt{bc}+a$, $\sqrt{ca+b} \geq \sqrt{ca}+b$. Thus it follows

$$\begin{aligned}\sqrt{ab+c}+\sqrt{bc+a}+\sqrt{ca+b} &\geq \sqrt{ab}+\sqrt{bc}+\sqrt{ca}+a+b+c \\ &= 1+\sqrt{ab}+\sqrt{bc}+\sqrt{ca}.\end{aligned}$$

■

7. Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that

$$\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+cd}{1+c}+\frac{1+da}{1+d} \geq 4.$$

Solution: Using $cd = 1/ab$ and $da = 1/bc$, the inequality to be proved is

$$(1+ab)\left\{\frac{1}{1+a}+\frac{1}{ab(1+c)}\right\}+(1+bc)\left\{\frac{1}{1+b}+\frac{1}{bc(1+d)}\right\} \geq 4.$$

But,

$$\begin{aligned}\frac{1}{1+a}+\frac{1}{ab(1+c)} &\geq \frac{4}{1+a+ab+abc} \\ \frac{1}{1+b}+\frac{1}{bc(1+d)} &\geq \frac{4}{1+b+bc+bcd}.\end{aligned}$$

Thus

$$\begin{aligned}\sum_{\text{cyclic}} \frac{1+ab}{1+a} &\geq \frac{4(1+ab)}{1+a+ab+abc}+\frac{4(1+bc)}{1+b+bc+bcd} \\ &= \frac{4(1+ab)}{1+a+ab+abc}+\frac{4a(1+bc)}{a+ab+abc+abcd} \\ &= \frac{4(1+ab)}{1+a+ab+abc}+\frac{4a(1+bc)}{1+a+ab+abc} \\ &= 4.\end{aligned}$$

■

8. If a, b, c, d are positive real numbers, prove that

$$\sqrt{\frac{a^2+b^2+c^2+d^2}{4}} \geq \sqrt[3]{\frac{abc+abd+acd+bcd}{4}}.$$

Solution: Using the AM-GM inequality, we have

$$\begin{aligned}
&\frac{abc+abd+acd+bcd}{4}\\
&\quad= \frac{1}{2}\left\{\frac{ab(c+d)}{2}+\frac{cd(a+b)}{2}\right\}\\
&\quad\le \frac{1}{2}\left\{\frac{(a+b)^2(c+d)}{8}+\frac{(a+b)(c+d)^2}{8}\right\}\\
&\quad= \frac{(a+b)}{2}\cdot\frac{(c+d)}{2}\cdot\frac{(a+b+c+d)}{4}\\
&\quad\le \left(\frac{a+b+c+d}{4}\right)^3.
\end{aligned}$$

This shows that

$$\sqrt[3]{\frac{abc+abd+acd+bcd}{4}} \le \frac{a+b+c+d}{4} \le \sqrt{\frac{a^2+b^2+c^2+d^2}{4}},$$

where we have used the root-mean-square inequality in the last step. ■

9. Let a, b, c be the sides of a triangle such that $a+b+c=2$. Prove that

$$a^2+b^2+c^2+2abc<2.$$

Solution: Note that $s=(a+b+c)/2=1$. Using $s-a>0$, it follows that $a<1$. Similarly, $b<1$ and $c<1$. We have

$$\begin{aligned}
a^2+b^2+c^2+2abc &= (a+b+c)^2-2(ab+bc+ca)+2abc\\
&= 2\bigl(2-(ab+bc+ca)+abc\bigr).
\end{aligned}$$

Thus, we have to prove that

$$2-(ab+bc+ca)+abc<1.$$

Equivalently,

$$1-(a+b+c)+(ab+bc+ca)-abc>0.$$

But this is precisely $(1-a)(1-b)(1-c)>0$, which follows from $a<1$, $b<1$, $c<1$.

Alternate Solution:

We use the transformation $a=y+z$, $b=z+x$, $c=x+y$. This is possible because a, b, c are the sides of a triangle. Then, x, y, z are positive real numbers such that $x+y+z=1$. The inequality gets transformed to

$$\sum_{\text{cyclic}} x^2+\sum_{\text{cyclic}} xy+\sum_{\text{cyclic}} x^2y+\sum_{\text{cyclic}} xy^2+2xyz<1.$$

This reduces to $1-xyz<1$ after using $x+y+z=1$. The result follows from $xyz>0$. ■

10. If a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 1$, prove that

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + a + b + c \geq 4\sqrt{3}.$$

Solution: Using the AM-GM inequality, we have

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{a+b+c}.$$

Thus, it is sufficient to prove that

$$\frac{9}{a+b+c} + a + b + c \geq 4\sqrt{3}.$$

Equivalently, we need to prove

$$(a+b+c)^2 - 4\sqrt{3}\,(a+b+c) + 9 \geq 0.$$

Or

$$\left(a+b+c-\sqrt{3}\right)\left(a+b+c-3\sqrt{3}\right) \geq 0.$$

However, we have

$$a+b+c \leq \sqrt{3}\,\sqrt{a^2+b^2+c^2} = \sqrt{3}.$$

Hence $a + b + c - 3\sqrt{3} \leq 0$ and the inequality follows. ■

11. Find all triplets (a, b, c) of positive real numbers which satisfy the system of equations:

$$\begin{aligned} a+b+c &= 6, \\ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} &= 2 - \frac{4}{abc}. \end{aligned}$$

Solution: By the AM-GM inequality, we have

$$abc \leq \left(\frac{a+b+c}{3}\right)^3 = 8.$$

Thus

$$2 - \frac{4}{abc} \leq 2 - \frac{4}{8} = \frac{3}{2}.$$

On the other hand,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{a+b+c} = \frac{3}{2}.$$

Thus,

$$\frac{3}{2} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 2 - \frac{4}{abc} \leq \frac{3}{2}.$$

This shows that equality holds in the AM-GM inequality. Hence $a = b = c$, giving $(a, b, c) = (2, 2, 2)$. ■

12. Let a, b, c be real numbers such that $a^2 + b^2 + c^2 = 1$. Prove that

$$\frac{a^2}{1+2bc} + \frac{b^2}{1+2ca} + \frac{c^2}{1+2ab} \geq \frac{3}{5}.$$

Solution: Using the AM-GM inequality, we have $2bc \leq b^2 + c^2 = 1 - a^2$ and similar bounds hold for $2ca$, $2ab$. Thus,

$$\begin{aligned}\frac{a^2}{1+2bc} + \frac{b^2}{1+2ca} + \frac{c^2}{1+2ab} &\geq \frac{a^2}{2-a^2} + \frac{b^2}{2-b^2} + \frac{c^2}{2-c^2} \\ &= -3 + 2\left(\frac{1}{2-a^2} + \frac{1}{2-b^2} + \frac{1}{2-c^2}\right).\end{aligned}$$

Thus, it is sufficient to prove that

$$\frac{1}{2-a^2} + \frac{1}{2-b^2} + \frac{1}{2-c^2} \geq \frac{1}{2}\left(\frac{3}{5} + 3\right) = \frac{9}{5}.$$

This follows from AM-HM inequality:

$$\frac{1}{2-a^2} + \frac{1}{2-b^2} + \frac{1}{2-c^2} \geq \frac{9}{6-(a^2+b^2+c^2)} = \frac{9}{5}.$$

■

13. Let a, b, c and α, β, γ be positive real numbers such that $\alpha + \beta + \gamma = 1$. Prove that

$$\alpha a + \beta b + \gamma c + 2\sqrt{(\alpha\beta + \beta\gamma + \gamma\alpha)(ab + bc + ca)} \leq a + b + c.$$

Solution: Introduce new variables

$$x = \frac{a}{a+b+c}, \quad y = \frac{b}{a+b+c}, \quad z = \frac{c}{a+b+c}.$$

The inequality may be written in the form

$$\alpha x + \beta y + \gamma z + 2\sqrt{(\alpha\beta + \beta\gamma + \gamma\alpha)(xy + yz + zx)} \leq 1.$$

Observe that $x + y + z = 1$. Using the AM-GM inequality

$$\begin{aligned}&\alpha x + \beta y + \gamma z + 2\sqrt{(\alpha\beta + \beta\gamma + \gamma\alpha)(xy + yz + zx)} \\ &\leq \frac{\alpha^2 + x^2}{2} + \frac{\beta^2 + y^2}{2} + \frac{\gamma^2 + z^2}{2} + (\alpha\beta + \beta\gamma + \gamma\alpha) + (xy + yz + zx) \\ &= \frac{1}{2}(\alpha + \beta + \gamma)^2 + \frac{1}{2}(x + y + z)^2 = 1.\end{aligned}$$

■

14. Prove that for all real numbers a, b, the inequality

$$a^2 + b^2 + 1 > a\sqrt{b^2+1} + b\sqrt{a^2+1}$$

holds.

Solution: By the Cauchy-Schwarz inequality,

$$\left(a\sqrt{b^2+1} + b\sqrt{a^2+1}\right)^2 \le \left(a^2+b^2\right)\left(a^2+b^2+2\right).$$

But it is easy to see that $x(x+2) < (x+1)^2$ for any positive x. Thus

$$\left(a^2+b^2\right)\left(a^2+b^2+2\right) < \left(a^2+b^2+1\right)^2,$$

which gives the desired inequality. ■

15. For a fixed positive integer n, compute the minimum value of the sum

$$x_1 + \frac{x_2^2}{2} + \frac{x_3^3}{3} + \cdots + \frac{x_n^n}{n},$$

where $x_1, x_2, x_3, \ldots, x_n$ are positive real numbers such that

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_n} = n.$$

Solution: Let S denote the sum to be minimised. Define

$$H = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \quad \text{and} \quad w_j = \frac{1}{jH}, 1 \le j \le n.$$

Then $w_j > 0$, for $1 \le j \le n$, and $w_1 + w_2 + \cdots + w_n = 1$. Using the weighted AM-GM inequality, we obtain

$$\begin{aligned} \frac{S}{H} = \sum_{j=1}^{n} w_j x_j^j &\ge \prod_{j=1}^{n} \left(x_j^j\right)^{w_j} \\ &= \left(\prod_{j=1}^{n} x_j\right)^{1/H}. \end{aligned}$$

Using the GM-HM inequality, we also get

$$\prod_{j=1}^{n} x_j \ge \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_n}} = 1.$$

It follows that $S \ge H$, and equality holds if and only if $x_1 = x_2 = \cdots = x_n$. Thus the least value of S is $H = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. ■

16. Let a, b, c, d be positive real numbers such that $a+b+c+d \leq 1$. Prove that
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a} \leq \frac{1}{64abcd}.$$

Solution: We have
$$\begin{aligned} \frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a} &= \frac{a^2cd+b^2da+c^2ab+d^2bc}{abcd} \\ &= \frac{(ab+cd)(ad+bc)}{abcd}. \end{aligned}$$
Thus, it is sufficient to prove that
$$(ab+cd)(ad+bc) \leq \frac{1}{64}.$$
The AM-GM inequality gives
$$(a+b)(c+d) \leq \left(\frac{a+b+c+d}{2}\right)^2 \leq \frac{1}{4},$$
and
$$\begin{aligned} (ac+bd)(ad+bc) &\leq \left(\frac{ac+bd+ad+bc}{2}\right)^2 \\ &= \left(\frac{a+b)(c+d)}{2}\right)^2 \\ &\leq \frac{1}{8^2} = \frac{1}{64}. \end{aligned}$$
■

17. Let a, b, c be positive real numbers, all less than 1, such that $a+b+c=2$. Prove that
$$abc \geq 8(1-a)(1-b)(1-c).$$

Solution: Introducing new variables x, y, z by $x = 1-a$, $y = 1-b$ and $z = 1-c$, we see that x, y, z are positive and $x+y+z=1$. The required inequality is
$$(1-x)(1-y)(1-z) \geq 8xyz.$$
Equivalently, we have to prove that
$$xy+yz+zx \geq 9xyz.$$
This may be written in the form
$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq 9.$$
Since $x+y+z=1$, this follows from the AM-HM inequality. ■

18. Let a, b, c be three positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}+\frac{(2b+c+a)^2}{2b^2+(c+a)^2}+\frac{(2c+a+b)^2}{2c^2+(a+b)^2}\le 8.$$

Solution: We introduce λ, μ and ν by $2\lambda = b+c$, $2\mu = c+a$ and $2\nu = a+b$. Then

$$a = \mu+\nu-\lambda, \quad b = \nu+\lambda-\mu, \quad c = \lambda+\mu-\nu.$$

Thus, we obtain

$$2a+b+c = 2(\mu+\nu), \quad 2b+c+a = 2(\nu+\lambda), \quad 2c+a+b = 2(\lambda+\mu-\nu).$$

The inequality to be proved is:

$$\sum_{\text{cyclic}} \frac{4(\mu+\nu)^2}{2(\mu+\nu-\lambda)^2+4\lambda^2}\le 8.$$

However, we observe that

$$\frac{4(\mu+\nu)^2}{2(\mu+\nu-\lambda)^2+4\lambda^2}=\frac{2(\mu+\nu)^2}{(\mu+\nu-\lambda)^2+2\lambda^2}\le\frac{4(\mu+\nu)^2}{(\mu+\nu)^2+2\lambda^2}.$$

Here we have used $(\mu+\nu)^2 \le \left(\mu+\nu-\lambda\right)^2+2\lambda^2$. Using $(\mu+\nu)^2 \le 2(\mu^2+\nu^2)$, we get

$$\frac{4(\mu+\nu)^2}{(\mu+\nu)^2+2\lambda^2}=\frac{4}{1+\left(2\lambda^2/(\mu+\nu)^2\right)}\le\frac{4}{1+\left(\lambda^2/(\mu^2+\nu^2)\right)}=\frac{4(\mu^2+\nu^2)}{\mu^2+\nu^2+\lambda^2}.$$

Thus,

$$\sum_{\text{cyclic}} \frac{4(\mu+\nu)^2}{2(\mu+\nu-\lambda)^2+4\lambda^2}\le\sum_{\text{cyclic}}\frac{4(\mu^2+\nu^2)}{\mu^2+\nu^2+\lambda^2}=8.$$

■

19. Three positive real numbers a, b, c are such that $(1+a)(1+b)(1+c) = 8$. Prove that $abc \le 1$.

Solution: We know that $1+a \ge 2\sqrt{a}$, $1+b \ge 2\sqrt{b}$ and $1+c \ge 2\sqrt{c}$. Thus, we get

$$8 = (1+a)(1+b)(1+c) \ge 8\sqrt{abc}.$$

It follows that $abc \le 1$. ■

20. If a, b, c are the sides of a triangle, prove that

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\ge 0.$$

Solution: Taking $s-a=x$, $s-b=y$ and $s-c=z$, we see that $a=y+z$, $b=z+x$ and $c=x+y$. The inequality is

$$\sum_{\text{cyclic}}(y+z)^2(z+x)(y-x)\geq 0.$$

Expanding the left hand side, this reduces to

$$xy^3+yz^3+zx^3\geq x^2yz+xy^2z+xyz^2.$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}x^2yz+xy^2z+xyz^2 &= \sum_{\text{cyclic}}\left(x^{3/2}z^{1/2}\right)\left(x^{1/2}yz^{1/2}\right)\\ &\leq \left(x^3z+y^3x+z^3y\right)^{1/2}\left(xy^2z+xyz^2+x^2yz\right)^{1/2}.\end{aligned}$$

This implies that

$$x^2yz+xy^2z+xyz^2\leq xy^3+yz^3+zx^3.$$

This also follows from Muirhead's inequality, since $(2,1,1)\prec(3,0,1)$. ■

21. Let $a_1,a_2,\ldots,a_n$ be $n(\geq 2)$ real numbers whose sum is 1. Prove that

$$\sum_{j=1}^{n}\frac{a_j}{2-a_j}\geq\frac{n}{2n-1}.$$

Solution: Using the Cauchy-Schwarz inequality, we see that

$$\sum_{j=1}^{n}\frac{a_j}{2-a_j}\geq\frac{(a_1+a_2+\cdots+a_n)^2}{\sum_{j=1}^{n}a_j(2-a_j)}=\frac{1}{2-\sum_{j=1}^{n}a_j^2}.$$

Thus, it is sufficient to prove that

$$\frac{1}{2-\sum_{j=1}^{n}a_j^2}\geq\frac{n}{2n-1}.$$

This is equivalent to $n\sum_{j=1}^{n}a_j^2\geq 1$, which follows from the Cauchy-Schwarz inequality:

$$1=(a_1+a_2+\cdots+a_n)^2\leq n(a_1^2+a_2^2+\cdots+a_n^2).$$

■

22. Let $a_1,a_2,\ldots,a_n$ be n positive real numbers whose sum is 1. Prove that

$$\sum_{j=1}^{n}\frac{a_j^2}{a_j+a_{j+1}}\geq\frac{1}{2}.$$

(Here $a_{n+1}=a_1$.)

Solution: We observe that

$$\sum_{j=1}^{n} \frac{a_j^2 - a_{j+1}^2}{a_j + a_{j+1}} = \sum_{j=1}^{n} \left(a_j - a_{j+1}\right) = 0.$$

Thus, we get

$$\sum_{j=1}^{n} \frac{a_j^2}{a_j + a_{j+1}} = \sum_{j=1}^{n} \frac{a_{j+1}^2}{a_j + a_{j+1}}.$$

Hence, if S is the required sum, we have

$$\begin{aligned} 4S &= \sum_{j=1}^{n} \frac{2\left(a_j^2 + a_{j+1}^2\right)}{a_j + a_{j+1}} \\ &\geq \sum_{j=1}^{n} \frac{\left(a_j + a_{j+1}\right)^2}{a_j + a_{j+1}} \\ &= \sum_{j=1}^{n} \left(a_j + a_{j+1}\right) = 2. \end{aligned}$$

It follows that $S \geq 1/2$. ■

23. Let a, b, c, d be four positive real numbers. Prove that

$$\frac{1}{a} + \frac{4}{b} + \frac{9}{c} + \frac{16}{d} \geq \frac{100}{a+b+c+d}.$$

Solution: This follows from the Cauchy-Schwarz inequality:

$$\begin{aligned} (1+2+3+4)^2 &= \left(\frac{\sqrt{a}}{\sqrt{a}} + 2\frac{\sqrt{b}}{\sqrt{b}} + 3\frac{\sqrt{c}}{\sqrt{c}} + 4\frac{\sqrt{d}}{\sqrt{d}}\right)^2 \\ &\leq \left(\frac{1}{a} + \frac{4}{b} + \frac{9}{c} + \frac{16}{d}\right)\left(a+b+c+d\right). \end{aligned}$$

Thus, it follows that

$$\frac{1}{a} + \frac{4}{b} + \frac{9}{c} + \frac{16}{d} \geq \frac{100}{a+b+c+d}.$$

■

24. Let $a_1, a_2, \ldots, a_n$ be $n(>2)$ positive real numbers such that $a_1 + a_2 + \cdots + a_n = 1$ and $a_j < 1/2$ for each j, $1 \leq j \leq n$. Prove that

$$\sum_{j=1}^{n} \frac{a_j^2}{1 - 2a_j} \geq \frac{1}{n-2}.$$

Solution: Since $a_j < 1/2$ for each j, we see that $1 - 2a_j > 0$, for $1 \le j \le n$. Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\left(\sum_{j=1}^{n} a_j\right)^2 &= \left(\sum_{j=1}^{n} \frac{a_j}{\sqrt{1-2a_j}}\sqrt{1-2a_j}\right)^2 \\
&\le \left(\sum_{j=1}^{n} \frac{a_j^2}{1-2a_j}\right)\left(\sum_{j=1}^{n}(1-2a_j)\right) \\
&= (n-2)\left(\sum_{j=1}^{n} \frac{a_j^2}{1-2a_j}\right).
\end{aligned}$$

Hence it follows that

$$\sum_{j=1}^{n} \frac{a_j^2}{1-2a_j} \ge \frac{1}{n-2},$$

because of $\sum_{j=1}^{n} a_j = 1$. ■

25. Let $x_1, x_2, \ldots, x_n,\ y_1, y_2, \ldots, y_n$ be $2n$ positive real numbers such that $x_1 + x_2 + \cdots + x_n \ge x_1y_1 + x_2y_2 + \cdots + x_ny_n$, where $n \ge 2$ is an integer. Prove that

$$x_1 + x_2 + \cdots + x_n \le \frac{x_1}{y_1} + \frac{x_2}{y_2} + \cdots + \frac{x_n}{y_n}.$$

Solution: The Cauchy-Schwarz inequality gives

$$(x_1 + x_2 + \cdots + x_n)^2 \le \left(\frac{x_1}{y_1} + \frac{x_2}{y_2} + \cdots + \frac{x_n}{y_n}\right)\left(x_1y_1 + x_2y_2 + \cdots + x_ny_n\right).$$

The inequality $x_1y_1 + x_2y_2 + \cdots + x_ny_n \le x_1 + x_2 + \cdots + x_n$ gives the result. ■

26. If $x_1, x_2, \ldots, x_n$ are n positive real numbers, prove that

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+x_2^2+\cdots+x_n^2} < \sqrt{n}.$$

Solution: Using the Cauchy-Schwarz inequality, it suffices to prove that

$$\frac{x_1^2}{\left(1+x_1^2\right)^2} + \frac{x_2^2}{\left(1+x_1^2+x_2^2\right)^2} + \cdots + \frac{x_n^2}{\left(1+x_1^2+x_2^2+\cdots+x_n^2\right)^2} < 1.$$

But for $j \geq 2$, we have

$$\begin{aligned} &\frac{x_j^2}{\left(1+x_1^2+x_2^2+\cdots+x_j^2\right)^2} \\ &\qquad \leq \frac{x_j^2}{\left(1+x_1^2+x_2^2+\cdots+x_{j-1}^2\right)\left(1+x_1^2+x_2^2+\cdots+x_j^2\right)} \\ &\qquad = \frac{1}{1+x_1^2+x_2^2+\cdots+x_{j-1}^2} - \frac{1}{1+x_1^2+x_2^2+\cdots+x_j^2} \,. \end{aligned}$$

We also have

$$\frac{x_1^2}{\left(1+x_1^2\right)^2} \leq 1 - \frac{1}{\left(1+x_1^2\right)^2}.$$

Summing all these, we get

$$\begin{aligned} &\frac{x_1^2}{\left(1+x_1^2\right)^2} + \frac{x_2^2}{\left(1+x_1^2+x_2^2\right)^2} + \cdots + \frac{x_n^2}{\left(1+x_1^2+x_2^2+\cdots+x_n^2\right)^2} \\ &\qquad \leq 1 - \frac{1}{1+x_1^2+x_2^2+\cdots+x_n^2} < 1. \end{aligned}$$

■

27. If a, b, c are positive real numbers, prove that

$$3\left(a^2b+b^2c+c^2a\right)\left(ab^2+bc^2+ca^2\right) \geq abc(a+b+c)^3.$$

Solution: Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\left(a^2b+b^2c+c^2a\right)\left(ab^2+bc^2+ca^2\right) \\ &\qquad \geq \left(a^2\sqrt{bc}+b^2\sqrt{ca}+c^2\sqrt{ab}\right)^2 \\ &\qquad = abc\left(a^{3/2}+b^{3/2}+c^{3/2}\right)^2. \end{aligned}$$

Thus, it is sufficient to prove that

$$3\left(a^{3/2}+b^{3/2}+c^{3/2}\right)^2 \geq \left(a+b+c\right)^3.$$

This follows from Hölder's inequality: taking $p = 3/2$ and $q = 3$, we obtain

$$(a+b+c) \leq \left(a^{3/2}+b^{3/2}+c^{3/2}\right)^{2/3}\left(1+1+1\right)^{1/3},$$

which gives the desired inequality. ■

28. Let $P(x) = ax^2+bx+c$ be a quadratic polynomial with non-negative coefficients and let α be a positive real number. Prove that

$$P(\alpha)P(1/\alpha) \geq P(1)^2.$$

Solution: We have

$$\begin{aligned}
P(\alpha)P(1/\alpha) &= \left(a\alpha^2+b\alpha+c\right)\left(\frac{a}{\alpha^2}+\frac{b}{\alpha}+c\right) \\
&= a^2+b^2+c^2 \\
&\qquad +(ab+bc)\left(\alpha+\frac{1}{\alpha}\right)+ac\left(\alpha^2+\frac{1}{\alpha^2}\right) \\
&\geq a^2+b^2+c^2+2ab+2bc+2ac \\
&= (a+b+c)^2 = P(1)^2;
\end{aligned}$$

we have used the AM-GM inequality. ■

29. If a, b, c, d, e are positive reals, prove the inequality

$$\sum \frac{a}{b+c} \geq \frac{5}{2},$$

where the sum is taken cyclically over a, b, c, d, e.

Solution: The inequality is equivalent to

$$\frac{a^2}{ab+ac}+\frac{b^2}{bc+bd}+\frac{c^2}{cd+ce}+\frac{d^2}{de+da}+\frac{e^2}{ea+eb} \geq \frac{5}{2}.$$

Using the Cauchy-Schwarz inequality, we see that

$$\begin{aligned}
&(a+b+c+d+e)^2 \\
&\quad \leq \left(\frac{a^2}{ab+ac}+\frac{b^2}{bc+bd}+\frac{c^2}{cd+ce}+\frac{d^2}{de+da}+\frac{e^2}{ea+eb}\right) \\
&\quad \times \left(ab+ac+bc+bd+cd+ce+de+da+ea+eb\right).
\end{aligned}$$

Thus, we obtain

$$\frac{a^2}{ab+ac}+\frac{b^2}{bc+bd}+\frac{c^2}{cd+ce}+\frac{d^2}{de+da}+\frac{e^2}{ea+eb} \geq \frac{(a+b+c+d+e)^2 2}{\sum_{\text{sym}} ab},$$

where

$$\sum_{\text{sym}} ab = ab+ac+ad+ae+bc+bd+be+cd+ce+de.$$

Hence, it is sufficient to prove that

$$(a+b+c+d+e)^2 \geq \frac{5}{2}\sum_{\text{sym}} ab.$$

This is equivalent to

$$2(a^2+b^2+c^2+d^2+e^2) \geq ab+ac+ad+ae+bc+bd+be+cd+ce+de.$$

After multiplying by 2 on both sides, we may write this in the form

$$(a-b)^2+(a-c)^2+(a-d)^2+(a-e)^2+(b-c)^2+(b-d)^2 \\ +(b-e)^2+(c-d)^2+(c-e)^2+(d-e)^2 \geq 0.$$

Hence the result follows. ■

30. If a, b, c are the sides of an acute-angled triangle, prove that

$$\sum_{\text{cyclic}} \sqrt{a^2+b^2-c^2}\sqrt{a^2-b^2+c^2} \leq ab+bc+ca.$$

Solution: Let us introduce $a^2+b^2-c^2=z$, $b^2+c^2-a^2=x$ and $c^2+a^2-b^2=y$. Then we have

$$\begin{aligned}
&\sum_{\text{cyclic}} \sqrt{a^2+b^2-c^2}\sqrt{a^2-b^2+c^2} \\
&\quad= \sqrt{xy}+\sqrt{yz}+\sqrt{zx} \\
&\quad= \frac{1}{2}\left\{\sqrt{xy}+\sqrt{yz}+\sqrt{yz}+\sqrt{zx}+\sqrt{zx}+\sqrt{xy}\right\} \\
&\quad= \frac{1}{2}\left\{\sqrt{xy}+\sqrt{xz}+\sqrt{yz}+\sqrt{yx}+\sqrt{zx}+\sqrt{zy}\right\} \\
&\quad\leq \frac{1}{2}\left\{\sqrt{(x+y)(x+z)}+\sqrt{(y+z)(y+x)}+\sqrt{(z+x)(z+y)}\right\} \\
&\quad= \frac{1}{2}\left\{\sqrt{2c^2\cdot 2b^2}+\sqrt{2a^2\cdot 2c^2}+\sqrt{2b^2\cdot 2a^2}\right\} \\
&\quad= ab+bc+ca.
\end{aligned}$$

■

31. Let a, b, c be non-negative real numbers such that

$$\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}=2.$$

Prove that $ab+bc+ca \leq 3/2$.

Solution: We have

$$\frac{a^2}{a^2+1}+\frac{b^2}{b^2+1}+\frac{c^2}{c^2+1}=3-\left(\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\right)=1.$$

Using the Cauchy-Schwarz inequality, we observe that

$$\left(\frac{a^2}{a^2+1}+\frac{b^2}{b^2+1}+\frac{c^2}{c^2+1}\right)\left(a^2+1+b^2+1+c^2+1\right) \geq \left(a+b+c\right)^2.$$

Thus, we get

$$a^2 + b^2 + c^2 + 3 \geq (a + b + c)^2.$$

This simplifies to

$$ab + bc + ca \leq \frac{3}{2}.$$

■

32. Suppose a, b, c are positive real numbers. Prove that

$$3\Big(a + b + c\Big) \geq 8\Big(abc\Big)^{1/3} + \left(\frac{a^3 + b^3 + c^3}{3}\right)^{1/3}.$$

When does equality hold?

Solution: Using the concavity of the function $f(x) = x^{1/3}$ on the interval $(0, \infty)$, we obtain

$$\begin{aligned} 8\Big(abc\Big)^{1/3} + \left(\frac{a^3 + b^3 + c^3}{3}\right)^{1/3} &\leq 9\left(\frac{8abc + \frac{a^3+b^3+c^3}{3}}{9}\right)^{1/3} \\ &= 3\Big(24abc + a^3 + b^3 + c^3\Big)^{1/3} \end{aligned}$$

Thus, it is sufficient to prove that

$$24abc + a^3 + b^3 + c^3 \leq (a + b + c)^3.$$

This reduces to

$$a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2 \geq 6abc,$$

which is a consequence of the AM-GM inequality. Equality holds if and only if $a = b = c$.

■

33. Let c_1,c_2,c_3,...,c_n be n real numbers such that either $0 \leq c_j \leq 1$ for all j or $c_j \geq 1$ for all j, $1 \leq j \leq n$. Prove that the inequality

$$\prod_{j=1}^{n} \left(1 - p + pc_j\right) \leq 1 - p + p \prod_{j=1}^{n} c_j$$

holds, for any real p with $0 \leq p \leq 1$.

Solution: We use induction on n. The statement is immediate for $n = 1$. Suppose it is true for all $j \le n-1$. Since $0 \le p \le 1$ and $c_j \ge 0$ imply $1 - p + pc_j \ge 0$, the induction hypothesis gives

$$\begin{aligned}\prod_{j=1}^{n}(1-p+pc_j) &= \prod_{j=1}^{n-1}(1-p+pc_j)\cdot(1-p+pc_n)\\ &\le \Big(1-p+p\prod_{j=1}^{n-1}c_j\Big)\cdot\Big(1-p+pc_n\Big).\end{aligned}$$

The induction step is complete once the inequality

$$\Big(1-p+p\prod_{j=1}^{n-1}c_j\Big)\cdot\Big(1-p+pc_n\Big) \le 1-p+p\prod_{j=1}^{n}c_j$$

is proved. This is equivalent to

$$\Big(p-p^2\Big)\Big\{c_n+\prod_{j=1}^{n-1}c_j-1-c_n\prod_{j=1}^{n-1}c_j\Big\} \le 0.$$

Since $p - p^2 \ge 0$, this is equivalent to

$$\Big(\prod_{j=1}^{n-1}c_j-1\Big)\Big(c_n-1\Big) \ge 0.$$

The given conditions on c_j's now imply the result.

■

34. Let x_1, x_2, x_3, x_4 be real numbers in the interval $(0, 1/2]$. Prove that

$$\begin{aligned}&\frac{x_1x_2x_3x_4}{(1-x_1)(1-x_2)(1-x_3)(1-x_4)}\\ &\qquad\le \frac{x_1^4+x_2^4+x_3^4+x_4^4}{(1-x_1)^4+(1-x_2)^4+(1-x_3)^4+(1-x_4)^4}.\end{aligned}$$

Solution: The inequality may be written in the form

$$\frac{\sum(1-x_j)^4}{\prod(1-x_j)} \le \frac{\sum x_j^4}{\prod x_j}.$$

By symmetry, we may assume that $x_1 \ge x_2 \ge x_3 \ge x_4$. If we set $1 - x_j = y_j$, then $y_1 \le y_2 \le y_3 \le y_4$. An easy computation shows that

$$\frac{\sum x_j^4}{\prod x_j} - 4 = \frac{\big(x_1^2-x_2^2\big)^2+\big(x_3^2-x_4^2\big)^2+2\big(x_1x_2-x_3x_4\big)^2}{x_1x_2x_3x_4}.$$

Thus, it is sufficient to prove that

$$\frac{\left(x_1^2-x_2^2\right)^2+\left(x_3^2-x_4^2\right)^2+2\left(x_1x_2-x_3x_4\right)^2}{x_1x_2x_3x_4} \geq \frac{\left(y_1^2-y_2^2\right)^2+\left(y_3^2-y_4^2\right)^2+2\left(y_1y_2-y_3y_4\right)^2}{y_1y_2y_3y_4}.$$

Since $x_1+x_2 \leq 1$, we have $x_1-x_2 \geq \left(x_1^2-x_2^2\right)$ which may be written in the form $x_1y_1 \geq x_2y_2$. Suppose p and q are real numbers such that $p \geq q \geq 1$. Then it is easy to see that

$$p+\frac{1}{p} \geq q+\frac{1}{q}$$

holds. Taking $p = x_1/x_2$ and $q = y_2/y_1$, we obtain

$$\frac{x_1}{x_2}+\frac{x_2}{x_1} \geq \frac{y_2}{y_1}+\frac{y_1}{y_2}.$$

This may be written as

$$\frac{x_1^2+x_2^2}{x_1x_2} \geq \frac{y_1^2+y_2^2}{y_1y_2},$$

or equivalently in the form

$$\frac{\left(x_1+x_2\right)^2}{x_1x_2} \geq \frac{\left(y_1+y_2\right)^2}{y_1y_2}.$$

But observe that $\left(x_1-x_2\right)^2 = \left(y_1-y_2\right)^2$ and $x_3x_4 \leq \left(1-x_3\right)\left(1-x_4\right) = y_3y_4$. Thus we obtain

$$\begin{aligned}\frac{\left(x_1^2-x_2^2\right)^2}{x_1x_2x_3x_4} &= \frac{\left(x_1+x_2\right)^2}{x_1x_2}\cdot\frac{\left(x_1-x_2\right)^2}{x_3x_4}\\ &\geq \frac{\left(y_1+y_2\right)^2}{y_1y_2}\cdot\frac{\left(y_1-y_2\right)^2}{y_3y_4}\\ &= \frac{\left(y_1^2-y_2^2\right)^2}{y_1y_2y_3y_4}.\end{aligned}$$

Similarly,

$$\frac{\left(x_3^2-x_4^2\right)^2}{x_1x_2x_3x_4} \geq \frac{\left(y_3^2-y_4^2\right)^2}{y_1y_2y_3y_4}$$

may be obtained. Taking $p = x_1x_2/x_3x_4$ and $q = y_3y_4/y_1y_2$, it can be easily verified that $p \geq q \geq 1$. We hence also obtain

$$\frac{2\left(x_1x_2-x_3x_4\right)^2}{x_1x_2x_3x_4} \geq \frac{2\left(y_1y_2-y_3y_4\right)^2}{y_1y_2y_3y_4}.$$

Combining all these, we get the result. Equality holds if and only if $x_1 = x_2 = x_3 = x_4$. ∎

35. Let $x_1,x_2,x_3,\ldots,x_n$ be n real numbers such that $0 < x_j \le 1/2$. Prove that

$$\left(\prod_{j=1}^{n} x_j \Big/ \left(\sum_{j=1}^{n} x_j\right)^n\right) \le \left(\prod_{j=1}^{n} (1-x_j) \Big/ \left(\sum_{j=1}^{n} (1-x_j)\right)^n\right).$$

Solution: (By Kshipra Bawalkar) Since $0 < x_j \le 1/2$, we have $1/2 \le 1-x_j < 1$ for, $1 \le j \le n$. Thus $\sum(1-x_j)$, $\sum x_j$, $\prod x_j$ and $\prod(1-x_j)$ are all positive. The inequality may be written in the form

$$\frac{\prod_{j=1}^{n} x_j}{\prod_{j=1}^{n}(1-x_j)} \le \frac{\left(\sum_{j=1}^{n} x_j\right)^n}{\left(\sum_{j=1}^{n}(1-x_j)\right)^n}.$$

Let $A = (x_1 + x_2 + \cdots + x_n)/n$. Let x_l and x_s denote respectively the largest and the smallest among $x_1,x_2,x_3,\ldots,x_n$. We show that

$$\frac{\prod_{j=1}^{n} x_j}{\prod_{j=1}^{n}(1-x_j)} \le \frac{x_1x_2\cdots A\cdots(x_l+x_s-A)\cdots x_n}{(1-x_1)(1-x_2)\cdots(1-A)\cdots(1-x_l-x_s+A)\cdots(1-x_n)},$$

where the right side is obtained by replacing x_l and x_s respectively by A and $x_l + x_s - A$. We need to prove that

$$\frac{x_lx_s}{(1-x_l)(1-x_s)} \le \frac{A(x_l+x_s-A)}{(1-A)(1-x_l-x_s+A)}.$$

We first observe that $0 < x_l + x_s \le 1$. The above inequality simplifies to

$$\left(A^2 - A(x_l+x_s) + x_lx_s\right)\left(1-x_l-x_s\right) \le 0.$$

Since $1 - x_l - x_s \ge 0$, this is equivalent to

$$A^2 - A(x_l+x_s) + x_lx_s \le 0.$$

But

$$A^2 - A(x_l+x_s) + x_lx_s = (A-x_l)(A-x_s) \le 0,$$

since $x_l \le A \le x_s$. Now consider the set

$$\left\{x_1, x_2 \ldots, A, \ldots, (x_l+x_s-A), \ldots, x_n\right\}$$

obtained by replacing x_l, x_s respectively by A, (x_l+x_s-A). This again satisfies the hypothesis of the problem, as $0 < A \le 1/2$ and $0 < x_l + x_s - A \le 1/2$

(since $x_s \geq A$ and $x_l \leq A$). Moreover the average of the set does not change by this process and it remains A. We may continue this process and after at most $n-1$ steps all the numbers in the set are equal to A, the average. Thus we obtain the inequality

$$\frac{\prod_{j=1}^{n} x_j}{\prod_{j=1}^{n}(1-x_j)} \leq \frac{A^n}{(1-A)^n}.$$

However

$$\begin{aligned}
\frac{A^n}{(1-A)^n} &= \frac{(nA)^n}{(n-nA)^n} \\
&= \frac{\left(\sum_{j=1}^{n} x_j\right)^n}{\left(n-\sum_{j=1}^{n} x_j\right)^n} \\
&= \frac{\left(\sum_{j=1}^{n} x_j\right)^n}{\left(\sum_{j=1}^{n}(1-x_j)\right)^n}.
\end{aligned}$$

It follows that

$$\frac{\prod_{j=1}^{n} x_j}{\prod_{j=1}^{n}(1-x_j)} \leq \frac{\left(\sum_{j=1}^{n} x_j\right)^n}{\left(\sum_{j=1}^{n}(1-x_j)\right)^n}.$$

Alternate Solution I:

We give here another solution based on induction on n. We use the classical technique of proving the result whenever n is a power of 2 and then fill up the remaining gaps by coming back. The case $n=2$ is simple. The inequality required is

$$\frac{x_1x_2}{(x_1+x_2)^2} \leq \frac{(1-x_1)(1-x_2)}{(2-x_1-x_2)^2}.$$

Simplification gives the equivalent inequality:

$$(x_1-x_2)^2(1-x_1-x_2) \geq 0.$$

Since $x_1+x_2 \leq 1$, the result follows for $n=2$. We also observe that equality holds if and only if $x_1 = x_2$. We use the case $n=2$ to prove the result for $n=4$. Consider positive numbers $x_1, x_2, x_3, x_4 \in (0, 1/2]$. Taking $y_1 = (x_1+x_2)/2$ and $y_2 = (x_3+x_4)/2$, the result for $n=2$ gives

$$\frac{y_1y_2}{(y_1+y_2)^2} \leq \frac{(1-y_1)(1-y_2)}{(2-y_1-y_2)^2}.$$

Substituting for y_1 and y_2, this reduces to

$$\frac{x_1x_2x_3x_4\big(x_1+x_2\big)^2\big(x_3+x_4\big)^2}{\Big(\sum_{j=1}^4 x_j\Big)^4} \le \frac{x_1x_2x_3x_4\big(2-x_1-x_2\big)^2\big(2-x_3-x_4\big)^2}{\Big(\sum_{j=1}^4 \big(1-x_j\big)\Big)^4}.$$

However, using the result for $n=2$, we also have

$$\begin{aligned} x_1x_2\big(2-x_1-x_2\big)^2 &\le \big(1-x_1\big)\big(1-x_2\big)\big(x_1+x_2\big)^2 \\ x_3x_4\big(2-x_3-x_4\big)^2 &\le \big(1-x_3\big)\big(1-x_4\big)\big(x_3+x_4\big)^2. \end{aligned}$$

Using these on the right side and effecting some cancellations $\big(x_1+x_2\big)^2\big(x_3+x_4\big)^2$, we get

$$\frac{\prod_{j=1}^4 x_j}{\Big(\sum_{j=1}^4 x_j\Big)^4} \le \frac{\prod_{j=1}^4 \big(1-x_j\big)}{\Big(\sum_{j=1}^4 \big(1-x_j\big)\Big)^4}.$$

This proves the result for $n=4$. Using induction on k, this proves the inequality for all n of the form $n=2^k$.

Take any n and fix k such that $2^k \le n < 2^{k+1}$. Let A be the average of these n numbers; $A = \big(x_1+x_2+\cdots+x_n\big)/n$. We consider 2^{k+1} numbers

$$x_1, x_2, \dots, x_n, A, A, \dots, A,$$

where A appears $2^{k+1}-n$ times in the above sequence. We apply the inequality for these 2^{k+1} numbers. (Note that $0 < A \le 1/2$.) Thus we obtain

$$\frac{x_1x_2\cdots x_nA^{2^{k+1}-n}}{\Big(\sum_{j=1}^n x_j + \big(2^{k+1}-n\big)A\Big)^{2^{k+1}}} \le \frac{\big(1-x_1\big)\big(1-x_2\big)\cdots\big(1-x_n\big)\big(1-A\big)^{2^{k+1}-n}}{\Big(\sum_{j=1}^n \big(1-x_j\big) + \big(2^{k+1}-n\big)\big(1-A\big)\Big)^{2^{k+1}}}.$$

This simplifies to

$$\frac{x_1x_2\cdots x_n}{A^n} \le \frac{\big(1-x_1\big)\big(1-x_2\big)\cdots\big(1-x_n\big)}{\big(1-A\big)^n},$$

which is the required inequality for n.

Alternate Solution II:

Here is another solution using Jensen's inequality for the convex function

$$f(x) = \log\left(\frac{1}{x} - 1\right),$$

defined on the interval $(0, 1/2)$. Writing $f(x) = \log(1-x) - \log x$, it is easy to check that

$$\begin{aligned} f'(x) &= -\frac{1}{1-x} - \frac{1}{x}, \\ f''(x) &= -\frac{1}{(1-x)^2} + \frac{1}{x^2} \\ &= \frac{1-2x}{x^2\left(1-x^2\right)} > 0, \end{aligned}$$

for $x \in (0, 1/2)$. Hence f is convex on $(0, 1/2)$. This implies that

$$\log\left(\frac{1}{\sum_{j=1}^{n}(x_j/n)} - 1\right) \leq \frac{1}{n}\sum_{j=1}^{n}\log\left(\frac{1}{x_j} - 1\right).$$

The exponentiation gives

$$\left(\frac{n}{\sum_{j=1}^{n} x_j} - 1\right)^n \leq \prod_{j=1}^{n}\left(\frac{1}{x_j} - 1\right).$$

Some simplification leads to the required inequality. ∎

36. Consider a sequence $\langle a_n \rangle$ of real numbers satisfying $a_{j+k} \leq a_j + a_k$. Prove that

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \geq a_n,$$

for all n.

Solution: We have for any n,

$$a_n \leq a_j + a_{n-j}, \quad 1 \leq j \leq (n-1).$$

Summing over j, we obtain

$$(n-1)a_n \leq 2\sum_{j=1}^{n-1} a_j.$$

We prove the given inequality by induction on n. It is obviously true for $n = 1$. Suppose it is true for all $k < n$. Thus, we have

$$b_k = a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_k}{k} \geq a_k,$$

for all $k < n$. Summing these over k, we get

$$\sum_{k=1}^{n-1} b_k \geq \sum_{k=1}^{n-1} a_k.$$

But, we have

$$\sum_{k=1}^{n-1} b_k = a_1 + \left(a_1 + \frac{a_2}{2}\right) + \left(a_1 + \frac{a_2}{2} + \frac{a_3}{3}\right) + \cdots + \left(a_1 + \frac{a_2}{2} + \cdots + \frac{a_{n-1}}{n-1}\right).$$

This simplifies to

$$\sum_{k=1}^{n-1} b_k = (n-1)a_1 + \frac{n-2}{2}a_2 + \cdots + \frac{n-(n-1)}{n-1}a_{n-1}.$$

We thus have

$$(n-1)a_1 + \frac{n-2}{2}a_2 + \cdots + \frac{n-(n-1)}{n-1}a_{n-1} \geq a_1 + a_2 + \cdots + a_{n-1}.$$

Adding $a_1 + a_2 + \cdots + a_n$ to both sides and simplifying, we get

$$n\left(a_1 + \frac{a_2}{2} + \cdots + \frac{a_{n-1}}{n-1}\right) + a_n \geq 2\left(a_1 + a_2 + \cdots + a_{n-1}\right) + a_n.$$

But we note that $2\left(a_1 + a_2 + \cdots + a_{n-1}\right) + a_n \geq (n-1)a_n + a_n = na_n$. Dividing by n, we get

$$a_1 + \frac{a_2}{2} + \cdots + \frac{a_{n-1}}{n-1} + \frac{a_n}{n} \geq a_n.$$

This completes the induction step and hence the inequality is valid for all n.

■

37. For positive real numbers x, y, z, prove the inequality

$$\sum \frac{x}{x + \sqrt{(x+y)(x+z)}} \leq 1,$$

where the sum is taken cyclically over x, y, z.

Solution: We observe that

$$\sqrt{(x+y)(x+z)} \geq \sqrt{xy} + \sqrt{xz}.$$

In fact, squaring on both the sides, this reduces to

$$x^2 + yz \geq 2x\sqrt{yz}.$$

which is immediate from the AM-GM inequality. Thus

$$\begin{aligned}\sum_{\text{cyclic}} \frac{x}{x+\sqrt{(x+y)(x+z)}} &\leq \sum_{\text{cyclic}} \frac{x}{x+\sqrt{xy}+\sqrt{xz}} \\ &= \sum_{\text{cyclic}} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{y}+\sqrt{z}} = 1.\end{aligned}$$

■

38. Let x, y be non-negative real numbers such that $x+y=2$. Prove the inequality

$$x^3y^3\left(x^3+y^3\right) \leq 2.$$

Solution: As a consequence of the AM-GM inequality, we have

$$xy \leq \left(\frac{x+y}{2}\right)^2 = 1.$$

Thus, we obtain $0 < xy \leq 1$. We write

$$\begin{aligned}x^3y^3\left(x^3+y^3\right) &= \left(xy\right)^3\left(x+y\right)\left(x^2-xy+y^2\right) \\ &= 2\left(xy\right)^3\left(\left(x+y\right)^2-3xy\right) \\ &= 2\left(xy\right)^3\left(4-3xy\right).\end{aligned}$$

Thus we need to prove that

$$\left(xy\right)^3\left(4-3xy\right) \leq 1.$$

Putting $z = xy$, this inequality reduces to

$$z^3\left(4-3z\right) \leq 1,$$

for $0 < z \leq 1$. We can prove this in different ways. We can put the inequality in the form

$$3z^4-4z^3+1 \geq 0.$$

Here the expression on the left hand side factors in to $(z-1)^2\left(3z^2+2z+1\right)$ and $\left(3z^2+2z+1\right)$ is positive since its discriminant $D=-8<0$. Or applying the AM-GM inequality to the positive reals $4-3z, z, z, z$, we obtain

$$z^3\left(4-3z\right) \leq \left(\frac{4-3z+3z}{4}\right)^4 \leq 1.$$

■

39. A convex quadrilateral $ABCD$ is inscribed in a unit circle. Suppose its sides satisfy the inequality $AB \cdot BC \cdot CD \cdot DA \geq 4$. Prove that the quadrilateral is a square.

Solution: Let $AB = a, BC = b, CD = c, DA = d$. We are given that $abcd \geq 4$. Using Ptolemy's theorem and the fact that each diagonal cannot exceed the diameter of the circle, we get $ac + bd = AC \cdot BD \leq 4$. But an application of the AM-GM inequality gives

$$ac + bd \geq 2\sqrt{abcd} \geq 2\sqrt{4} = 4.$$

We conclude that $ac+bd = 4$. This forces $AC \cdot BD = 4$, giving $AC = BD = 2$. Each of AC and BD is thus a diameter. This implies that $ABCD$ is a rectangle. Note that

$$(ac - bd)^2 = (ac + bd)^2 - 4abcd \leq 16 - 16 = 0$$

and hence $ac = bd = 2$. Thus we get $a = c = \sqrt{ac} = \sqrt{2}$, and similarly $b = d = \sqrt{2}$. It now follows that $ABCD$ is a square. ■

40. Let $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ be two sets of reals such that $0 < h \leq a_j \leq H$ and $0 < m \leq b_j \leq M$ for some reals h, H, m, M. Prove that

$$1 \leq \frac{(\sum a_j{}^2)(\sum b_j{}^2)}{(\sum a_j b_j)^2} \leq \frac{1}{4}\left(\sqrt{\frac{HM}{hm}} + \sqrt{\frac{hm}{HM}}\right)^2.$$

Solution: The first inequality is simply a restatement of the Cauchy-Schwarz inequality. Define $c_1 = \min\{a_1, a_2, \ldots, a_n\}$ and having defined $c_1, c_2, \ldots,$ c_{j-1}, define

$$c_j = \min\Big\{\{a_1, a_2, \ldots, a_n\} \setminus \{c_1, c_2, \ldots, c_{j-1}\}\Big\},$$

for $2 \leq j \leq n$. Then $c_1 \leq c_2 \leq \cdots \leq c_n$ and $\{c_1, c_2, \ldots, c_n\}$ is simply the rearrangement of $\{a_1, a_2, \ldots, a_n\}$ in increasing order. Similarly, we get the rearrangement $\{d_1, d_2, \ldots, d_n\}$ of the elements in $\{b_1, b_2, \ldots, b_n\}$ in decreasing order: $d_1 \geq d_2 \geq \cdots \geq d_n$. Using the rearrangement inequality, we have

$$\sum_{j=1}^{n} c_j d_j \leq \sum_{j=1}^{n} a_j b_j.$$

We may assume that not all the a_j's are equal and not all the b_j's are equal so that $c_1 < c_n$ and $d_n < d_1$. Hence we obtain

$$c_n d_1 - c_1 d_n = (c_n - c_1)d_1 + c_1(d_1 - d_n) > 0.$$

If $k \geq 2$, define u_k and v_k by

$$c_k^2 = u_k c_1^2 + v_k c_n^2, \quad d_k^2 = u_k d_1^2 + v_k d_n^2.$$

Then $u_k \geq 0$, $v_k \geq 0$ and

$$c_k d_k = \left(u_k c_1^2 + v_k c_n^2\right)^{1/2}\left(u_k d_1^2 + v_k d_n^2\right)^{1/2} \geq u_k c_1 d_1 + v_k c_n d_n,$$

using the Cauchy-Schwarz inequality. We observe that $u_k = 0$ for some k implies that $c_k = c_{k+1} = \cdots = c_n$, $d_k = d_{k+1} = \cdots = d_n$ and $v_k = 1$. Similar is the case when $v_k = 0$. If $u_k > 0$ and $v_k > 0$, then $c_k d_k > u_k c_1 d_1 + v_k c_n d_n$. Hence

$$1 \le \frac{\left(\sum {c_j}^2\right)\left(\sum {d_j}^2\right)}{\left(\sum c_j d_j\right)^2} \le \frac{\left(Pc_1^2 + Qc_n^2\right)\left(Pd_1^2 + Qd_n^2\right)}{\left(Pc_1 d_1 + Qc_n d_n\right)^2}$$

where $P = 1 + u_2 + \cdots + u_n$ and $Q = v_1 + v_2 + \cdots + v_{n-1} + 1$. The expression on the right hand side is equal to

$$1 + PQ\left(\frac{c_n d_1 - c_1 d_n}{Pc_1 d_1 + Qc_n d_n}\right)^2.$$

Using $Pc_1 d_1 + Qc_n d_n \ge 2\sqrt{PQc_1 d_1 c_n d_n}$, we obtain

$$\begin{aligned} 1 + PQ\left(\frac{c_n d_1 - c_1 d_n}{Pc_1 d_1 + Qc_n d_n}\right)^2 \quad &\le \quad \left(\frac{\sqrt{\frac{c_n d_1}{c_1 d_n}} + \sqrt{\frac{c_1 d_n}{c_n d_1}}}{2}\right)^2 \\ &\le \quad \left(\frac{\sqrt{\frac{HM}{hm}} + \sqrt{\frac{hm}{HM}}}{2}\right)^2. \end{aligned}$$

Since $\sum a_j^2 = \sum c_j^2$, $\sum b_j^2 = \sum d_j^2$ and

$$\sum_{j=1}^{n} c_j d_j \le \sum_{j=1}^{n} a_j b_j,$$

we get the desired inequality. ■

41. Let $f : [0, a] \to \mathbb{R}$ be a convex function. Consider n points $x_1, x_2, x_3, \ldots, x_n$ in $[0, a]$ such that $\sum_{j=1}^{n} x_j$ is also in $[0, a]$. Prove that

$$\sum_{j=1}^{n} f(x_j) \le f\left(\sum_{j=1}^{n} x_j\right) + (n-1) f(0).$$

Solution: Using the convexity of f, we have

$$f\left(\frac{x_1(x_1 + x_2) + x_2 \cdot 0}{x_1 + x_2}\right) \le \frac{x_1}{x_1 + x_2} f(x_1 + x_2) + \frac{x_2}{x_1 + x_2} f(0).$$

Thus it follows that

$$f(x_1) \le \frac{x_1}{x_1 + x_2} f(x_1 + x_2) + \frac{x_2}{x_1 + x_2} f(0).$$

Interchanging x_1 and x_2, we also obtain

$$f(x_2) \leq \frac{x_2}{x_1+x_2} f(x_1+x_2) + \frac{x_1}{x_1+x_2} f(0).$$

Adding these two, we get

$$f(x_1) + f(x_2) \leq f(x_1+x_2) + f(0).$$

Now we use the induction on n. Suppose the result holds for $(n-1)$; say

$$f(x_1) + f(x_2) + \cdots + f(x_{n-1}) \leq f(x_1 + x_2 + \cdots + x_{n-1}) + (n-2) f(0),$$

for all points $x_1, x_2, \ldots, x_{n-1}$ in $[0, a]$ such that their sum is also in $[0, a]$. Now consider any n points x_1,x_2,x_3,. . .,x_n in $[0, a]$ such that $\sum_{j=1}^{n} x_j$ is also in $[0, a]$. Then

$$\begin{aligned} & f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + f(x_n) \\ & \quad \leq \quad f(x_1 + x_2 + \cdots + x_{n-1}) + (n-2) f(0) + f(x_n) \\ & \quad \leq \quad f(x_1 + x_2 + \cdots + x_{n-1} + x_n) + f(0) + (n-2) f(0) \\ & \quad = \quad f(x_1 + x_2 + \cdots x_{n-1} + x_n) + (n-1) f(0). \end{aligned}$$

This completes induction and hence the proof as well. ■

42. For any natural number n, prove

$$\binom{2n}{n} \sqrt{3n} < 4^n.$$

Solution: We show by induction that

$$\binom{2n}{n} \sqrt{3n+1} \leq 4^n.$$

For $n = 1$, the result is obvious. Suppose it holds for all $k \leq n$. Then we have

$$\begin{aligned} \binom{2n+2}{n+1} \sqrt{3(n+1)+1} &= \frac{2(2n+1)}{n+1} \binom{2n}{n} \sqrt{\frac{3n+4}{3n+1}} \cdot \sqrt{3n+1} \\ &\leq \frac{2(2n+1)}{n+1} \sqrt{\frac{3n+4}{3n+1}} \cdot 4^n, \end{aligned}$$

by induction hypothesis. However, we can show that

$$\frac{2n+1}{n+1} \sqrt{\frac{3n+4}{3n+1}} \leq 2.$$

In fact, this statement is equivalent to

$$(2n+1)^2 (3n+4) \leq 4(n+1)^2 (3n+1).$$

This further reduces to

$$12n^3 + 28n^2 + 19n + 4 \leq 12n^3 + 28n^2 + 20n + 4,$$

which is true for all natural numbers n. Hence we obtain

$$\binom{2n+2}{n+1}\sqrt{3(n+1)+1} \leq 4^{n+1},$$

which completes induction. Since $\sqrt{3n} < \sqrt{3n+1}$, we get the required inequality. ■

43. Let a, b, c be positive real numbers and let x be a non-negative real number. Prove that

$$a^{x+2} + b^{x+2} + c^{x+2} \geq a^x bc + ab^x c + abc^x.$$

Solution: We observe that

$$a^x b^2 + a^2 b^x \leq a^{x+2} + b^{x+2}.$$

This follows easily, by writing the inequality in the form $\left(a^x - b^x\right)\left(a^2 - b^2\right) \geq 0$. Thus we have,

$$\begin{aligned}
&2\left(a^{x+2} + b^{x+2} + c^{x+2}\right) \\
&\quad= \left(a^{x+2} + b^{x+2}\right) + \left(b^{x+2} + c^{x+2}\right) + \left(c^{x+2} + a^{x+2}\right) \\
&\quad\geq \left(a^x b^2 + b^x a^2\right) + \left(b^x c^2 + c^x b^2\right) + \left(c^x a^2 + a^x c^2\right) \\
&\quad= a^x\left(b^2 + c^2\right) + b^x\left(c^2 + a^2\right) + c^x\left(a^2 + b^2\right) \\
&\quad\geq 2\left(a^x bc + b^x ca + c^x ab\right).
\end{aligned}$$

Hence the result follows. ■

44. Let $(a_1, a_2, \ldots, a_n)$, $(b_1, b_2, \ldots, b_n)$, and $(c_1, c_2, \ldots, c_n)$ be three sequences of positive real numbers. Prove that

$$\sum_{j=1}^{n} a_j b_j c_j \leq \left(\sum_{j=1}^{n} a_j^3\right)^{1/3} \left(\sum_{j=1}^{n} b_j^3\right)^{1/3} \left(\sum_{j=1}^{n} c_j^3\right)^{1/3}.$$

Solution: Using Hölder's inequality with exponents $p = 3$ and $q = 3/2$, we get

$$\sum_{j=1}^{n} a_j b_j c_j \leq \left(\sum_{j=1}^{n} a_j^3\right)^{1/3} \left(\sum_{j=1}^{n} \left(b_j c_j\right)^{3/2}\right)^{2/3}.$$

Using the Cauchy-Schwarz inequality, we also get

$$\sum_{j=1}^{n} \left(b_j c_j\right)^{3/2} \leq \left(\sum_{j=1}^{n} b_j^3\right)^{1/2} \left(\sum_{j=1}^{n} c_j^3\right)^{1/2}.$$

Combining these two, we get the desired inequality. ■

45. Prove for any three real numbers a, b, c, the inequality

$$3\left(a^2-a-1\right)\left(b^2-b-1\right)\left(c^2-c+1\right) \geq \left(abc\right)^2 - abc + 1.$$

Solution: Consider the function $f(a) = 3\left(a^2-a+1\right)^3 - \left(a^6+a^3+1\right)$ for $a \in \mathbb{R}$. Some computation shows that $f(1) = 0$, $f'(1) = 0$, $f''(1) = 0$, $f'''(1) = 0$, but $f^{(iv)}(1) \neq 0$. Thus $(a-1)^4$ divides $f(a)$ but not any higher power of $(a-1)$. Expanding $f(a)$, we obtain

$$f(a) = (a-1)^4\left(2a^2 - a + 2\right).$$

But the discriminant of $2a^2 - a + 2$ is -15 and hence $2a^2 - a + 2 \geq 0$ for all real a. It follows that

$$3\left(a^2-a+1\right)^3 \geq a^6 + a^3 + 1,$$

for all real a. Using the previous problem, we obtain

$$\begin{aligned} \left((xyz)^2 + (xyz) + 1\right)^3 &= \left(x^2 \cdot y^2 \cdot z^2 + x \cdot y \cdot z + 1 \cdot 1 \cdot 1\right)^3 \\ &\leq \left(x^6 + x^3 + 1\right)\left(y^6 + y^3 + 1\right)\left(z^6 + z^3 + 1\right). \end{aligned}$$

Hence we get

$$\left((xyz)^2 + (xyz) + 1\right)^3 \leq 27\left(x^2 - x + 1\right)^3\left(y^2 - y + 1\right)^3\left(z^2 - z + 1\right)^3.$$

But for arbitrary reals a, b, c, we have

$$(abc)^2 - (abc) + 1 \leq |abc|^2 + |abc| + 1.$$

Thus it follows that

$$\left((abc)^2 - (abc) + 1\right)^3 \leq 27\left(a^2 - a + 1\right)\left(b^2 - b + 1\right)\left(c^2 - c + 1\right).$$

■

46. In a triangle ABC, show that

$$\frac{1}{\sin A} + \frac{1}{\sin B} \geq \frac{8}{3 + 2\cos C}.$$

Find the conditions for equality.

Solution: The function $f(x) = 1/\sin x$ is convex on $[0, \pi]$. Jensen's theorem shows that

$$\frac{1}{\sin A} + \frac{1}{\sin B} \geq \frac{2}{\sin\left(\frac{A+B}{2}\right)} = \frac{2}{\cos(C/2)}.$$

Thus, it suffices to prove that

$$\frac{1}{\cos(C/2)} \geq \frac{4}{3 + 2\cos C}.$$

Using $\cos C = 2\cos^2(C/2) - 1$, this reduces to

$$\Big(2\cos(C/2) - 1\Big)^2 \geq 0.$$

Equality holds here only if $\cos(C/2) = 1/2$ which is equivalent to $C = 120°$. Equality holds in Jensen's inequality if and only if $A = B$. Thus equality holds in the inequality only for that triangle with $C = 120°$ and $A = B = 30°$. ■

47. Consider a real polynomial of the form

$$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + 1,$$

where $a_j \geq 0$ for $1 \leq j \leq (n-1)$. Suppose $P(x) = 0$ has n real roots. Prove that $P(2) \geq 3^n$.

Solution: Since the coefficients are all non-negative, we see that $P(t) \geq 1$, for any $t \geq 0$. Thus $P(x) = 0$ has only negative roots; let these be $-\alpha_1$, $-\alpha_2$, $\ldots$, $-\alpha_n$, where α_j's are all positive. We have

$$P(x) = (x + \alpha_1)(x + \alpha_2)\cdots(x + \alpha_n).$$

We also observe that

$$\alpha_1\alpha_2\cdots\alpha_n = 1.$$

Using the AM-GM inequality, we obtain

$$2 + \alpha_j = 1 + 1 + \alpha_j \geq 3\big(\alpha_j\big)^{1/3}, \text{ for } 1 \leq j \leq n.$$

Hence

$$\begin{aligned} P(2) &= (2+\alpha_1)(2+\alpha_2)\cdots(2+\alpha_n) \\ &\geq 3^n\Big(\alpha_1\alpha_2\cdots\alpha_n\Big)^{1/3} \\ &= 3^n. \end{aligned}$$

In fact, this may be generalised as follows: for any $t \geq 0$, the inequality $p(t) \geq (1+t)^n$. We use the weighted AM-GM inequality. As in the above solution, we have

$$P(t) = (t + \alpha_1)(t + \alpha_2)\cdots(t + \alpha_n).$$

Consider a general term $(t+\alpha_k)$. We have

$$\begin{aligned} t+\alpha_k &= \frac{t\cdot 1+1\cdot\alpha_k}{t+1}(t+1) \\ &\geq (t+1)(\alpha_k)^{1/(t+1)}, \end{aligned}$$

by the weighted AM-GM inequality. Thus

$$P(t) \geq (t+1)^n(\alpha_1\alpha_2\cdots\alpha_n)^{1/(t+1)} = (t+1)^n.$$

■

48. Let $a_1<a_2<a_3<\ldots<a_n$ be n positive integers. Prove that

$$\big(a_1+a_2+a_3+\cdots+a_n\big)^2 \leq a_1^3+a_2^3+a_3^3+\cdots+a_n^3.$$

Solution: We prove this by induction on n. The inequality is clear for $n=1$: $a_1>0$ implies that $a_1\geq 1$ and hence $a_1^2\leq a_1^3$. Suppose it holds for all distinct positive integers $0<a_1<a_2<\cdots<a_n$ and $a_{n+1}>a_n$. Then we see that

$$a_{n+1}\geq 1+a_n, \quad a_{n+1}\geq a_{n-1}+2, \quad \ldots, a_{n+1}\geq a_1+n.$$

Adding all these inequalities, we get

$$na_{n+1}\geq \big(a_1+a_2+\cdots+a_n\big)+\frac{n(n+1)}{2}.$$

Thus

$$\big(a_1+a_2+\cdots+a_n\big)\leq na_{n+1}-\frac{n(n+1)}{2}.$$

Hence

$$\begin{aligned} \left(\sum_{j=1}^{n+1}a_j\right)^2 &= \left(\sum_{j=1}^{n}a_j+a_{n+1}\right)^2 \\ &= \left(\sum_{j=1}^{n}a_j\right)^2+2a_{n+1}\left(\sum_{j=1}^{n}a_j\right)+a_{n+1}^2 \\ &\leq \sum_{j=1}^{n}a_j^3+2a_{n+1}\left(na_{n+1}-\frac{n(n+1)}{2}\right)+a_{n+1}^2 \\ &= \sum_{j=1}^{n}a_j^3+\big(2n+1\big)a_{n+1}^2-n(n+1)a_{n+1}. \end{aligned}$$

Thus it is sufficient to prove that

$$\begin{aligned} &\big(2n+1\big)a_{n+1}^2-n(n+1)a_{n+1}\leq a_{n+1}^3 \\ \Longleftrightarrow\quad &\big(2n+1\big)a_{n+1}-n(n+1)\leq a_{n+1}^2 \\ \Longleftrightarrow\quad &a_{n+1}^2-\big(2n+1\big)a_{n+1}+n(n+1)\geq 0 \\ \Longleftrightarrow\quad &\big(a_{n+1}-n\big)\big(a_{n+1}-n-1\big)\geq 0. \end{aligned}$$

Since a_j are integers and $0 < a_1 < a_2 < \cdots < a_n < a_{n+1}$, it follows that $a_j \geq j$, for $1 \leq j \leq n+1$. Hence $a_{n+1} \geq n+1$, and thus

$$\big(a_{n+1} - n\big)\big(a_{n+1} - n - 1\big) \geq 0.$$

■

49. Consider a sequence a_1,a_2,a_3,. . .,a_n of positive real numbers which add up to 1, where $n \geq 2$ is an integer. Prove that for any positive real numbers x_1,x_2,x_3,. . .,x_n with $\sum_j^n x_j = 1$, the inequality

$$2\sum_{j<k} x_j x_k \leq \frac{n-2}{n-1} + \sum_{j=1}^{n} \frac{a_j x_j^2}{1-a_j},$$

holds.

Solution: We have

$$1 = \left(\sum_{j=1}^{n} x_j\right)^2 = \sum_{j=1}^{n} x_j^2 + 2\sum_{j<k} x_j x_k.$$

Thus we have to prove that

$$1 - \sum_{j=1}^{n} x_j^2 \leq \frac{n-2}{n-1} + \sum_{j=1}^{n} \frac{a_j x_j^2}{1-a_j}.$$

This is equivalent to

$$\frac{1}{n-1} \leq \sum_{j=1}^{n} \frac{x_j^2}{1-a_j}.$$

This follows from the Cauchy-Schwarz inequality:

$$\begin{aligned} 1 = \left(\sum_{j=1}^{n} x_j\right)^2 &\leq \left(\sum_{j=1}^{n} \frac{x_j^2}{1-a_j}\right)\left(\sum_{j=1}^{n}(1-a_j)\right) \\ &= (n-1)\left(\sum_{j=1}^{n} \frac{x_j^2}{1-a_j}\right). \end{aligned}$$

■

50. Let x_1, x_2, x_3, x_4 be four positive real numbers such that $x_1x_2x_3x_4 = 1$. Prove that

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 \geq \min\left\{x_1 + x_2 + x_3 + x_4, \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right\}.$$

Solution: Let us put $x_1^3+x_2^3+x_3^3+x_4^3 = A$ and $a_j = A - x_j^3$. Then it is easy to check that

$$A = \frac{1}{3}\Big(a_1+a_2+a_3+a_4\Big).$$

Moreover,

$$\frac{1}{3}a_1 = \frac{x_2^3+x_3^3+x_4^3}{3} \geq x_2x_3x_4 = \frac{1}{x_1}.$$

Similarly, we get

$$\frac{1}{3}a_2 \geq \frac{1}{x_2}, \quad \frac{1}{3}a_3 \geq \frac{1}{x_3}, \quad \frac{1}{3}a_4 \geq \frac{1}{x_4}.$$

Thus it follows that

$$A = \frac{1}{3}\Big(a_1+a_2+a_3+a_4\Big) \geq \frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}.$$

Using Chebyshev's inequality, we also have

$$\frac{x_1^3+x_2^3+x_3^3+x_4^3}{4} \geq \frac{x_1^2+x_2^2+x_3^2+x_4^2}{4}\cdot\frac{x_1+x_2+x_3+x_4}{4}.$$

But

$$\frac{x_1^2+x_2^2+x_3^2+x_4^2}{4} \geq 1,$$

by the AM-GM inequality. Thus we also obtain

$$A \geq x_1+x_2+x_3+x_4.$$

It now follows that

$$A \geq \max\left\{x_1+x_2+x_3+x_4, \frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right\}.$$

■

51. Let $\{x\}$ denote the fractional part of x; i.e., $\{x\} = x - [x]$. Prove for any positive integer n,

$$\sum_{j=1}^{n}\{\sqrt{j}\} \leq \frac{n^2-1}{2}.$$

Solution: We use induction on n. For $n = 1$, the result is clear. Suppose it holds for some n. We prove its validity for $n+1$ also. We observe that $n < \sqrt{n^2+j} < n+1$ for $1 \leq j \leq 2n$. Thus

$$\{\sqrt{n^2+j}\} = \sqrt{n^2+j} - n < \sqrt{n^2+j+(j/2n)^2} - n = \frac{j}{2n}.$$

Hence

$$\begin{aligned}\sum_{j=1}^{(n+1)^2}\{\sqrt{j}\} &= \sum_{j=1}^{n^2}\{\sqrt{j}\}+\sum_{j=n^2+1}^{(n+1)^2}\{\sqrt{j}\}\\ &\le \frac{n^2-1}{2}+\frac{1}{2n}\sum_{j=1}^{2n} j+0\\ &= \frac{n^2-1}{2}+\frac{1}{2n}\cdot\frac{2n(2n+1)}{2}\\ &= \frac{(n+1)^2-1}{2}.\end{aligned}$$

This completes the inductive step and hence the proof as well. ■

52. If a, b, c are positive real numbers, prove that

$$\frac{a^2}{(a+b)(a+c)}+\frac{b^2}{(b+c)(b+a)}+\frac{c^2}{(c+a)(c+b)}\ge\frac{3}{4}.$$

Solution: After expanding, the inequality takes the form

$$4\big(a^2b+b^2c+c^2a\big)+4\big(ab^2+bc^2+ca^2\big)\ge 3\big(a+b\big)\big(b+c\big)\big(c+a\big).$$

This reduces to

$$\big(a^2b+b^2c+c^2a\big)+\big(ab^2+bc^2+ca^2\big)\ge 6abc,$$

which is a consequence of the AM-GM inequality. ■

53. Suppose a, b, c are the sides of a triangle. Prove that

$$a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c)\le 3abc.$$

Solution: Introducing x, y, z by

$$b+c-a=2x,\quad c+a-b=2y,\quad a+b-c=2z,$$

we get $a=y+z$, $b=z+x$, $c=x+y$ and the inequality takes the form

$$2x(y+z)^2+2y(z+x)^2+2z(x+y)^2\le 3(x+y)(y+z)(z+x).$$

Note that x, y, z are positive. The inequality may be reduced to

$$6xyz\le x^2y+y^2z+z^2x+xy^2+yz^2+zx^2,$$

which directly follows from the AM-GM inequality. ■

54. Let x, y, z be positive real numbers such that $xyz\ge xy+yz+zx$. Prove that

$$xyz\ge 3(x+y+z).$$

Solution: We have

$$\begin{aligned}
&\left(xy+yz+zx\right)^2-3xyz(x+y+z)\\
&\quad= (xy)^2+(yz)^2+(zx)^2-(xy)(yz)-(yz)(zx)-(zx)(xy)\\
&\quad= \frac{1}{2}\left\{(xy-yz)^2+(yz-zx)^2+(zx-xy)^2\right\}.
\end{aligned}$$

Hence it follows that

$$3xyz(x+y+z)\le\left(xy+yz+zx\right)^2\le(xyz)^2.$$

Since $xyz>0$, we obtain $3(x+y+z)\le xyz$. ■

55. Let b_1,b_2,b_3,...,b_n be n non-negative real numbers and let b denote the sum of these numbers. Prove that

$$\sum_{j=1}^{n-1}b_jb_{j+1}\le\frac{b^2}{4}.$$

Solution: Let b_k denote the largest among b_j's. Then we have

$$\begin{aligned}
\sum_{j=1}^{n-1}b_jb_{j+1} &= \sum_{j=1}^{k-1}b_jb_{j+1}+\sum_{j=k}^{n-1}b_jb_{j+1}\\
&\le b_k\sum_{j=1}^{k-1}b_j+b_k\sum_{j=k}^{n-1}b_{j+1}\\
&= b_k\left(b-b_k\right)\\
&= \frac{b^2}{4}-\left(\frac{b}{2}-b_k\right)^2\le\frac{b^2}{4}.
\end{aligned}$$

■

56. Let a,b,c,d be complex numbers such that $ac\neq 0$. Prove that

$$\frac{\max\left\{|ac|,|ad+bc|,|bd|\right\}}{\max\left\{|a|,|b|\right\}\left\{|c|,|d|\right\}}\ge\frac{-1+\sqrt{5}}{2}.$$

Solution: Let us take $r=b/a$, $s=d/c$ and $k=\left(\sqrt{5}-1\right)/2$. Then $k^2=1-k$ and $0<k<1$. We have to show that

$$\max\left\{1,|r+s|,|rs|\right\}\ge k\max\left\{1,|r|\right\}\max\left\{1,|s|\right\}.$$

We consider several cases.

Case 1. $|r|\ge 1$ and $|s|\ge 1$.

In this case

$$\max\left\{1, |r+s|, |rs|\right\} \geq |rs| > k|r||s| = k\max\left\{1, |r|\right\}\max\left\{1, |s|\right\}.$$

Case 2. $|r| < 1$ and $|s| < 1$.

We obtain

$$\max\left\{1, |r+s|, |rs|\right\} \geq 1 > k = k\max\left\{1, |r|\right\}\max\left\{1, |s|\right\}.$$

Case 3. $|r| < 1$ and $|s| \geq 1$.

We have to show that

$$\max\left\{1, |r+s|, |rs|\right\} \geq k|s|.$$

If either $|r+s| \geq k|s|$ or $1 \geq k|s|$, the result follows. Suppose $|r+s| < k|s|$ and $1 < k|s|$. Then

$$|r| + |r+s| = |-r| + |r+s| \geq |s|,$$

and hence

$$|r| \geq |s| - |r+s| > |s| - k|s| = k^2|s|.$$

Thus, we obtain

$$|r||s| > k^2|s|^2 > k|s|$$

since $k|s| > 1$. This shows that

$$\max\left\{1, |r+s|, |rs|\right\} \geq k|s|.$$

By symmetry, the result follows in the case $|r| \geq 1$ and $|s| < 1$. ■

57. Let x, y, z be three real numbers in the interval $[0,1]$ such that $xyz = (1-x)(1-y)(1-z)$. Find the least possible value of $x(1-z)+y(1-x)+z(1-y)$.

Solution: We show that

$$x(1-z) + y(1-x) + z(1-y) \geq \frac{3}{4}.$$

If $x = 0$, we see that either $y = 1$ or $z = 1$ and the inequality follows. Similar is the case when $y = 0$ or $z = 0$. We may hence assume that $x > 0$, $y > 0$ and $z > 0$. Now put

$$\frac{1-x}{x} = a, \quad \frac{1-y}{y} = b, \quad \frac{1-z}{z} = c.$$

Then $abc = 1$ and we have to prove

$$\sum_{\text{cyclic}} \frac{c}{(1+a)(1+c)} \geq \frac{3}{4}.$$

This is equivalent to

$$4\left(\sum_{\text{cyclic}} a + \sum_{\text{cyclic}} ab\right) \geq 3\left(1 + \sum_{\text{cyclic}} a + \sum_{\text{cyclic}} ab + abc\right).$$

This reduces to

$$\sum_{\text{cyclic}} a + \sum_{\text{cyclic}} ab \geq 6.$$

However, we have

$$\begin{aligned} a+b+c &\geq 3\Big(abc\Big)^{1/3} = 3, \\ ab+bc+ca &\geq 3\Big(abc\Big)^{2/3} = 3. \end{aligned}$$

These two together give

$$\sum_{\text{cyclic}} a + \sum_{\text{cyclic}} ab \geq 6.$$

Equality holds if and only if $a = b = c$, which is equivalent to $x = y = z$. ■

58. Let $x_1, x_2, x_3, \ldots, x_n$ be non-negative real numbers such that

$$\sum_{j=1}^{n} \frac{1}{1+x_j} \leq 1.$$

Prove that $x_1 x_2 x_3 \cdots x_n \geq (n-1)^n$.

Solution: Introduce $y_j = 1/(1+x_j)$, for $1 \leq j \leq n$. Then we get $x_j = \big(1-y_j\big)/y_j$, $1 \leq j \leq n$. The condition translates to $\sum_{j=1}^{n} y_j \leq 1$ and the conclusion we have to derive is

$$\prod_{j=1}^{n} \left(\frac{1-y_j}{y_j}\right) \geq (n-1)^n.$$

If we use the AM-HM inequality, we get

$$\sum_{j=1}^{n} \frac{1}{y_j} \geq \frac{n^2}{\sum_{j=1}^{n} y_j} \geq n.$$

Using the AM-GM inequality, we get

$$\begin{aligned}
\prod_{j=1}^{n}\left(\frac{1-y_j}{y_j}\right) &\geq \left(\frac{1}{n}\sum_{j=1}^{n}\frac{1-y_j}{y_j}\right)^n \\
&= \left(\sum_{j=1}^{n}\frac{1}{n}\left(\frac{1}{y_j}-1\right)\right)^n \\
&= \left(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{y_j}-1\right)^n \\
&\geq \left(\frac{n^2}{n}-1\right)^n \\
&= (n-1)^n.
\end{aligned}$$

Alternate solution:

We introduce the polynomial

$$P(z) = \big(z+x_1\big)\big(z+x_2\big)\cdots\big(z+x_n\big).$$

Then

$$\sum_{j=1}^{n}\frac{1}{1+x_j} = \frac{P'(1)}{P(1)}.$$

The given condition is $P'(1) \leq P(1)$. We now write

$$P(z) = z^n + \sigma_1 z^{n-1} + \sigma_2 z^{n-2} + \cdots + \sigma_n,$$

where σ's are elementary symmetric functions in $x_1, x_2, x_3, \ldots, x_n$. Then the derivative is given by

$$P'(z) = nz^{n-1} + (n-1)\sigma_1 z^{n-2} + (n-2)\sigma_2 z^{n-3} + \cdots + \sigma_{n-1}.$$

The condition $P'(1) \leq P(1)$ is now equivalent to

$$\sigma_n \geq (n-1) + (n-2)\sigma_1 + (n-3)\sigma_2 + \cdots + \sigma_{n-2}.$$

Now we use the inequalities

$$\sigma_j \geq \binom{n}{j}\big(\sigma_n\big)^{j/n}, \quad 1 \leq j \leq n.$$

Taking $r = \big(\sigma_n\big)^{1/n}$, we obtain

$$r^n - \binom{n}{n-2}r^{n-2} - 2\binom{n}{n-3}r^{n-3} - \cdots - (n-2)\binom{n}{1}r - (n-1) \geq 0.$$

Consider the polynomial

$$Q(x) = x^n - \binom{n}{n-2}x^{n-2} - 2\binom{n}{n-3}x^{n-3} - \cdots - (n-2)\binom{n}{1}x - (n-1).$$

We observe that $Q(x)$ has only one change of sign for $x > 0$. Hence the number of positive roots of $Q(x) = 0$ cannot exceed 1. However, it is an easy exercise to check that $Q(n-1) = 0$. Thus, it follows that $Q(x) \geq 0$ in $(0, \infty)$ if and only if $x \geq (n-1)$. Since $Q(r) \geq 0$, we conclude that $r \geq (n-1)$. This gives $\sigma_n \geq (n-1)^n$. ■

59. For positive real numbers a, b, c, prove the inequality

$$3\left(a + \sqrt{ab} + \sqrt[3]{abc}\right) \leq 4(a+b+c).$$

Solution: Here is a beautiful solution to the problem. We write

$$a + \sqrt{ab} + \sqrt[3]{abc} = a + \frac{1}{2}\sqrt{a \cdot 4b} + \frac{1}{4}\sqrt[3]{a \cdot 4b \cdot 16c}.$$

Using the AM-GM inequality, we have

$$\begin{aligned} a + \frac{1}{2}\sqrt{a \cdot 4b} + \frac{1}{4}\sqrt[3]{a \cdot 4b \cdot 16c} & \\ &\leq a + \frac{1}{4}(a+4b) + \frac{1}{12}(a+4b+16c) \\ &= \frac{4}{3}(a+b+c). \end{aligned}$$

■

60. Show that for any two natural numbers m, n, the inequality

$$\frac{1}{m+n+1} - \frac{1}{(m+1)(n+1)} \leq \frac{4}{45}$$

holds.

Solution: Suppose we introduce

$$f(m,n) = \frac{1}{m+n+1} - \frac{1}{(m+1)(n+1)}.$$

Then it is easy to see that $f(1,1) = f(1,2) = 1/12 < 4/45$. Now it is not hard to prove that

$$\frac{1}{(m+1)(n+1)} \geq \frac{4}{(m+n+2)^2}.$$

In fact, this is just equivalent to $(m-n)^2 \geq 0$. Thus if $k = m+n+2$, then

$$\begin{aligned} f(m,n) &= \frac{1}{m+n+1} - \frac{1}{(m+1)(n+1)} \\ &\leq \frac{1}{m+n+1} - \frac{4}{(m+n+2)^2} \\ &= \frac{1}{k-1} - \frac{4}{k^2}. \end{aligned}$$

Consider the function $g(k)$ defined for natural numbers $k \geq 2$ by

$$g(k) = \frac{1}{k-1} - \frac{4}{k^2}.$$

We show that it is a non-increasing function for $k \geq 6$. In fact for $k \geq l$,

$$\begin{aligned} g(k) - g(l) &= \frac{1}{k-1} - \frac{4}{k^2} - \frac{1}{l-1} + \frac{4}{l^2} \\ &= \frac{l-k}{(l-1)(k-1)} - 4\frac{(l-k)(l+k)}{l^2k^2} \\ &= (l-k)\frac{(lk-2l-2k)^2 - 4l - 4k}{(l-1)(k-1)l^2k^2}. \end{aligned}$$

Since $l-k \leq 0$, we have to show that $lk - 2(l+k) \geq 2\sqrt{l+k}$ for $l \geq 6$ and $k \geq 6$. For $l \geq 6$ and $k \geq 6$, we have $(l-4) \geq 2$ and $(k-4) \geq 2$ and hence $(l-4)(k-4) \geq (l-4)+(k-4)$. This gives $lk \geq 5(l+k) - 24$. Thus

$$lk - 2(l+k) \geq 3(l+k) - 24.$$

It is sufficient to prove that $3(l+k) - 24 \geq 2\sqrt{l+k}$, whenever $l \geq 6$ and $k \geq 6$. Consider the quadratic function

$$q(t) = 3t^2 - 2t - 24.$$

Then $q(t) \geq 0$ if and only if $t \geq (1+\sqrt{73})/3$. Since

$$\sqrt{l+k} \geq \sqrt{12} \geq \frac{1+\sqrt{73}}{3},$$

for $l \geq 6$ and $k \geq 6$, it follows that

$$3(l+k) - 2\sqrt{l+k} - 24 \geq 0.$$

This shows that $f(m,n) \leq g(6) = 4/45$ for all m, n such that $m+n+2 \geq 6$.

■

61. If a, b are two positive real numbers, prove that

$$a^b + b^a > 1.$$

Solution: If either $a > 1$ or $b > 1$, then the result holds. Thus it is sufficient to consider the case $0 < a < 1$ and $0 < b < 1$. Define $f(x) = a^x + x^a - 1$ for $x \in (0,1)$, where $a \in (0,1)$ is fixed. Then

$$f(0) = 0, f(1) = a > 0, f'(x) = a^x \ln a + ax^{a-1}.$$

Suppose for some $x \in (0,1)$, the inequality $a^x + x^a \leq 1$ holds. Then $f(x) \leq 0$, $f(0) = 0$ and $f(1) > 0$. Using Rolle's theorem, we can find $b \in (0,1)$ such that $f(b) \leq 0$ and $f'(b) = 0$. This implies that

$$a^b \ln a + ab^{a-1} = 0, \quad a^b + b^a - 1 \leq 0.$$

Simplification gives

$$1 - \frac{b}{a} \ln a - a^{-b} \leq 0.$$

We show that this is not true for any $a, b \in (0,1)$. Consider the function

$$g(x) = 1 - \frac{\ln a}{a} x - a^{-x},$$

where $a \in (0,1)$ is fixed. It is easy to compute

$$\begin{aligned} g'(x) &= -\frac{\ln a}{a} + a^{-x} \ln a = \ln a \left(a^{-x} - \frac{1}{a} \right), \\ g''(x) &= -a^x \left(\ln a \right)^2. \end{aligned}$$

Thus $g''(x) < 0$ for all $x \in (0,1)$. This implies that $g'(x)$ is a decreasing function in $(0,1)$. Hence $g'(x) > g'(1) = 0$ for all $x \in (0,1)$. But then $g(x)$ is an increasing function on $(0,1)$. Hence $g(b) > g(0) = 0$. Thus, there is no $b \in (0,1)$ such that $g(b) \leq 0$. ∎

62. Let a, b be positive real numbers such that $a + b = 1$ and let p be a positive real. Prove that

$$\left(a + \frac{1}{a} \right)^p + \left(b + \frac{1}{b} \right)^p \geq \frac{5^p}{2^{p-1}}.$$

Solution: Suppose $p > 1$. If q is the conjugate index of p, i.e., $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then Hölder's inequality gives

$$\left(a + \frac{1}{a} + b + \frac{1}{b} \right) \leq \left\{ \left(a + \frac{1}{a} \right)^p + \left(b + \frac{1}{b} \right)^p \right\}^{1/p} 2^{1/q}.$$

However, $a + b = 1$ and

$$\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{1}{ab} \geq \frac{1}{\left(\frac{a+b}{2} \right)^2} = 4.$$

Thus, we obtain

$$5 \leq \left\{ \left(a + \frac{1}{a} \right)^p + \left(b + \frac{1}{b} \right)^p \right\}^{1/p} 2^{(p-1)/p},$$

since $q = (p-1)/p$. This proves that

$$\left(a + \frac{1}{a} \right)^p + \left(b + \frac{1}{b} \right)^p \geq \frac{5^p}{2^{p-1}},$$

for $p > 1$. If $p = 1$, then the inequality to be proved is

$$a + \frac{1}{a} + b + \frac{1}{b} \geq 5,$$

which is immediate using $a + b = 1$ and $\dfrac{1}{a} + \dfrac{1}{b} \geq 4$.

Suppose $0 < p < 1$. Consider the function

$$f(x) = \left(x + \frac{1}{x} \right)^p + \left(1 - x + \frac{1}{1-x} \right)^p,$$

for $x \in (0, 1)$. Its derivative is

$$f'(x) = p\left(1 - \frac{1}{x^2}\right)\left(x + \frac{1}{x}\right)^{p-1} + p\left(-1 + \frac{1}{(1-x)^2}\right)\left(1 - x + \frac{1}{1-x}\right)^{p-1}.$$

For $0 < x < 1/2$, we observe that

$$x + \frac{1}{x} > 1 - x + \frac{1}{1-x}.$$

In fact, this is equivalent to $(1 - 2x)\big(-1 + 1/x(1-x)\big) > 0$, which may be seen to be true since $1 - 2x > 0$ and $x(1-x) \leq 1/4$. Since $0 < p < 1$, it follows that

$$\left(x + \frac{1}{x} \right)^{p-1} < \left(1 - x + \frac{1}{1-x} \right)^{p-1}.$$

This gives

$$\begin{aligned} f'(x) &< p\left(1 - x + \frac{1}{1-x}\right)^{p-1} \left\{ 1 - \frac{1}{x^2} - 1 + \frac{1}{(1-x)^2} \right\} \\ &= p\left(1 - x + \frac{1}{1-x}\right)^{p-1} \frac{2x-1}{x^2(1-x)^2} \\ &< 0. \end{aligned}$$

Similarly, we can prove that $f'(x) > 0$ for $1/2 < x < 1$. It follows that $f(x)$ is decreasing in $(0, 1/2)$ and increasing in $(1/2, 1)$. Thus $f(x) \geq f(1/2)$ for all $x \in (0, 1)$. But

$$f(1/2) = (5/2)^p + (5/2)^p = \frac{5^p}{2^{p-1}}.$$

We conclude that

$$f(x) \geq \frac{5^p}{2^{p-1}},$$

for all $x \in (0,1)$. ■

63. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Solution: A solution using Schur's inequality was given in chapter 2 (Example 2.29). Here we give two more solutions.

Alternate Solution 1.

Since a, b, c are positive real numbers such that $abc = 1$, we can find positive real numbers x, y, z such that $a = x/y$, $b = y/z$ and $c = z/x$. The inequality reduces to

$$(z + x - y)(x + y - z)(y + z - x) \leq xyz.$$

This is symmetric in x, y, z. Hence it is sufficient to consider the case $x \leq y \leq z$. It follows that $y+z-x > 0$ and $z+x-y > 0$. If $x+y-z \leq 0$, then the left-side is non-positive and the right-side is positive; hence the inequality holds. Suppose $x + y - z > 0$. Then x, y, z are the sides of a triangle. Let its area, circum-radius, in-radius, semi-perimeter be respectively Δ, R, r, s. The inequality may be written in the equivalent form:

$$(x + y + z)(z + x - y)(x + y - z)(y + z - x) \leq xyz(x + y + z).$$

But the left-side is $16\Delta^2$ and the right-side is $8R\Delta s$. Thus the equivalent inequality is

$$16\Delta^2 \leq 8R\Delta s.$$

Using $\Delta = rs$, this reduces to $2r \leq R$, which is a standard result(Euler's inequality).

Alternate Solution 2.

If $a - 1 + (1/b) \leq 0$, then we see that $a < 1$ and $b > 1$. Hence we obtain

$$b - 1 + \frac{1}{c} > 0, \quad c - 1 + \frac{1}{a} > 0.$$

Thus, the product on the left hand side of the inequality becomes negative and forces the inequality. So is the case when either $b - 1 + (1/c) \leq 0$ or $c-1+(1/a) \leq 0$. We may thus assume that $a-1+(1/b) > 0$, $b-1+(1/c) > 0$ and $c - 1 + (1/a) > 0$. We observe that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) = (ab - b + 1)(bc - c + 1) + (ca - a + 1).$$

Suppose
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \le a+b+c.$$
This implies that $ab+bc+ca \le a+b+c$. Using the AM-GM inequality, we obtain
$$\begin{aligned}(ab-b+1)(bc-c+1)+(ca-a+1) &\le \left(\frac{ab+bc+ca-a-b-c+3}{3}\right)^3 \\ &\le 1.\end{aligned}$$

If
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} > a+b+c,$$
we introduce the variables $\alpha = 1/a$, $\beta = 1/b$ and $\gamma = 1/c$. We see that α, β and γ are positive reals with $\alpha\beta\gamma = 1$. Moreover, these satisfy
$$\frac{1}{\alpha}+\frac{1}{\gamma}+\frac{1}{\beta} \le \alpha+\gamma+\beta.$$
Applying the above reasoning to α, γ and β, we obtain
$$\left(\alpha-1+\frac{1}{\gamma}\right)\left(\gamma-1+\frac{1}{\beta}\right)\left(\beta-1+\frac{1}{\alpha}\right) \le 1.$$
This reduces to
$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) \le 1.$$

■

64. Let x, y, z be real numbers in the interval $[-1, 2]$ such that $x+y+z=0$. Prove that
$$\sqrt{\frac{(2-x)(2-y)}{(2+x)(2+y)}}+\sqrt{\frac{(2-y)(2-z)}{(2+y)(2+z)}}+\sqrt{\frac{(2-z)(2-x)}{(2+z)(2+x)}} \ge 3.$$

Solution: The inequality we need to prove is
$$\sum_{\text{cyclic}} \frac{\sqrt{4-x^2}\sqrt{4-y^2}}{(2+x)(2+y)} \ge 3. \tag{64.1}$$
Using the condition $x+y+z=0$, we obtain
$$\begin{aligned}3(4-x^2) = 12-3x^2 &= 12-2x^2-(y+z)^2 \\ &= (y-z)^2+2\left(6-\sum_{\text{cyclic}} x^2\right) \\ &= (y-z)^2+4\left(3+\sum_{\text{cyclic}} xy\right).\end{aligned}$$

We observe that

$$3+\sum_{\text{cyclic}} xy=\sum_{\text{cyclic}}(1+x)(1+y)\geq 0.$$

Thus, we get

$$\begin{aligned}
&\sqrt{4-x^2}\sqrt{4-y^2}\\
&= \frac{1}{3}\left(\sqrt{(y-z)^2+4\left(3+\sum_{\text{cyclic}} xy\right)}\right)\left(\sqrt{(x-z)^2+4\left(3+\sum_{\text{cyclic}} xy\right)}\right)\\
&\geq \frac{1}{3}\left[(y-z)(x-z)+4\left(3+\sum_{\text{cyclic}} xy\right)\right]\\
&= \frac{1}{3}(12+5xy+3xz+3yz+z^2)\\
&= \frac{1}{3}(12+5xy+2xz+2yz)\\
&= (4+xy)+\frac{2}{3}\left(\sum_{\text{cyclic}} xy\right),
\end{aligned}$$

where we have used the Cauchy-Schwarz inequality and the given relation $x+y+z=0$. Using this estimate we obtain

$$\begin{aligned}
&\sum_{\text{cyclic}}\frac{\left(4-x^2\right)^{\frac{1}{2}}\left(4-y^2\right)^{\frac{1}{2}}}{(2+x)(2+y)}\\
&\qquad= \frac{\sum_{\text{cyclic}}\left(4-x^2\right)^{\frac{1}{2}}\left(4-y^2\right)^{\frac{1}{2}}(2+z)}{\prod(2+x)}\\
&\qquad\geq \frac{1}{\prod(2+x)}\left\{\sum_{\text{cyclic}}(4+xy)(2+z)+\frac{2}{3}\Big(\sum_{\text{cyclic}} xy\Big)\Big(\sum_{\text{cyclic}}(2+z)\Big)\right\}\\
&\qquad= \frac{1}{\prod(2+x)}\left\{\sum_{\text{cyclic}}(8+4z+2xy+xyz)+\frac{2}{3}\Big(\sum_{\text{cyclic}} xy\Big)6\right\}\\
&\qquad= \frac{1}{\prod(2+x)}\left\{24+2\Big(\sum_{\text{cyclic}} xy\Big)+3xyz+4\Big(\sum_{\text{cyclic}} xy\Big)\right\}\\
&\qquad= \frac{1}{\prod(2+x)}\left\{24+6(\sum_{\text{cyclic}} xy)+3xyz\right\},
\end{aligned}$$

where $\prod(2+x)=(2+x)(2+y)(2+z)$. However

$$24+6\left(\sum_{\text{cyclic}} xy\right)+3xyz=3(2+x)(2+y)(2+z).$$

We get the desired inequality (64.1). ■

65. Let $\langle a_n \rangle$ be a sequence of distinct positive integers. Prove that

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \geq \sum_{k=1}^{n} \frac{1}{k},$$

for every positive integer n.

Solution: We use the rearrangement theorem (Theorem 4 on page 21). Fix n and consider the first n elements $a_1, a_2, \ldots, a_n$. Let $\sigma(1), \sigma(2), \ldots, \sigma(n)$ be a permutation of $1, 2, \ldots, n$ such that

$$a_{\sigma(1)} < a_{\sigma(2)} < \cdots < a_{\sigma(n)}.$$

Observe that

$$\frac{1}{1^2} \geq \frac{1}{2^2} \geq \cdots \geq \frac{1}{n^2}.$$

Using theorem 4, it follows that

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \geq \sum_{k=1}^{n} \frac{a_{\sigma(k)}}{k^2}.$$

However, $a_{\sigma(k)}$ is a positive integer for each k. Since $\langle a_{\sigma(k)} \rangle$ is an increasing sequence, it may be concluded that $a_{\sigma(k)} \geq k$ for $1 \leq k \leq n$. This gives

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \geq \sum_{k=1}^{n} \frac{k}{k^2} = \sum_{k=1}^{n} \frac{1}{k}.$$

■

66. Let x, y, z be non-negative real numbers such that $x + y + z = 1$. Prove that

$$0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}.$$

Solution: Let us introduce $f(x, y, z) = xy + yz + zx - 2xyz$. Assume $x \leq y \leq z$, which is permissible by the symmetry of the expression. Using $x + y + z = 1$, it may be concluded that $x \leq 1/2$. Write

$$f(x, y, z) = x(y + z) + yz(1 - 2x).$$

Since $1 - 2x \geq 0$, it follows that $f(x, y, z) \geq 0$. On the other hand

$$(1 - x)^2 = (y + z)^2 \geq 4yz,$$

by the AM-GM inequality. Thus

$$\begin{aligned} f(x,y,z)-\frac{7}{27} &= x(y+z)+yz(1-2x)-\frac{7}{27} \\ &\le x(1-x)+\frac{(1-x)^2(1-2x)}{4}-\frac{7}{27} \\ &= -\frac{(3x-1)^2(6x+1)}{108}\le 0. \end{aligned}$$

This proves the other part of the inequality.

Alternate Solution:
As in the previous solution, we may assume $x \le y \le z$. And the same method shows that $f(x,y,z) \ge 0$. Introducing new variables a, b, c by

$$x = a+\frac{1}{3}, \quad y = b+\frac{1}{3}, \quad z = c+\frac{1}{3},$$

the expression $f(x,y,z)$ takes the form

$$f(x,y,z)=\frac{7}{27}+\frac{1}{3}\Big(bc-a^2-6abc\Big).$$

The conditions on a, b, c are

$$a \le b \le c, \quad a+b+c=0, \quad -\frac{1}{3}\le a,b,c\le\frac{2}{3}.$$

Thus it follows that either $a \le b \le 0 \le c$ or $-(1/3) \le a \le 0 \le b \le c$. In the first case $bc \le 0$, and $abc \ge 0$. Hence $bc - a^2 - 6abc \le 0$. In the second case, we rewrite the expression:

$$bc-a^2-6abc=-(b-c)^2-3bc(1+2a).$$

Again $bc \ge 0$ and $1+2a \ge 0$ since $a \ge -(1/3) \ge -(1/2)$. Thus it follows that $f(x,y,z) \le 7/27$.

(For yet another proof, refer to example 2.41.) ■

67. Let $x_1, x_2, \ldots, x_n$ be n positive real numbers. Prove that

$$\sum_{j=1}^{n}\frac{x_j^3}{x_j^2+x_jx_{j+1}+x_{j+1}^2}\ge\frac{1}{3}\sum_{j=1}^{n}x_j\ ,$$

where $x_{n+1}=x_1$.

Solution: Let us introduce

$$A = \sum_{j=1}^{n} \frac{x_j^3}{x_j^2 + x_j x_{j+1} + x_{j+1}^2}.$$

Consider

$$B = \sum_{j=1}^{n} \frac{x_{j+1}^3}{x_j^2 + x_j x_{j+1} + x_{j+1}^2}.$$

Observe that

$$\begin{aligned} A - B &= \sum_{j=1}^{n} \frac{x_j^3 - x_{j+1}^3}{x_j^2 + x_j x_{j+1} + x_{j+1}^2} \\ &= \sum_{j=1}^{n} \left(x_j - x_{j+1}\right) \\ &= 0. \end{aligned}$$

This shows that $A = B$ and hence $A = (A+B)/2$. However, $A+B$ is symmetric and it is advantageous to deal with it. Observe that

$$\frac{x_j^3 + x_{j+1}^3}{x_j^2 + x_j x_{j+1} + x_{j+1}^2} \geq \frac{x_j + x_{j+1}}{3}.$$

Thus

$$\begin{aligned} A = \frac{1}{2}\left(A + B\right) &= \frac{1}{2} \sum_{j=1}^{n} \frac{x_j^3 + x_{j+1}^3}{x_j^2 + x_j x_{j+1} + x_{j+1}^2} \\ &\geq \frac{1}{2} \times \frac{1}{3} \sum_{j=1}^{n} \left(x_j + x_{j+1}\right) \\ &= \frac{1}{3} \sum_{j=1}^{n} x_j. \end{aligned}$$

This proves the inequality. ■

68. Suppose x, y, z are non-negative real numbers. Prove the inequality

$$x(x-z)^2 + y(y-z)^2 \geq (x-z)(y-z)(x+y-z).$$

Find conditions for equality.

Solution: Expanding and rearranging, the inequality may be written in the form

$$x^3 + y^3 + z^3 + 3xyz \geq x^2y + y^2z + z^2x + xy^2 + yz^2 + zx^2.$$

This reduces to

$$x(x-y)(x-z) + y(y-x)(y-z) + z(z-x)(z-y) \geq 0,$$

which follows from Schur's inequality. Equality holds if and only if $x = y = z$; $x = 0$, $y = z$; $y = 0$, $z = x$; $z = 0$, $x = y$. ■

69. Prove that for any positive reals a, b, c, the inequality,

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{c+a}{c+b} + \frac{a+b}{a+c} + \frac{b+c}{b+a}$$

holds.

Solution: Put

$$\frac{a}{b} = x, \quad \frac{b}{c} = y, \quad \frac{c}{a} = z.$$

Then $xyz = 1$ and it is easy to calculate

$$\begin{aligned}
\frac{c+a}{c+b} &= 1 + x - \frac{x+y}{1+y}, \\
\frac{a+b}{a+c} &= 1 + y - \frac{y+z}{1+z}, \\
\frac{b+c}{b+a} &= 1 + z - \frac{z+x}{1+x}.
\end{aligned}$$

Thus, the inequality reduces to

$$\frac{x+y}{1+y} + \frac{y+z}{1+z} + \frac{z+x}{1+x} \geq 3.$$

This may be written in the form

$$x^2 + y^2 + z^2 + x^2z + y^2x + z^2y - (x+y+z) - 3 \geq 0.$$

Using $xyz = 1$ and the AM-GM inequality, we get

$$x^2z + y^2x + z^2y \geq 3xyz = 3.$$

Using the Cauchy-Schwarz inequality, we also have

$$\begin{aligned}
x + y + z &\leq \sqrt{3}\left(x^2 + y^2 + z^2\right)^{1/2} \\
&= \left(3(x^2y^2z^2)^{1/3}\right)^{1/2}\left(x^2 + y^2 + z^2\right)^{1/2} \\
&\leq \left(x^2 + y^2 + z^2\right)^{1/2}\left(x^2 + y^2 + z^2\right)^{1/2} \\
&= x^2 + y^2 + z^2.
\end{aligned}$$

Hence the result follows. ■

70. If a, b are real numbers, prove that

$$a^2 + ab + b^2 \geq 3(a + b - 1).$$

Solution: This may be written in the form

$$(a + b - 1)^2 + (a - 2)^2 + (b - 2)^2 \geq 3.$$

By the Cauchy-Schwarz inequality, we have

$$\left|(a + b - 1) - (a - 2) - (b - 2)\right|^2 \leq 3\big((a + b - 1)^2 + (a - 2)^2 + (b - 2)^2\big).$$

Hence the required inequality follows. ■

71. Define a sequence $\langle x_n \rangle$ by

$$x_1 = 2, \quad x_{n+1} = \frac{x_n^4 + 9}{10x_n}.$$

Prove that $\dfrac{4}{5} < x_n \leq \dfrac{5}{4}$ for all $n > 1$.

Solution: First observe that all the terms of the sequence are positive. Moreover,

$$\begin{aligned} x_{n+1} = \frac{x_n^4 + 9}{10x_n} &= \frac{x_n^3}{10} + \frac{3}{10x_n} + \frac{3}{10x_n} + \frac{3}{10x_n} \\ &\geq 4\left(\frac{x_n^3}{10} \cdot \frac{3}{10x_n} \cdot \frac{3}{10x_n} \cdot \frac{3}{10x_n}\right)^{1/4} \\ &= \frac{2}{5}\left(\sqrt[4]{27}\right) > \frac{4}{5}. \end{aligned}$$

Thus $x_n > 4/5$ for all n. To prove the other inequality, observe that $x_2 = 5/4$. If we show that $x_{n+1} \leq x_n$ for all $n \geq 2$, it follows that $x_n \leq 5/4$ for all $n \geq 2$. Thus we have to check that

$$\frac{x_n^4 + 9}{10x_n} \leq x_n,$$

for $n \geq 2$. Equivalently, we need to prove that $x_n^4 - 10x_n^2 + 9 \leq 0$, for all $n \geq 2$. If $1 \leq x_n \leq 5/4$, we have $1 \leq x_n^2 < 9$ and hence $x_n^4 - 10x_n^2 + 9 \leq 0$. If $x_n < 1$, we see that

$$x_{n+1} = \frac{x_n^4 + 9}{10x_n} < \frac{10}{10x_n} = \frac{1}{x_n} < \frac{5}{4},$$

since $x_n > 4/5$, by the first part. ■

72. Let a, b, c be positive real numbers such that $a^2 - ab + b^2 = c^2$. Prove that $(a - c)(b - c) \leq 0$.

Solution: We have to show that $ab - ac - bc + c^2 \leq 0$. This reduces to

$$a^2 + b^2 \leq c(a+b).$$

Squaring both sides, this may be written in the form

$$\left(a^2 + b^2\right)^2 \leq \left(a+b\right)\left(a^3 + b^3\right).$$

This reduces to the standard inequality $2ab \leq a^2 + b^2$, which is true. ■

73. Let a, b, c be positive real numbers. Prove that

$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{a^2 + ac + c^2}.$$

Solution: Consider the coordinate plane with $O = (0,0)$ as the origin. Let $A = (a, 0)$, $B = (b/2, b\sqrt{3}/2)$ and $C = (-c/2, c\sqrt{3}/2)$. Then we see that

$$AB = \sqrt{a^2 + b^2 - ab}, \quad BC = \sqrt{b^2 + c^2 - bc}, \quad CA = \sqrt{a^2 + b^2 + ac}.$$

Moreover A, B, C are the vertices of a triangle (may be degenerate) in the plane. Hence triangle inequality gives

$$\sqrt{a^2 + b^2 + ac} \leq \sqrt{a^2 + b^2 - ab} + \sqrt{b^2 + c^2 - bc}.$$

■

74. For all real numbers a, show that

$$(a^3 - a + 2)^2 \geq 4a^2(a^2 + 1)(a - 2)$$

holds.

Solution: Consider the quadratic function

$$f(x) = a^2(a-2)x^2 - (a^3 - a + 2)x + (a^2 + 1).$$

We observe that $f(0) = a^2 + 1 > 0$ and $f(1) = -(a^2 - a + 1) < 0$ for all a. Hence $f(x) = 0$ has a real root. But then the discriminant of $f(x)$ must be non-negative. Thus we get

$$(a^3 - a + 2)^2 \geq 4a^2(a^2 + 1)(a - 2).$$

■

75. Let a, b, c be distinct real numbers. Prove that

$$\left(\frac{2a - b}{a - b}\right)^2 + \left(\frac{2b - c}{b - c}\right)^2 + \left(\frac{2c - a}{c - a}\right)^2 \geq 5.$$

Solution: Put

$$x = \frac{a}{a-b}, \quad y = \frac{b}{b-c}, \quad z = \frac{c}{c-a}.$$

Then it is easy to see that

$$(x-1)(y-1)(z-1) = xyz.$$

This reduces to

$$x + y + z = xy + yz + zx + 1.$$

Hence

$$\begin{aligned}
&\left(\frac{2a-b}{a-b}\right)^2 + \left(\frac{2b-c}{b-c}\right)^2 + \left(\frac{2c-a}{c-a}\right)^2 \\
&= (1+x)^2 + (1+y)^2 + (1+z)^2 \\
&= 3 + x^2 + y^2 + z^2 + 2x + 2y + 2z \\
&= 3 + x^2 + y^2 + z^2 + 2(xy + yz + zx + 1) \\
&= 5 + (x+y+z)^2 \geq 5.
\end{aligned}$$

■

76. Let α, β, $x_1,x_2,x_3,\ldots,x_n$ be positive reals such that $\alpha + \beta = 1$, and $x_1+x_2+x_3+\cdots+x_n = 1$. Prove that

$$\sum_{j=1}^{n} \frac{x_j^{2m+1}}{\alpha x_j + \beta x_{j+1}} \geq \frac{1}{n^{2m-1}},$$

for every positive integer m, where $x_{n+1} = x_1$.

Solution: Using the Cauchy-Schwarz inequality, we get

$$\left(\sum_{j=1}^{n} x_j^{m+1}\right)^2 \leq \left(\sum_{j=1}^{n} \frac{x_j^{2m+1}}{\alpha x_j + \beta x_{j+1}}\right)\left(\sum_{j=1}^{n} x_j\big(\alpha x_j + \beta x_{j+1}\big)\right).$$

However,

$$\begin{aligned}
\sum_{j=1}^{n} x_j\big(\alpha x_j + \beta x_{j+1}\big) &= \sum_{j=1}^{n} \alpha x_j^2 + \sum_{j=1}^{n} \beta x_j x_{j+1} \\
&\leq (\alpha+\beta)\sum_{j=1}^{n} x_j^2.
\end{aligned}$$

Moreover, Hölder's inequality gives

$$\left(\sum_{j=1}^{n} x_j^2\right)^{m+1} \leq n^{m-1}\left(\sum_{j=1}^{n} x_j^{m+1}\right)^2.$$

It follows that

$$\sum_{j=1}^{n} \frac{x_j^{2m+1}}{\alpha x_j + \beta x_{j+1}} \geq \frac{\left(\sum_{j=1}^{n} x_j^2\right)^{m+1}}{n^{m-1}(\alpha+\beta)\left(\sum_{j=1}^{n} x_j^2\right)} = \frac{\left(\sum_{j=1}^{n} x_j^2\right)^{m}}{n^{m-1}}.$$

Again the Cauchy-Schwarz inequality shows that

$$n\left(\sum_{j=1}^{n} x_j^2\right) \geq \left(\sum_{j=1}^{n} x_j\right)^2 = 1.$$

Combining, we get

$$\sum_{j=1}^{n} \frac{x_j^{2m+1}}{\alpha x_j + \beta x_{j+1}} \geq \frac{1}{n^{2m-1}}.$$

■

77. Given positive reals a, b, c, d, prove that

$$\sqrt{(a+c)^2+(b+d)^2} \leq \sqrt{a^2+b^2} + \sqrt{c^2+d^2} \leq \sqrt{(a+c)^2+(b+d)^2} + \frac{2|ad-bc|}{\sqrt{(a+c)^2+(b+d)^2}}.$$

Solution: We observe that

$$\begin{aligned}
&\sqrt{(a+c)^2+(b+d)^2} \leq \sqrt{a^2+b^2} + \sqrt{c^2+d^2} \\
&\iff (a+c)^2+(b+d)^2 \leq a^2+b^2+c^2+d^2+2\sqrt{\left(a^2+b^2\right)\left(c^2+d^2\right)} \\
&\iff 2ac+2bd \leq 2\sqrt{\left(a^2+b^2\right)\left(c^2+d^2\right)} \\
&\iff \left(ac+bd\right)^2 \leq \left(a^2+b^2\right)\left(c^2+d^2\right) \\
&\iff 2abcd \leq a^2d^2+b^2c^2,
\end{aligned}$$

which follows from the AM-GM inequality. The second inequality is equivalent to

$$\left(\sqrt{a^2+b^2}+\sqrt{c^2+d^2}\right)\left(\sqrt{(a+c)^2+(b+d)^2}\right) \leq (a+c)^2+(b+d)^2+2|ad-bc|.$$

In view of the first part, it suffices to prove that

$$\left(\sqrt{a^2+b^2}+\sqrt{c^2+d^2}\right)^2 \leq (a+c)^2+(b+d)^2+2|ad-bc|.$$

Squaring, and simplifying, this reduces to

$$0 \leq 2(ac+bd)|ad-bc|,$$

which is true by the positivity of a, b, c, d. Equality holds here if and only if $ad = bc$.

■

78. In a triangle ABC, prove that

$$\sqrt{s-a}+\sqrt{s-b}+\sqrt{s-c} \leq \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{2}}.$$

Solution: We use the standard transformations: $s-a=x$, $s-b=y$ and $s-c=z$. The inequality is equivalent to

$$\sqrt{x}+\sqrt{y}+\sqrt{z} \leq \frac{\sqrt{y+z}+\sqrt{z+x}+\sqrt{x+y}}{\sqrt{2}}.$$

This follows from the root-mean-square inequality:

$$\sum_{\text{cyclic}} \sqrt{2x} = \sum_{\text{cyclic}} \frac{\sqrt{2x}+\sqrt{2y}}{2} \leq \sum_{\text{cyclic}} \sqrt{\frac{2x+2y}{2}} = \sum_{\text{cyclic}} \sqrt{x+y}.$$

Note that equality holds if and only if $x = y = z$, which is equivalent to $a = b = c$ or that ABC is an equilateral triangle. ■

79. Let ABC be an acute-angled triangle with altitudes AD, BE, CF and ortho-centre H. Prove that

$$\frac{HD}{HA}+\frac{HE}{HB}+\frac{HF}{HC} \geq \frac{3}{2}.$$

Solution: Let us introduce $x = [BHC]$, $y = [CHA]$ and $z = [AHB]$. Then we see that

$$\frac{HD}{HA} = \frac{[BDH]}{[BHA]} = \frac{[CDH]}{[CHA]} = \frac{[BDH]+[CDH]}{[BHA]+[CHA]} = \frac{[BHC]}{[BHA]+[CHA]} = \frac{x}{y+z}.$$

Similarly, we may obtain

$$\frac{HE}{HB} = \frac{y}{z+x}, \quad \frac{HF}{HC} = \frac{z}{x+y}.$$

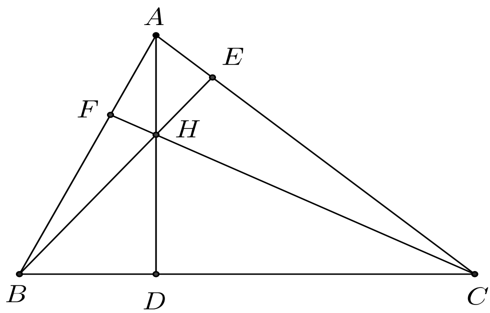

Fig. 6.1

Thus, we obtain

$$\begin{aligned}\frac{HD}{HA}+\frac{HE}{HB}+\frac{HF}{HC} &= \frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\\ &\geq \frac{3}{2};\end{aligned}$$

see Nesbitt's inequality. ■

80. For any point P inside a triangle ABC, let r_1, r_2, r_3 denote the distances of P from the lines BC, CA, AB respectively. Find all points P for which $a/r_1 + b/r_2 + c/r_3$ is minimal.

Solution: Let Δ denote the area of ABC. Then it is easy to see that

$$2\Delta = ar_1 + br_2 + cr_3.$$

Let $\lambda = a/r_1 + b/r_2 + c/r_3$. Then we see that

$$\begin{aligned}2\lambda\Delta &= \Big(ar_1 + br_2 + cr_3\Big)\left(\frac{a}{r_1}+\frac{b}{r_2}+\frac{c}{r_3}\right)\\ &= a^2+b^2+c^2+\sum_{\text{cyclic}} ab\left(\frac{r_1}{r_2}+\frac{r_2}{r_1}\right)\\ &\geq a^2+b^2+c^2+2ab+2bc+2ca=(a+b+c)^2.\end{aligned}$$

This shows that

$$\lambda \geq \frac{(a+b+c)^2}{2\Delta},$$

for any point P. Equality holds if and only if $r_1 = r_2 = r_3$. This is precisely when P is the in-centre of ABC. Thus, the given expression is minimal only if P is the in-centre of ABC. ■

81. Let $ABCDEF$ be a convex hexagon in which AB, BC, CD are respectively parallel to DE, EF, FA. Let R_A, R_B, R_C be the circum-radii of the triangles FAB, BCD, DEF respectively, and let p denote the perimeter of the hexagon. Prove that

$$R_A + R_B + R_C \geq \frac{p}{2}.$$

Solution: Inscribe the hexagon in a rectangle $MNPQ$ as shown in the figure. Note that $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$.

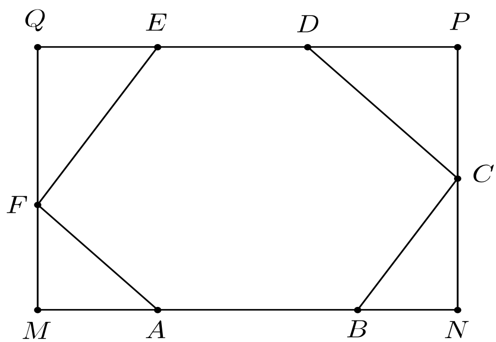

Fig. 6.2

In triangle BCD, we have

$$4R_B = \frac{2BD}{\sin C} \geq \frac{MQ + NP}{\sin C}.$$

But $MQ = MF + FQ$ and $NP = NC + CP$. Again sine rule gives

$$\begin{aligned} MF &= FA \sin \angle MAF = FA \sin A, \\ FQ &= EF \sin E = EF \sin B, \\ NC &= BC \sin B, \\ CP &= CD \sin D = CD \sin A. \end{aligned}$$

Thus we get,

$$4R_B \geq FA\frac{\sin A}{\sin C} + EF\frac{\sin B}{\sin C} + BC\frac{\sin B}{\sin C} + CD\frac{\sin A}{\sin C}.$$

Similarly, we can prove that

$$\begin{aligned} 4R_C &\geq BC\frac{\sin C}{\sin B} + AB\frac{\sin A}{\sin B} + EF\frac{\sin C}{\sin B} + DE\frac{\sin A}{\sin B}, \\ 4R_A &\geq DE\frac{\sin B}{\sin A} + CD\frac{\sin C}{\sin A} + AB\frac{\sin B}{\sin A} + FA\frac{\sin C}{\sin A}. \end{aligned}$$

Adding these, we get

$$\begin{aligned}
&4\big(R_A+R_B+R_C\big)\\
&\quad\geq\ AB\left(\frac{\sin A}{\sin B}+\frac{\sin B}{\sin A}\right)+BC\left(\frac{\sin C}{\sin B}+\frac{\sin B}{\sin C}\right)\\
&\qquad+CD\left(\frac{\sin C}{\sin A}+\frac{\sin A}{\sin C}\right)+DE\left(\frac{\sin A}{\sin B}+\frac{\sin B}{\sin A}\right)\\
&\qquad+EF\left(\frac{\sin C}{\sin B}+\frac{\sin B}{\sin C}\right)+FA\left(\frac{\sin C}{\sin A}+\frac{\sin A}{\sin C}\right)\\
&\quad\geq\ 2\big(AB+BC+CD+DE+EF+FA\big)\\
&\quad=\ 2p.
\end{aligned}$$

It follows that

$$R_A+R_B+R_C\geq\frac{p}{2}.$$

■

82. Let h_a, m_b, w_c denote respectively the altitude from A to BC, the median from B to CA and the internal angle bisector of angle C. Prove that

$$h_a+m_b+w_c\leq\frac{\sqrt{3}}{2}(a+b+c).$$

Solution: We first prove an auxiliary result.
Lemma: Suppose $a+b\leq 2c$. Then

$$m_a+m_b+w_c\leq\frac{\sqrt{3}}{2}(a+b+c),$$

and equality holds if and only if $a=b=c$.

Proof of Lemma: Let $a=u+x$, $b=u-x$, and $c=2v$ so that $-v\leq x\leq v$ and $v\leq u\leq 2v$. Then

$$w_c^2=ab\left(1-\frac{c^2}{(a+b)^2}\right)=\left(u^2-x^2\right)\left(1-\frac{v^2}{u^2}\right)\leq u^2-x^2. \tag{82.1}$$

Equality holds if and only if $x=0$; i.e., $a=b$. We also have

$$2m_a=\sqrt{2b^2+2c^2-a^2}=\sqrt{(3u-x)^2+8v^2-8u^2}=f(x),$$

say. Then $2m_b=f(-x)$. It is easy to verify that $f''(x)\leq 0$. Hence $f(x)$ is concave. This gives

$$m_a+m_b=\frac{1}{2}f(x)+\frac{1}{2}f(-x)\leq f(0)=\sqrt{u^2+8v^2}. \tag{82.2}$$

Thus it is sufficient to prove that

$$\sqrt{u^2+8v^2}+\sqrt{u^2-v^2} \le \sqrt{3}(u+v), \tag{82.3}$$

and equality holds if and only if $u = 2v$. Taking $y = u/v$, the inequality (82.3) may be written in the form

$$2\sqrt{(y^2+8)(y^2-1)} \le y^2+6y-4.$$

Since $1 < y \le 2$, this is equivalent to

$$4(y^2+8)(y^2-1) \le \left(y^2+6y-4\right)^2,$$

which reduces to

$$(2-y)^3(2+y) \ge 0.$$

This proves the inequality (82.3) and hence the lemma.

Coming back to the solution of the problem, if $a+b \le 2c$, then the lemma gives

$$m_a+m_b+w_c \le \frac{\sqrt{3}}{2}(a+b+c).$$

Using $h_t \le w_t \le m_t$ for $t = a, b, c$, we get

$$h_a+w_b+m_c \le m_a+m_b+w_c \le \frac{\sqrt{3}}{2}(a+b+c).$$

Similarly, if $b+c \le 2a$, then the proof of lemma shows that

$$m_b+m_c+w_a \le \frac{\sqrt{3}}{2}(a+b+c),$$

and hence

$$h_a+m_b+w_c \le w_a+m_b+m_c \le \frac{\sqrt{3}}{2}(a+b+c).$$

We may, therefore, assume that $a+b \ge 2c$, $b+c \ge 2a$. Hence it follows that

$$a+2b+c \ge 2(a+c),$$

so that $a+c \le 2b$. This implies that

$$m_a+m_c+w_b \le \frac{\sqrt{3}}{2}(a+b+c).$$

Moreover, we also have

$$\begin{aligned} 4a+2c &\le 2(b+c)+(a+b) = a+3b+2c, \\ 2a+4c &\le (b+c)+2(a+b) = 2a+3b+c. \end{aligned}$$

From these two we get $a \leq b$ and $c \leq b$. Thus

$$b < 2a + 2c - b \leq 2a + (a+b) - b = 3a,$$

and similarly $b < 3c$. Thus we obtain

$$a \leq b < 3a, \quad c \leq b < 3c, \quad a + b \geq 2c, \quad b + c \geq 2a.$$

Suppose we show that

$$h_a - h_b \leq m_a - m_b.$$

Then

$$h_a + m_b + w_c \leq m_a + h_b + w_c \leq m_a + w_b + m_c \leq \frac{\sqrt{3}}{2}(a+b+c),$$

as observed earlier. Thus all we need to prove is $h_a - h_b \leq m_a - m_b$.

We may assume $b = 1$. Thus we have

$$\frac{1}{3} < a \leq 1, \quad \frac{1}{3} < c \leq 1, \quad 2a \leq 1 + c, \quad 2c \leq 1 + a, \quad a + c > 1.$$

Heron's formula gives

$$\begin{aligned} 16\Delta^2 &= (1+a+c)(1+a-c)(1+c-a)(c+a-1) \\ &= 2(1+a^2)c^2 - c^4 - (1-a^2)^2 = g(c), \end{aligned}$$

say. Note that g is a strictly increasing function of c in the interval $1/3 \leq c \leq 1$. Since $2c \leq 1 + a$, it follows that $g(c) \leq g\big((1+a)/2\big)$. This gives

$$16\Delta^2 \leq \frac{3}{16}(3-a)(3a-1)(1+a)^2.$$

Thus

$$\begin{aligned} h_a - h_b &= 2\Delta\left(\frac{1}{a} - 1\right) \\ &= \frac{2\Delta(1-a)}{a} \\ &\leq \frac{\sqrt{3}}{8}\left(\frac{1-a^2}{a}\right)\sqrt{(3-a)(3a-1)}. \end{aligned}$$

But we know that

$$2m_a = \sqrt{2 + 2c^2 - a^2}, \quad 2m_b = \sqrt{2c^2 + 2a^2 - 1}.$$

Moreover, for any positive $s > t$, we have

$$\sqrt{s} - \sqrt{t} \geq \frac{s-t}{\sqrt{2(s+t)}}.$$

Since $a < b$, we have $m_a > m_b$ and hence

$$\begin{aligned} 2(m_a - m_b) &= \sqrt{2 + 2c^2 - a^2} - \sqrt{2c^2 + 2a^2 - 1} \\ &\geq 3(1 - a^2)\left(4c^2 + a^2 + 1\right)^{-1/2}. \end{aligned}$$

Note that the right side is a decreasing function of c. Hence, replacing c by $(1 + a)/2$, we get

$$m_a - m_b \geq \frac{3}{4}(1 - a^2)\left(1 + a + a^2\right)^{-1/2}.$$

Thus it is sufficient to prove that

$$\frac{\sqrt{3}(1 - a^2)}{8a}\sqrt{(3 - a)(3a - 1)} \leq \frac{3}{4}(1 - a^2)\left(1 + a + a^2\right)^{-1/2}.$$

This is simply

$$(1 - a)^2(3a^2 - a + 3) \geq 0,$$

which is true. ■

83. Let ABC be a triangle with centroid G. Prove that

$$\sin \angle ABG + \sin \angle ACG \leq \frac{2}{\sqrt{3}}.$$

Solution: Let AD, BE, CF be the medians of triangle ABC and let G be its centroid. First we solve a special case of this problem when the circum-circle of triangle AGB is tangent to BC at B. In this case $DB^2 = DG \cdot DA$. This gives

$$3a^2 = 4m_a^2 = 2b^2 + 2c^2 - a^2.$$

Hence $b^2 + c^2 = 2a^2$. This gives that $4m_b^2 = 2a^2 + 2c^2 - b^2 = 3c^2$ and similarly, $4m_c^2 = 3b^2$.

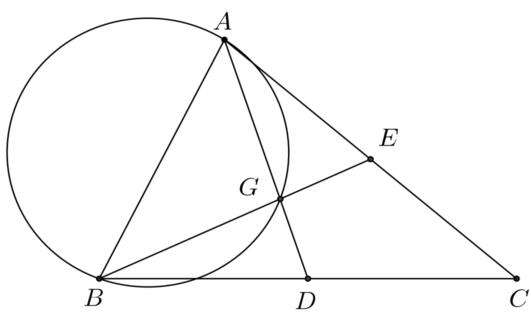

Fig. 6.3

Thus

$$\begin{aligned} \sin \angle ABG + \sin \angle ACG &= \left(\frac{b}{2m_b} + \frac{c}{2m_c} \right) \sin A \\ &= \left(\frac{b}{\sqrt{3}\, c} + \frac{c}{\sqrt{3}\, b} \right) \sin A \\ &= \frac{b^2 + c^2}{\sqrt{3}\, bc} \sin A. \end{aligned}$$

Using cosine rule, we get

$$2bc \cos A = b^2 + c^2 - a^2 = 2a^2 - a^2 = a^2.$$

Using this, we obtain

$$\sin \angle ABG + \sin \angle ACG = \frac{2a^2 \sin A}{\sqrt{3}\, a^2/2 \cos A} = \frac{2}{\sqrt{3}} \sin 2A \leq \frac{2}{\sqrt{3}}$$

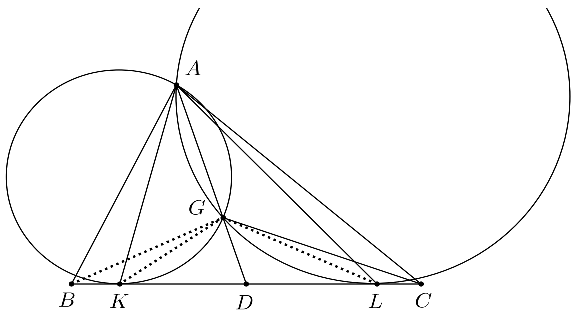

Fig. 6.4

In the general case, there are two circles through A and G touching BC, say at K and L. Here K lies between B and D; L between D and C. Note that G is also the centroid of the triangle AKL. This follows from $DL^2 = DG \cdot DA = DK^2$ so that $DK = DL$ showing AD is the median of AKL from A; and $AG : GD = 2 : 1$. Moreover $\angle ABG \leq \angle AKG$, $\angle ACG \leq \angle ALG$. Suppose $\angle AKG \leq 90°$, and $\angle ALG \leq 90°$. Using the special case above, we get

$$\sin \angle ABG + \sin \angle ACG \leq \sin \angle AKG + \angle ALG \leq \frac{2}{\sqrt{3}}.$$

In the other case, we may assume that $\angle AKG > 90°$ and $\angle ALG \leq 90°$. (Obviously, both cannot be larger than $90°$.) Hence $AG^2 > AK^2 + KG^2$. Let

us introduce $KL = a_1$, $LA = b_1$ and $AK = c_1$, the sides of triangle AKL. Then

$$\begin{aligned} AG^2 = \frac{4}{9}AD^2 &= \frac{1}{9}\left(2b_1^2 + 2c_1^2 - a_1^2\right), \\ KG^2 &= \frac{1}{9}\left(2c_1^2 + 2a_1^2 - b_1^2\right). \end{aligned}$$

Hence we obtain

$$\frac{1}{9}\left(2b_1^2 + 2c_1^2 - a_1^2\right) > c_1^2 + \frac{1}{9}\left(2c_1^2 + 2a_1^2 - b_1^2\right).$$

By the property of tangency, we have $DK^2 = DG \cdot DA$. Using $DK = a_1/2$, $DG = m_a/3$, $DA = m_a$ and m_a also gives median length of triangle AKL, we see that

$$3a_1^2 = 4m_a^2 = 2b_1^2 + 2c_1^2 - a_1^2.$$

This reduces to $2a_1^2 = b_1^2 + c_1^2$. Thus we get

$$\frac{a_1^2}{3} > c_1^2 + \frac{c_1^2}{3}.$$

It follows that $a_1^2 > 4c_1^2$. Similarly, we get $b_1^2 > 7c_1^2$. Now we have

$$\sin\angle ALG = \frac{c_1}{2m_{c_1}}\sin\angle LAK = \frac{c_1}{\sqrt{3}b_1}\sqrt{1 - \cos^2\angle LAK}.$$

However $a_1^2 = b_1^2 + c_1^2 - 2b_1c_1\cos\angle LAK$. This gives $\cos\angle LAK = (b_1^2 + c_1^2)/(4b_1c_1)$ and hence

$$\sin\angle ALG = \frac{c_1}{\sqrt{3}b_1}\sqrt{1 - \left(\frac{b_1^2 + c_1^2}{4b_1c_1}\right)^2}.$$

Putting $c_1^2/b_1^2 = x$, We get

$$\sin\angle ALG = \frac{1}{4\sqrt{3}}\sqrt{14x - 1 - x^2}.$$

Using $x < 1/7$, this gives $\sin\angle ALG < 1/7$. Thus

$$\sin\angle AKG + \sin\angle ALG < 1 + \frac{1}{7} < \frac{2}{\sqrt{3}}.$$

This implies that

$$\sin\angle ABG + \sin\angle ACG < \frac{2}{\sqrt{3}}.$$

$\blacksquare$

84. Let ABC be a triangle with in-radius r. Let Γ_1, Γ_2, Γ_3 be three circles inscribed inside ABC such that each touches other circles and also two of the sides. (Such a configuration is called *Malafatti* circles.) Let O_1, O_2, O_3 be respectively the centres of the circles Γ_1, Γ_2, Γ_3. If r' denotes the in-radius of $O_1O_2O_3$, prove that

$$\frac{r}{r'} \geq 1 + \sqrt{3}.$$

Find conditions for equality.

Solution: We have $BD = s - b$. Hence $BE/BD = O_2E/ID = r_2/r$ and this gives $BE = r_2(s-b)/r$. Thus $ED = BD - BE = (r - r_2)(s-b)/r$. Similarly, we obtain $DF = (r - r_3)(s-c)/r$. Now $PO_3 = r_3 - r_2$, $O_2O_3 = r_2 + r_3$ and hence $O_2P = 2\sqrt{r_2 r_3}$. Thus we have

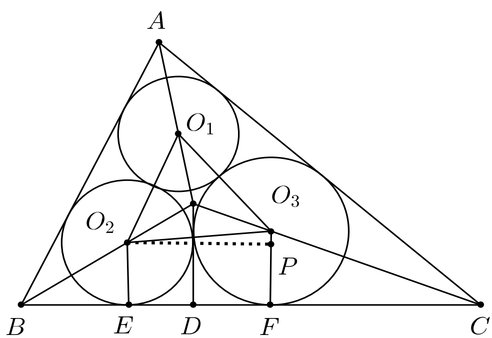

Fig. 6.5

$$2\sqrt{r_2r_3} = O_2P = EF = ED + DF = \frac{(r-r_2)(s-b) + (r-r_3)(s-c)}{r}.$$

We therefore get

$$2r\sqrt{r_2r_3} = (r-r_2)(s-b) + (r-r_3)(s-c).$$

Similarly, we may obtain relations

$$\begin{aligned} 2r\sqrt{r_3r_1} &= (r-r_3)(s-c) + (r-r_1)(s-a) \\ 2r\sqrt{r_1r_2} &= (r-r_1)(s-a) + (r-r_2)(s-b). \end{aligned}$$

Solving these, we obtain

$$\begin{aligned}
s-a &= \frac{r}{r-r_1}\Big(-\sqrt{r_2r_3}+\sqrt{r_3r_1}+\sqrt{r_1r_2}\Big)\\
s-b &= \frac{r}{r-r_2}\Big(\sqrt{r_2r_3}-\sqrt{r_3r_1}+\sqrt{r_1r_2}\Big)\\
s-c &= \frac{r}{r-r_3}\Big(\sqrt{r_2r_3}+\sqrt{r_3r_1}-\sqrt{r_1r_2}\Big).
\end{aligned}$$

Using $rs = [ABC] = \sqrt{s(s-a)(s-b)(s-c)}$, we may write

$$r^2\{(s-a)+(s-b)+(s-c)\} = (s-a)(s-b)(s-c).$$

Substituting for $(s-a)$, $(s-b)$, $(s-c)$, we get a quadratic equation in r:

$$\left(\frac{1}{\sqrt{r_1}}+\frac{1}{\sqrt{r_2}}+\frac{1}{\sqrt{r_3}}\right)r^2 - 2\Big(\sqrt{r_1}+\sqrt{r_2}+\sqrt{r_3}\Big)r + 2\sqrt{r_1r_2r_3} = 0.$$

This leads to

$$r = \frac{2\sqrt{r_1r_2r_3}}{\sqrt{r_1}+\sqrt{r_2}+\sqrt{r_3}-\sqrt{r_1+r_2+r_3}}.$$

Now the triangle $O_1O_2O_3$ has sides r_1+r_2, r_2+r_3, r_3+r_1. Hence, as in the case of ABC, two ways of expressing the area of a triangle give

$$r'(r_1+r_2+r_3) = [O_1O_2O_3] = \sqrt{r_1r_2r_3(r_1+r_2+r_3)}.$$

We therefore obtain

$$\frac{r}{r'} = \frac{2\sqrt{r_1+r_2+r_3}}{\sqrt{r_1}+\sqrt{r_2}+\sqrt{r_3}-\sqrt{r_1+r_2+r_3}}.$$

But $\sqrt{r_1}+\sqrt{r_2}+\sqrt{r_3} \le \sqrt{3}\sqrt{r_1+r_2+r_3}$, where we have used the Cauchy-Schwarz inequality. Hence

$$\frac{r}{r'} \ge \frac{2}{\sqrt{3}-1} = \sqrt{3}+1.$$

■

85. Let Ω be a Brocard point of a triangle ABC. Let $A\Omega$, $B\Omega$,$C\Omega$ extended meet the circum-circle of ABC in K, L, M respectively. Prove that

$$\frac{A\Omega}{\Omega K}+\frac{B\Omega}{\Omega L}+\frac{C\Omega}{\Omega M} \ge 3.$$

Solution: We observe that $\angle KMC = \angle KAC$ and $\angle CML = \angle CBL = \omega$, the Brocard angle. Thus we get

$$\angle KML = \angle KMC + \angle CML = \angle KAC + \angle BAK = \angle BAC.$$

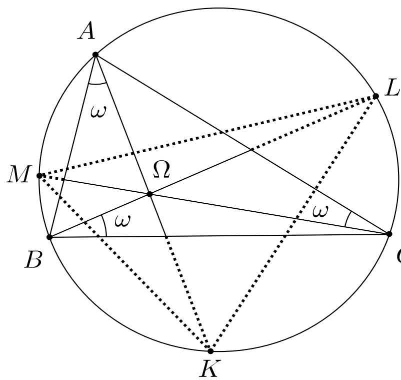

Fig. 6.6

Similarly, we obtain $\angle LKM = \angle CBA$ and $\angle MLK = \angle ACB$. These relations show that the triangles ABC and MKL are similar. Since both have the same circum-circle, they are indeed congruent. We also observe that

$$\angle A\Omega C = 180^\circ - \angle \Omega CA - \angle \Omega AC = 180^\circ - \omega - \Big(\angle BAC - \omega\Big) = 180^\circ - \angle A.$$

Using the sine rule in triangle $A\Omega C$, we get

$$\frac{A\Omega}{\sin\omega} = \frac{AC}{\sin\angle A\Omega C} = \frac{b}{\sin A}.$$

Thus, $A\Omega = b\sin\omega/\sin A$. Similarly, $B\Omega = c\sin\omega/\sin B$, $C\Omega = a\sin\omega/\sin C$. Using similar arguments in the triangle MKL, we obtain $K\Omega = a\sin\omega/\sin B$, $L\Omega = b\sin\omega/\sin C$ and $M\Omega = c\sin\omega/\sin A$. Thus

$$\frac{A\Omega}{\Omega K} + \frac{B\Omega}{\Omega L} + \frac{C\Omega}{\Omega M} = \sum_{\text{cyclic}} \frac{b\sin B}{a\sin A} = \sum_{\text{cyclic}} \frac{b^2}{a^2} \geq 3,$$

by the AM-GM inequality. Observe that equality holds if and only if $a = b = c$, which corresponds to the case of an equilateral triangle. ■

86. Let P be a point inside a triangle ABC. Let r_A, r_B, r_C denote the in-radii of triangles PBC, PCA, PAB respectively. Prove that

$$\frac{a}{r_A} + \frac{b}{r_B} + \frac{c}{r_C} \geq 6\big(2+\sqrt{3}\big).$$

Solution: We use the steps developed in example **3.6.8**. Using the estimates there, we get

$$\begin{aligned}
\frac{a}{r_A} &\geq \frac{(x+y+z)}{2\Delta}\frac{a^2}{x} + \frac{ah_b(x+z)}{2\Delta x} + \frac{ah_c(y+x)}{2\Delta x} \\
\frac{b}{r_B} &\geq \frac{(x+y+z)}{2\Delta}\frac{b^2}{y} + \frac{bh_c(y+x)}{2\Delta y} + \frac{bh_a(z+y)}{2\Delta y} \\
\frac{c}{r_C} &\geq \frac{(x+y+z)}{2\Delta}\frac{c^2}{z} + \frac{ch_a(z+y)}{2\Delta z} + \frac{ch_b(x+z)}{2\Delta z}.
\end{aligned}$$

Thus

$$\sum \frac{a}{r_A} \geq \frac{(x+y+z)}{2\Delta}\sum \frac{a^2}{x} + \sum \frac{h_b}{2\Delta}\left\{a+c+\frac{az}{x}+\frac{cx}{z}\right\},$$

where the sum is cyclically over a, b, c and x, y, z. However, the AM-GM inequality gives

$$a+c+\frac{az}{x}+\frac{cx}{z} \geq a+c+2\sqrt{ac},$$

and the Cauchy-Schwarz inequality gives

$$(x+y+z)\left(\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z}\right) \geq (a+b+c)^2.$$

Thus, we have

$$\begin{aligned}
\sum \frac{a}{r_A} &\geq \frac{(a+b+c)^2}{2\Delta} + \frac{h_b}{2\Delta}(a+c+2\sqrt{ac}) \\
&= \frac{(a+b+c)^2}{2\Delta} + \sum \frac{a+c+2\sqrt{ac}}{b} \\
&\geq \frac{9}{2\Delta}(abc)^{2/3} + \sum\left\{\frac{a}{b}+\frac{b}{a}\right\} + 2\sum\frac{\sqrt{ac}}{b} \\
&\geq \frac{9}{2\Delta}(abc)^{2/3} + 12.
\end{aligned}$$

However $(abc)^2 \geq \left(4\Delta/\sqrt{3}\right)$. (See **3.6.8**.) Thus, we obtain

$$\sum \frac{a}{r_A} \geq \frac{18}{\sqrt{3}} + 12 = 6\left(2+\sqrt{3}\right).$$

■

87. Suppose $A_1A_2\cdots A_7$, $B_1B_2\cdots B_7$, $C_1C_2\cdots C_7$ are three regular heptagons which are such that $A_1A_2 = B_1B_3 = C_1C_4$. If Δ_1, Δ_2, Δ_3 denote respectively their areas, prove that

$$\frac{1}{2} < \frac{\Delta_2+\Delta_3}{\Delta_1} < 2-\sqrt{2}.$$

Solution: Let the septagons be inscribed in three concentric circles of radii R_A, R_B, R_C respectively. (See figure.) Then we have

$$\begin{aligned}\frac{C_1C_4}{A_1A_4} &= \frac{R_C}{R_A} = \frac{C_1C_2}{A_1A_2}\\ \frac{B_1B_3}{A_1A_3} &= \frac{R_B}{R_A} = \frac{B_1B_2}{A_1A_2}.\end{aligned}$$

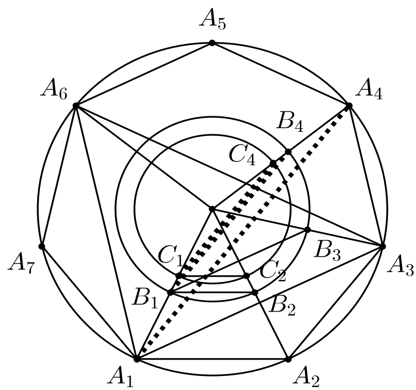

Fig. 6.7

Using $A_1A_2 = B_1B_3 = C_1C_4$, we get

$$C_1C_2 = \frac{(A_1A_2)^2}{A_1A_4}, \quad B_1B_2 = \frac{(A_1A_2)^2}{A_1A_3}.$$

Let $A_1A_2 = a$, $A_1A_3 = b$ and $A_1A_4 = c$. Then we obtain $C_1C_2 = a^2/c$ and $B_1B_2 = a^2/b$. Using Ptolemy's theorem, we also have

$$A_1A_2 \cdot A_3A_6 + A_2A_3 \cdot A_1A_6 = A_1A_3 \cdot A_2A_6.$$

Using the regularity of septagons, we see that $A_3A_6 = c = A_2A_6$, $A_1A_6 = b$ and $A_2A_3 = a$. Thus $ac + ab = bc$ or

$$\frac{1}{b} + \frac{1}{c} = \frac{1}{a}.$$

We also observe that

$$\begin{aligned}\frac{\Delta_2}{\Delta_1} &= \frac{(B_1B_2)^2}{(A_1A_2)^2} = \frac{a^2}{b^2},\\ \frac{\Delta_3}{\Delta_1} &= \frac{(C_1C_2)^2}{(A_1A_2)^2} = \frac{a^2}{c^2}.\end{aligned}$$

Thus

$$\frac{\Delta_2 + \Delta_3}{\Delta_1} = \frac{a^2}{b^2} + \frac{a^2}{c^2} > \frac{a^2}{2}\left(\frac{1}{b} + \frac{1}{c}\right)^2 = \frac{1}{2}.$$

We also have

$$\frac{\Delta_2+\Delta_3}{\Delta_1}=a^2\left\{\left(\frac{1}{b}+\frac{1}{c}\right)^2-\frac{2}{bc}\right\}=1-\frac{2a^2}{bc}.$$

However,

$$a=4R_A^2\sin^2(\pi/7),\quad b=4R_A^2\sin^2(2\pi/7),\quad c=4R_A^2\sin^2(3\pi/7).$$

Thus

$$\begin{aligned}\frac{a^2}{bc} &= \frac{\sin^2(\pi/7)}{\sin(2\pi/7)\sin(3\pi/7)}\\ &= \frac{\sin(\pi/7)}{2\cos(\pi/7)\sin(4\pi/7)}\\ &= \frac{1}{8\cos^2(\pi/7)\cos(2\pi/7)}\\ &= \frac{1}{4\cos(2\pi/7)\Big(1+\cos(2\pi/7)\Big)}.\end{aligned}$$

But $\cos(2\pi/7)<\cos(2\pi/8)=\cos(\pi/4)=1/\sqrt{2}$. Hence

$$\frac{a^2}{bc}>\frac{1}{2(1+\sqrt{2})}=\frac{\sqrt{2}-1}{2}.$$

We thus obtain

$$\frac{\Delta_2+\Delta_3}{\Delta_1}=1-2\frac{a^2}{bc}<1-2\left(\frac{\sqrt{2}-1}{2}\right)=2-\sqrt{2}.$$

■

88. Let ABC be a triangle, and let D, E be points on BC, CA such that the in-centre of ABC lies on DE. Prove that $[ABC]\geq 2r^2$.

Solution: Choose D' on BC and E' on CA such that $D'IE'$ is perpendicular to CI. We may assume that $CD\geq CE$. Since CI bisects $\angle C$, we have

$$\frac{ID}{IE}=\frac{CD}{CE}\geq 1.$$

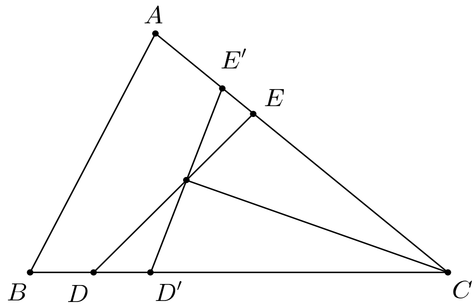

Fig. 6.8

Hence

$$\begin{aligned}[DID'] &= \frac{1}{2} ID' \cdot ID \cdot \sin \angle DID' \\ &\geq \frac{1}{2} IE' \cdot IE \cdot \sin \angle EIE' = [EIE'];\end{aligned}$$

here we have used $D'I = IE'$. Thus it follows that $[CDE] \geq [CD'E']$. But note that

$$CD' = \frac{ID'}{\sin(C/2)}, \quad r = ID' \cos(C/2).$$

Thus

$$[CD'E'] = 2[CD'I] = r \cdot CD' = \frac{r^2}{\sin(C/2)\cos(C/2)} = \frac{2r^2}{\sin C} \geq 2r^2.$$

We hence obtain $[CDE] \geq [CD'E'] \geq 2r^2$. ■

89. Let A, X, D be points on a line with X between A and D. Let B be a point such that $\angle ABX \geq 120°$ and let C be a point between B and X. Prove that $2AD \geq \sqrt{3}\left(AB + BC + CD\right)$.

Solution: Construct an equilateral triangle ATX on the segment AX. Using the cosine rule, we have

$$AX^2 = AB^2 + BX^2 - 2AB \cdot BX \cos \angle ABX.$$

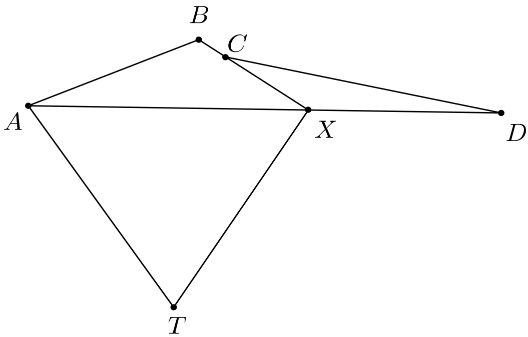

Fig. 6.9

But note that $\cos\angle ABX \le -1/2$. Thus $AX^2 \ge AB^2 + BX^2 + AB\cdot BX$. This leads to

$$\begin{aligned} 4AX^2 &\ge 4AB^2 + 4BX^2 + 4AB\cdot BX \\ &= 3(AB+BX)^2 + (AB-BX)^2 \\ &\ge 3(AB+BX)^2. \end{aligned}$$

Thus $2AX \ge \sqrt{3}\,(AB+BX)$. Finally,

$$\begin{aligned} 2AD &= 2AX + 2XD \\ &\ge \sqrt{3}\,(AB+BX) + 2XD \\ &= \sqrt{3}\,(AB+BC) + \sqrt{3}\,CX + 2XD \\ &\ge \sqrt{3}\,(AB+BC) + \sqrt{3}\,(CX+XD) \\ &\ge \sqrt{3}\,(AB+BC+CD). \end{aligned}$$

■

90. Let the internal bisectors of the angles A, B, C of a triangle ABC meet the sides BC, CA, AB in D, E, F and the circum-circle in L, M, N respectively. Prove that

$$\frac{AD}{DL} + \frac{BE}{EM} + \frac{CF}{FN} \ge 9.$$

Solution: Using the property of angle bisector, we have

$$BD = \frac{ca}{b+c}, \quad , DC = \frac{ba}{b+c}, \quad AD^2 = \frac{4bc}{(b+c)^2}s(s-a).$$

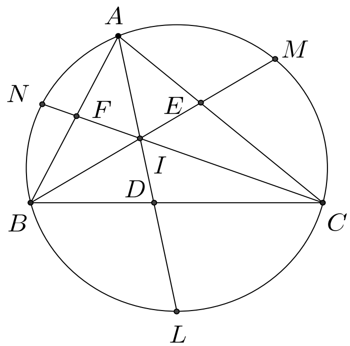

Fig. 6.10

Thus,we get

$$\frac{AD}{DK} = \frac{AD^2}{AD \cdot DK} = \frac{AD^2}{BD \cdot DC} = \frac{4s(s-a)}{a^2}.$$

Similarly, we may obtain

$$\frac{BE}{EL} = \frac{4s(s-b)}{b^2}, \quad \frac{CF}{FM} = \frac{4s(s-c)}{c^2}.$$

Hence, we obtain

$$\begin{aligned}
\frac{AD}{DK} + \frac{BE}{EL} + \frac{CF}{FM} &= 4s\left\{\frac{(s-a)}{a^2} + \frac{(s-b)}{b^2} + \frac{(s-c)}{c^2}\right\} \\
&= 4s^2\left\{\sum \frac{(s-a)}{s}\frac{1}{a^2}\right\} \\
&\geq 4s^2\left\{\sum \frac{(s-a)}{s} \cdot \frac{1}{a}\right\}^2 \\
&= 4s^2\left\{\sum \frac{1}{a} - \frac{3}{s}\right\}^2 \\
&= \left\{2s\sum \frac{1}{a} - 6\right\}^2 \\
&\geq (9-6)^2 = 9,
\end{aligned}$$

where we have used the convexity of the function $f(x) = x^2$ on the real line, and the AM-HM inequality. ■

91. Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$, $DE = EF = FA$, and $\angle BCD = \angle EFA = \pi/3$. Suppose G and H are two interior points of the hexagon such that $\angle AGB = \angle DHE = 2\pi/3$. Prove that

$$AG + GB + GH + DH + HE \geq CF.$$

Solution: Join BD and EA. Note that BDC and EFA are equilateral triangles. Hence $BA = BD$ and $EA = ED$. It follows that BE bisects $\angle ABD$ and $\angle AED$.

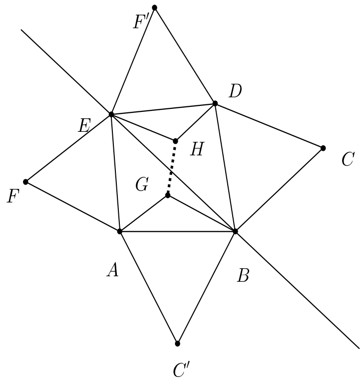

Fig. 6.11

Construct external equilateral triangles ABC' with AB as base and $EF'D$ with ED as base. Then C' is the reflection of C in BE and F' is that of F in BE. Now $AC'BG$ is a cyclic quadrilateral. Hence, using Ptolemy's theorem and the equilaterality of $AC'B$, we get $C'G = AG + GB$. Similarly, we may obtain $HF' = DH + EH$. Hence

$$AG + GB + GH + DH + HE = C'G + GH + HF' \geq C'F' = CF,$$

by the property of reflection. Note that equality holds if and only if G and H both lie on $C'F'$. ■

92. Let O and G be respectively the circum-centre and centroid of a triangle ABC. If R and r are its circum-radius and in-radius, prove that $OG \leq \sqrt{R(R-2r)}$.

Solution: We have $OG^2 = R^2 - (1/9)(a^2 + b^2 + c^2)$. Using $rs = \Delta = abc/4R$, we get $2rR = abc/(a + b + c)$. Thus

$$R^2 - OG^2 = \frac{a^2 + b^2 + c^2}{9} \geq \frac{abc}{a + b + c} = 2rR.$$

This gives $OG^2 \leq R^2 - 2Rr$, which is the needed inequality. ■

93. Let M be the point of intersection of two diagonals of a cyclic quadrilateral. Let N be the point of intersection of two lines joining the midpoints of opposite pair of sides. If O is the centre of the circumscribing circle, prove that $OM \geq ON$.

Solution: For any point T, let $\vec{T}$ denote the position vector of T with respect to some coordinate frame in the plane. Then we see that

$$\vec{S} = \frac{\vec{A}+\vec{D}}{2}, \quad \vec{Q} = \frac{\vec{B}+\vec{C}}{2},$$

so that

$$\frac{\vec{S}+\vec{Q}}{2} = \frac{\vec{A}+\vec{B}+\vec{C}+\vec{D}}{4}.$$

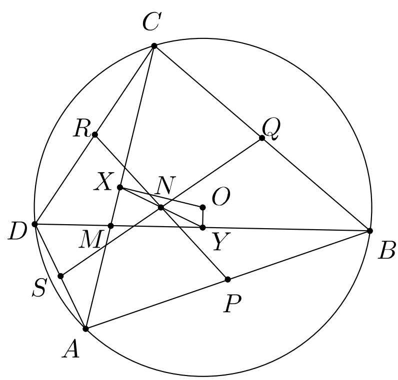

Fig. 6.12

On the other-hand,

$$\vec{Y} = \frac{\vec{D}+\vec{B}}{2}, \quad \vec{X} = \frac{\vec{A}+\vec{C}}{2},$$

so that

$$\frac{\vec{Y}+\vec{X}}{2} = \frac{\vec{A}+\vec{B}+\vec{C}+\vec{D}}{4}.$$

This shows that N is also the mid-point of X and Y. Note that OX and OY are perpendicular to AC and BD respectively. Hence, the circle with OM as a diameter passes through X and Y. Now ON is the median of triangle XOY. It follows that $ON \leq$ diameter $= OM$. ∎

94. Let ABC be a triangle with internal angle bisectors AD, BE, CF. Suppose AD, BE, CF when extended meet the circum-circle again in K, L, M respectively. If $l_a = AD/AK$, $l_b = BE/BL$ and $l_c = CF/CM$, prove that

$$\frac{l_a}{\sin^2 A} + \frac{l_b}{\sin^2 B} + \frac{l_c}{\sin^2 C} \geq 3.$$

Solution: Observe that $AK \le 2R$ and $AD \ge h_a$. Thus

$$l_a = \frac{AD}{AK} \ge \frac{h_a}{2R}.$$

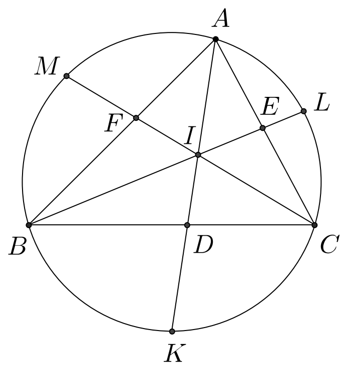

Fig. 6.13

Using similar estimates for the other two fractions, we get

$$\begin{aligned}
\sum_{\text{cyclic}} \frac{l_a}{\sin^2 A} &\ge \sum_{\text{cyclic}} \frac{h_a}{2R\sin^2 A} \\
&= \sum_{\text{cyclic}} \frac{h_a}{a \sin A} \\
&\ge 3\left(\frac{h_a h_b h_c}{abc \sin A \sin B \sin C}\right)^{1/3} \\
&= 3.
\end{aligned}$$

(Here we have used $h_a = b\sin C$, $h_b = c\sin A$ and $h_c = a\sin B$.) ■

95. Let O be the circum-circle of a triangle ABC. Suppose AO, BO, CO when extended meet the circum-circles of triangles BOC, COA, AOB in K, L, M respectively. Prove that

$$\frac{AK}{OK} + \frac{BL}{OL} + \frac{CM}{OM} \ge \frac{9}{2}.$$

Solution: Let us invert the configuration with respect to the circum-circle Γ of the triangle ABC. For any point $X \ne O$, let X' denote the inversion of X in Γ. The inversion of the circum-circle of the triangle OBC is the line BC. Hence $K' = D$, the point of intersection of AK and BC. The property of inversion shows that for any two points $X, Y \ne O$,

$$X'Y' = \frac{R^2 \cdot XY}{OX \cdot OY}.$$

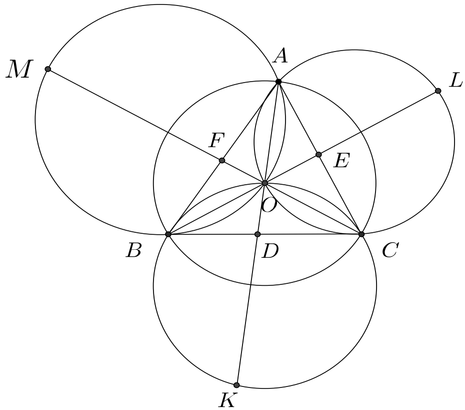

Fig. 6.14

Thus,

$$AK = \frac{R^2 \cdot A'K'}{OA' \cdot OK'}.$$

This gives

$$\begin{aligned}\frac{AK}{OK} = \frac{R^2 \cdot A'K'}{OA' \cdot OK' \cdot OK} &= \frac{A'K'}{OA'} \text{ (since } OK \cdot OK' = R^2\text{)}\\ &= \frac{AK'}{OA} \text{ (since } A' = A\text{)}\\ &= \frac{AD}{OA}.\end{aligned}$$

Let $OD : DC = x : y$, $CE : EA = z : x$, $AF : FB = y : z$. Then it is easy to calculate that $AD/OA = (x+y+z)/(x+y)$. This gives $AK/OK = (x+y+z)/(x+y)$. Similarly, $BL/OL = (x+y+z)/(y+z)$, $CM/OM = (x+y+z)/(z+x)$. Thus we get

$$\begin{aligned}\frac{AK}{OK} + \frac{BL}{OL} + \frac{CM}{OM} &= (x+y+z)\left(\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x}\right)\\ &\geq \frac{9}{2},\end{aligned}$$

by the AM-GM inequality. ■

96. Show that in any triangle ABC

$$\frac{1}{m_a m_b} + \frac{1}{m_b m_c} + \frac{1}{m_c m_a} \leq \frac{\sqrt{3}}{[ABC]}.$$

Solution: Let AD, BE, CF denote the medians of ABC so that $m_a = AD$, $m_b = BE$, $m_c = CF$. Let G be the centroid of ABC. Extend GD to K such that $GD = DK$. We observe that

$$GB = \frac{2}{3}m_b, \quad BK = \frac{2}{3}m_c, \quad KG = \frac{2}{3}m_a.$$

We also observe that

$$[GBK] = 2[GBD] = \frac{1}{3}[ABC].$$

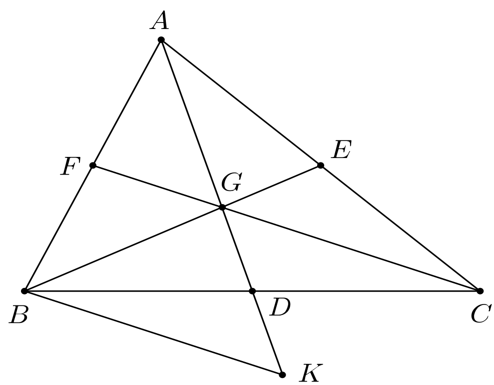

Fig. 6.15

Hence the area Δ_m of a triangle whose sides are m_a, m_b, m_c is

$$\Delta_m = \frac{9}{4}[GBK] = \frac{3}{4}[ABC]$$

Thus, we have to prove that

$$\frac{1}{m_a m_b} + \frac{1}{m_b m_c} + \frac{1}{m_c m_a} \leq \frac{3\sqrt{3}}{4\Delta_m}.$$

Since this relates the area of a triangle with its sides, all we need to prove is that in a triangle with sides a, b, c and area Δ the inequality

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \leq \frac{3\sqrt{3}}{4\Delta}$$

holds. But, this is the standard inequality

$$\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{4},$$

in a triangle ABC.

■

97. In any triangle ABC, prove that

$$w_a + w_b + w_c \le \sqrt{3}s.$$

Solution: We have

$$w_a^2 = \frac{4bc}{(b+c)^2}s(s-a) \le s(s-a),$$

and similar inequalities for w_b^2 and w_c^2. Using Cauchy-Schwarz inequality, we get

$$\begin{aligned}\big(w_a + w_b + w_c\big)^2 &\le 3\big(w_a^2 + w_b^2 + w_c^2\big)\\ &\le 3\big(s(s-a) + s(s-b) + s(s-c)\big)\\ &= 3s^2.\end{aligned}$$

Hence the inequality follows. ■

98. Let ABC be a triangle with points D, E, F respectively on the sides BC, CA, AB. Let the lines AD, BE, CF, when produced meet the circum-circle respectively in K, L, M. Prove that

$$\frac{AD}{DK} + \frac{BE}{EL} + \frac{CF}{FM} \ge 9.$$

Solution: Let $BD : DC = x : y$. Using Stewart's theorem, we get

$$xb^2 + yc^2 = \big(x+y\big)\big(AD^2 + BD \cdot DC\big).$$

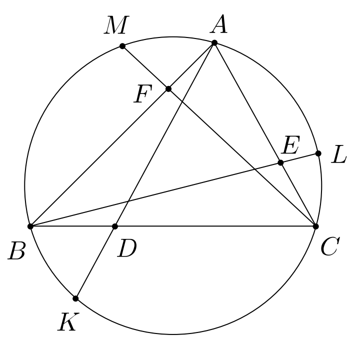

Fig. 6.16

This implies that

$$\frac{xb^2 + yc^2}{\big(x+y\big)BD \cdot DC} - 1 = \frac{AD^2}{BD \cdot DC} = \frac{AD^2}{AD \cdot DK}.$$

Thus,

$$\frac{AD}{DK} = \frac{xb^2 + yc^2}{(x+y)BD \cdot DC} - 1.$$

But, we know that

$$BD = \frac{xa}{x+y}, \quad DC = \frac{ya}{x+y}.$$

Thus, we get

$$\begin{aligned} \frac{AD}{DK} &= \frac{\left(xb^2 + yc^2\right)\left(x+y\right)}{xya^2} - 1 \\ &\geq \frac{\left(\sqrt{x}\, b\sqrt{y} + \sqrt{y}\, c\sqrt{x}\right)^2}{xya^2} - 1 \\ &= \frac{(b+c)^2}{a^2} - 1; \end{aligned}$$

we have used the Cauchy-Schwarz inequality. Using similar lower-bounds for the other two ratios, we obtain

$$\begin{aligned} \frac{AD}{DK} + \frac{BE}{EL} + \frac{CF}{FM} &\geq \sum_{\text{cyclic}} \frac{(b+c)^2}{a^2} - 3 \\ &\geq \frac{1}{3}\left(\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}\right)^2 - 3 \\ &\geq \frac{6^2}{3} - 3 = 9. \end{aligned}$$

■

99. Show that in a triangle ABC,

$$\max\left\{am_a, bm_b, cm_c\right\} \leq sR.$$

Solution: Let D, E, F be the mid-points of BC, CA, AB respectively. Let P and Q be the reflections of D in AB and AC respectively. Join DP and DQ. Let these intersect AB, AC in L, M respectively. We have

$$2LM = PQ \leq PF + FE + EQ = DF + FE + ED = \frac{1}{2}(a+b+c) = s.$$

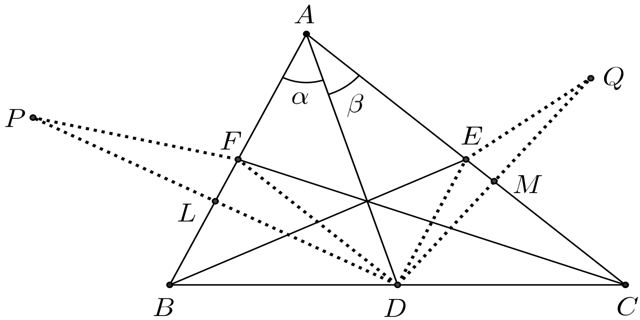

Fig. 6.17

But $ALDM$ is a cyclic quadrilateral. Hence, Ptolemy's theorem gives

$$LM \cdot AD = AL \cdot MD + AM \cdot LD.$$

Thus we obtain

$$\begin{aligned} LM &= AD\left\{\frac{AL}{AD}\cdot\frac{MD}{AD}+\frac{AM}{MD}\cdot\frac{LD}{AD}\right\} \\ &= AD\left\{\cos\alpha\sin\beta+\cos\beta\sin\alpha\right\} \\ &= AD\sin(\alpha+\beta) = AD\sin A. \end{aligned}$$

This shows that $LM = m_a \sin A$. Hence

$$s \geq 2LM = 2m_a \sin A = 2m_a \cdot \frac{a}{2R} = \frac{am_a}{R}.$$

Thus $am_a \leq sR$. Similarly, we may obtain $bm_b \leq sR$ and $cm_c \leq sR$. These three estimates give

$$\max\left\{am_a, bm_b, cm_c\right\} \leq sR.$$

■

100. Let $ABCD$ be a convex quadrilateral of area 1 unit. Prove that

$$AB + BC + CD + DA + AC + BD \geq 4 + \sqrt{8}.$$

Solution: If θ denotes one of the angles between the two diagonals, then we know that the area of $ABCD$ is $AC \cdot BD \cdot \sin\theta/2$. Thus

$$1 = \frac{1}{2}AC \cdot BD \cdot \sin\theta \leq \frac{1}{2}AC \cdot BD.$$

This shows that $AC \cdot BD \geq 2$. Now we know

$$[ABC] = \frac{1}{2} AB \cdot BC \cdot \sin B \leq \frac{1}{2} AB \cdot BC$$

$$[ADC] = \frac{1}{2} CD \cdot DA \cdot \sin D \leq \frac{1}{2} CD \cdot DA.$$

Adding, we get

$$1 = [ABCD] \leq \frac{1}{2}\bigl(AB \cdot BC + CD \cdot DA\bigr).$$

Similarly, we get

$$1 = [ABCD] \leq \frac{1}{2}\bigl(AB \cdot DA + CD \cdot BC\bigr).$$

Adding these two, we get

$$\bigl(AB + CD\bigr)\bigl(BC + DA\bigr) \geq 4.$$

This implies that

$$\bigl(AB + BC + CD + DA\bigr)^2 \geq 4\bigl(AB + CD\bigr)\bigl(BC + DA\bigr) \geq 16,$$

and hence $AB + BC + CD + DA \geq 4$. Again

$$\bigl(AC + BD\bigr)^2 \geq 4AC \cdot BD \geq 8.$$

It follows that $AC + BD \geq \sqrt{8}$. Hence

$$AB + BC + CD + DA + AC + BD \geq 4 + \sqrt{8}.$$

■

101. Let $ABCD$ be a square inscribed in circle. If M is a point on the arc AB (arc not containing C and D), prove that

$$MC \cdot MD \geq (3 + 2\sqrt{2}) MA \cdot MB.$$

Solution: Join M to A, B, C, D. Draw perpendiculars OL and MK to AB. We have

$$[MAB] = \frac{1}{2} MA \cdot MB \cdot \sin \angle AMB,$$

$$[MCD] = \frac{1}{2} MC \cdot MD \cdot \sin \angle DMC.$$

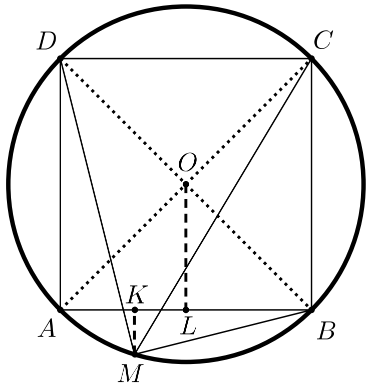

Fig. 6.17

But

$$\angle DMC = \frac{1}{2}\angle DOC = \frac{1}{2}\angle AOB = 180^\circ - \angle AMB,$$

and hence $\sin\angle DMC = \sin\angle AMB$. Thus

$$\frac{MC \cdot MD}{MA \cdot MB} = \frac{[MCD]}{[MAB]} = \frac{h+a}{h},$$

where $h = MK$ and a denotes the side length of $ABCD$. Note that $h \leq r - OL = r - (a/2)$. We hence obtain,

$$\frac{h+a}{h} \geq \frac{r-(a/2)+a}{r-(a/2)} = \frac{2r+a}{2r-a}.$$

However $r = a/\sqrt{2}$. Thus

$$\frac{h+a}{h} \geq \frac{2+\sqrt{2}}{2-\sqrt{2}} = 3 + 2\sqrt{2}.$$

■

102. Let a, b, c be the sides of a triangle ABC with in-radius r. Prove that

$$a\sin A + b\sin B + c\sin C \geq 9r.$$

Solution: We may write this in the equivalent form

$$a^2 + b^2 + c^2 \geq 18Rr.$$

Using two different expressions for the area of a triangle, we have

$$\frac{abc}{4R} = rs = \frac{r(a+b+c)}{2}.$$

This gives $Rr = abc/2(a+b+c)$ and hence the inequality is equivalent to

$$9abc \leq (a+b+c)(a^2+b^2+c^2).$$

However, observe that

$$\begin{aligned}(abc)^{2/3} &\leq \frac{a^2+b^2+c^2}{3},\\ (abc)^{1/3} &\leq \frac{a+b+c}{3}.\end{aligned}$$

Multiplying these, we get the desired inequality. ■

103. Suppose ABC is an acute-angled triangle with area Δ and in-radius r. Prove that

$$\left(\sqrt{\cot A}+\sqrt{\cot B}+\sqrt{\cot C}\right)^2 \leq \frac{\Delta}{r^2}.$$

Solution: We use the expression

$$\cot A = \frac{\cos A}{\sin A} = \frac{b^2+c^2-a^2}{2bc\sin A} = \frac{R(b^2+c^2-a^2)}{abc},$$

and similar expressions for $\cot B$ and $\cot C$. Thus the inequality to be proved is:

$$\left(\sqrt{b^2+c^2-a^2}+\sqrt{c^2+a^2-b^2}+\sqrt{a^2+b^2-c^2}\right)^2 \leq \frac{\Delta abc}{r^2 R} = (a+b+c)^2.$$

Using the concavity of $f(x)=\sqrt{x}$ on $(0,\infty)$, we observe that

$$\frac{1}{2}\sqrt{b^2+c^2-a^2}+\frac{1}{2}\sqrt{c^2+a^2-b^2} \leq c,$$

and we have similar expressions for other sums. Thus it follows that

$$\sqrt{b^2+c^2-a^2}+\sqrt{c^2+a^2-b^2}+\sqrt{a^2+b^2-c^2} \leq a+b+c,$$

and that completes the proof. ■

104. Let ABC be a triangle having the circum-radius R. Let P be an interior point of ABC. Prove that

$$\frac{AP}{BC^2}+\frac{BP}{CA^2}+\frac{CP}{AB^2} \geq \frac{1}{R}.$$

Solution: Draw perpendiculars from P on to BC, CA, AB to meet them in X, Y, Z respectively. Draw perpendiculars from Y and Z on to the extended line XP to meet it in N and M respectively. Since $PZAY$ is a cyclic quadrilateral, it is easy to see that $ZY = PA\sin\angle A$.

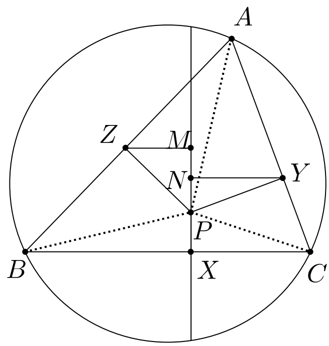

Fig. 6.18

But $ZY \geq ZM + YN$. We also note that $BXPZ$ is a cyclic quadrilateral. Hence

$$\angle XBZ = 180^\circ - \angle XPZ = \angle ZPM.$$

It follows that $\angle ZPM = B$ and hence $ZM = ZP\sin B$. Similarly, it may be proved that $YN = YP\sin C$. Thus we get

$$PA\sin A \geq PZ\sin B + PY\sin C.$$

This implies that $a \cdot PA \geq b \cdot PZ + c \cdot PY$. Similarly, we may prove that

$$b \cdot PB \geq c \cdot PX + a \cdot PZ, \quad c \cdot PC \geq a \cdot PY + b \cdot PX.$$

Thus we obtain

$$\begin{aligned}
\frac{AP}{BC^2} + \frac{BP}{CA^2} + \frac{CP}{AB^2} &\geq PX\left(\frac{c}{b^2} + \frac{b}{c^2}\right) + PY\left(\frac{a}{c^2} + \frac{c}{a^2}\right) + PZ\left(\frac{b}{a^2} + \frac{a}{b^2}\right) \\
&\geq \frac{2PX}{bc} + \frac{2PY}{ca} + \frac{2PZ}{ab} \\
&= \frac{4[ABC]}{abc} = \frac{1}{R},
\end{aligned}$$

where $[ABC]$ denotes the area of triangle ABC. ■

105. With every natural number n, associate a real number a_n by

$$a_n = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k},$$

where $\{p_1, p_2, \ldots, p_k\}$ is the set of all prime divisors of n. Show that for any natural number $N \geq 2$,

$$\sum_{n=2}^{N} a_2 a_3 \cdots a_n < 1.$$

Solution: We observe that

$$\sum_{n=2}^{N} a_k = \sum_{\substack{p \leq N \\ p \text{ a prime}}} \frac{1}{p}\left[\frac{N}{p}\right].$$

On the other hand

$$\begin{aligned}
\sum_{\substack{p \leq N \\ p \text{ a prime}}} \frac{1}{p}\left[\frac{N}{p}\right] &\leq \sum_{\substack{p \leq N \\ p \text{ a prime}}} \frac{1}{p}\left(\frac{N}{p}\right) \\
&= N \sum_{\substack{p \leq N \\ p \text{ a prime}}} \frac{1}{p^2} \\
&< N\left(\frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2}\right) \\
&< N\left(\frac{1}{4} + \frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{k(k+1)}\right) \\
&= \frac{N}{2}.
\end{aligned}$$

Thus we see that

$$\sum_{k=2}^{N} a_k < \frac{N}{2}.$$

Using the AM-GM inequality, we obtain

$$\begin{aligned}
a_2 a_3 \cdots a_N &< \left(\frac{a_2 + a_3 + \cdots + a_N}{N-1}\right)^{N-1} \\
&< \frac{1}{2^{N-1}}\left(1 + \frac{1}{N-1}\right)^{N-1} \\
&< \frac{3}{2^{N-1}}.
\end{aligned}$$

Hence we get

$$\begin{aligned}
\sum_{n=2}^{N} a_2a_3\cdots a_n &= a_2+a_2a_3+a_2a_3a_4+a_2a_3a_4a_5+\cdots \\
&< \frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{60}+3\left(\frac{1}{2^5}+\frac{1}{2^6}+\cdots\right) \\
&= \frac{46}{60}+\frac{6}{32} \\
&= \frac{229}{240}<1.
\end{aligned}$$

■

106. Let n be a fixed integer, with $n \geq 2$.

(a) Determine the least constant C such that the inequality

$$\sum_{1\leq i<j\leq n} x_ix_j(x_i^2+x_j^2) \leq C\left(\sum_{1\leq i\leq n} x_i\right)^4$$

holds for all real numbers $x_1,\ldots,x_n \geq 0$.

(b) For this constant C, determine when equality holds.

Solution: We show that $C = 1/8$ is the least constant and there is a configuration for which this is attained. We have

$$\begin{aligned}
\left(\sum_{j=1}^{n} x_j\right)^4 &= \left(\sum_{j=1}^{n} x_j^2+2\sum_{1\leq j<k\leq n} x_jx_k\right)^2 \\
&\geq 4\left(\sum_{j=1}^{n} x_j^2\right)\left(2\sum_{1\leq j<k\leq n} x_jx_k\right) \\
&= 8\left(\sum_{j=1}^{n} x_j^2\right)\left(\sum_{1\leq j<k\leq n} x_jx_k\right) \\
&\geq 8\left(\sum_{1\leq j<k\leq n} x_jx_k(x_j^2+x_k^2)\right).
\end{aligned}$$

The second inequality is an equality only if $n-2$ of x_j's are zero. If we take $x_3 = x_4 = \cdots = x_n = 0$, then the first inequality becomes an equality only if $x_1 = x_2$. Thus if we take $x_1 = x_2 = a$ for some real number a and $x_3 = x_4 = \cdots = x_n = 0$, then we get equality with constant $C = 1/8$.

Alternate Solution:

It is sufficient to prove that

$$\sum_{1\le i<j\le n} x_i x_j (x_i^2 + x_j^2) \le \frac{1}{8},$$

under the assumption that $\sum x_j = 1$.

We can make use of Newton's identities; if for $k \le n$

$$S_k = \sum_{j=1}^{n} x_j^k, \qquad \text{and} \qquad \alpha_k = \sum_{1\le j_1\le j_2\le\cdots\le j_k\le n} x_{j_1} x_{j_2} \cdots x_{j_k},$$

and $\alpha_k = 0$ for $k > n$, then

$$S_2 - \alpha_1 S_1 + 2\alpha_2 = 0,$$
$$S_4 - \alpha_1 S_3 + \alpha_2 S_2 - \alpha_3 S_1 + 4\alpha_4 = 0.$$

We thus have

$$\begin{aligned}
\sum_{1\le i<j\le n} x_i x_j (x_i^2 + x_j^2) &= \left(\sum_{j=1}^{n} x_j^3\right)\left(\sum_{j=1}^{n} x_j\right) - \sum_{j=1}^{n} x_j^4 \\
&= S_3 - S_4 = \alpha_2 S_2 - \alpha_3 + 4\alpha_4 \\
&= \alpha_2(1-\alpha_2) + 4\alpha_4 - \alpha_3.
\end{aligned}$$

Here we have used $\alpha_1 = S_1 = 1$. However, we know that $\alpha_2(1-2\alpha_2) \le 1/8$. On the other hand

$$\begin{aligned}
4\alpha_4 - \alpha_3 &= \sum_{j_1<j_2<j_3} x_{j_1} x_{j_2} x_{j_3} \left(\sum_{k\notin\{j_1,j_2,j_3\}} x_k\right) \\
&\qquad - \sum_{j_1<j_2<j_3} x_{j_1} x_{j_2} x_{j_3} \\
&= \sum_{j_1<j_2<j_3} x_{j_1} x_{j_2} x_{j_3} \left(\sum_{k\notin\{j_1,j_2,j_3\}} x_k - 1\right) \\
&= -\sum_{j_1<j_2<j_3} x_{j_1} x_{j_2} x_{j_3} (x_{j_1} + x_{j_2} + x_{j_3}) \le 0.
\end{aligned}$$

Thus we get

$$\sum_{1\le i<j\le n} x_i x_j (x_i^2 + x_j^2) \le \frac{1}{8},$$

for $n \ge 4$. ∎

107. Let a, b, c, d be real numbers such that

$$\left(a^2 + b^2 - 1\right)\left(c^2 + d^2 - 1\right) > \left(ac + bd - 1\right)^2.$$

Prove that $a^2 + b^2 - 1 > 0$ and $c^2 + d^2 - 1 > 0$.

Solution: Suppose, if possible, one of a^2+b^2, c^2+d^2 is less than 1. (Observe that none can be equal to 1.) Since $\left(ac+bd-1\right)^2 \geq 0$, the product $\left(a^2+b^2-1\right)\left(c^2+d^2-1\right)$ is positive. Hence it follows that $a^2+b^2-1<0$ and $c^2+d^2-1<0$. Let us put $x=1-a^2-b^2$ and $y=1-c^2-d^2$. Then $0<x,y<1$ and hence

$$\begin{aligned} 4xy &= \left(2-2a^2-2b^2\right)\left(2-2c^2-2d^2\right) \\ &> \left(2ac+2bd-2\right)^2 \\ &= \left(a^2+b^2+x+c^2+d^2+y-2ac-2bd\right)^2 \\ &= \left((a-c)^2+(b-d)^2+x+y\right)^2 \\ &\geq (x+y)^2. \end{aligned}$$

This shows that $(x-y)^2<0$, which is impossible. We conclude that $a^2+b^2>1$ and $c^2+d^2>1$.

Here is a generalisation used in the US Selection Tests-2004.

Let $a_1,a_2,a_3,\ldots,a_n$ *and* $b_1,b_2,b_3,\ldots,b_n$ *real numbers such that*

$$\left(a_1^2+a_2^2+\cdots+a_n^2-1\right)\left(b_1^2+b_2^2+\cdots+b_n^2-1\right) > \left(a_1b_1+a_2b_2+\cdots+a_nb_n-1\right)^2.$$

Show that $a_1^2+a_2^2+\cdots+a_n^2>1$, *and* $b_1^2+b_2^2+\cdots+b_n^2>1$.

The argument of the proof is essentially the same. But we prove an inequality due to Aczel which will help us to resolve the problem quickly. Let $x_1,x_2,x_3,\ldots,x_m$ and $y_1,y_2,y_3,\ldots,y_m$ be real numbers such that

$$x_1^2 > x_2^2+x_3^2+\cdots+x_m^2.$$

Then

$$\begin{aligned} &\left(x_1y_1-x_2y_2-x_3y_3-\cdots-x_my_m\right)^2 \\ &\qquad \geq \left(x_1^2-x_2^2-x_3^2-\cdots-x_m^2\right)\left(y_1^2-y_2^2-y_3^2-\cdots-y_m^2\right). \end{aligned} \tag{$\star$}$$

This follows from the property of the following quadratic polynomial:

$$\begin{aligned} f(t) &= \left(x_1t+y_1\right)^2-\sum_{j=2}^{m}\left(x_jt+y_j\right)^2 \\ &= \left(x_1^2-\sum_{j=2}^{m}x_j^2\right)t^2+2\left(x_1y_1-\sum_{j=2}^{m}x_jy_j\right)t+\left(y_1^2-\sum_{j=2}^{m}y_j^2\right). \end{aligned}$$

Note that the leading coefficient is positive. Moreover,

$$f\left(-\frac{y_1}{x_1}\right)=\sum_{j=2}^{m}\left(-\frac{x_jy_1}{x_1}+y_j\right)^2 \leq 0.$$

Hence the discriminant must be non-negative, which implies $(\star)$.

We apply $(\star)$ to the sequences $1, a_1, a_2, \ldots a_n$ and $1, b_1, b_2, \ldots b_n$. Since

$$\left(\sum_{j=1}^{n} a_j^2 - 1\right)\left(\sum_{j=1}^{n} b_j^2 - 1\right) > \left(\sum_{j=1}^{n} a_j b_j - 1\right)^2 > 0,$$

it follows that both $\sum_{j=1}^{n} a_j^2 - 1$ and $\sum_{j=1}^{n} b_j^2 - 1$ have same sign. If both are negative, then Aczel's inequality shows that

$$\begin{aligned}\left(1 - \sum_{j=1}^{n} a_j b_j\right)^2 &\geq \left(\sum_{j=1}^{n} a_j^2 - 1\right)\left(\sum_{j=1}^{n} b_j^2 - 1\right) \\ &> \left(1 - \sum_{j=1}^{n} a_j b_j\right)^2.\end{aligned}$$

This contradiction proves that $\sum_{j=1}^{n} a_j^2 - 1$ and $\sum_{j=1}^{n} b_j^2 - 1$ are both positive. ■

108. Let x_1,x_2,x_3,. . .,x_{100} be 100 positive integers such that

$$\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_{100}}} = 20.$$

Prove that at least two of x_j's are equal.

Solution: Suppose all the x_j's are distinct; say $x_1 < x_2 < x_3 < \cdots < x_{100}$. Then $x_j \geq j$ for $1 \leq j \leq 100$. Thus

$$20 = \frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_{100}}} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{100}}.$$

But for any natural number k, it is easy to see that

$$\frac{1}{\sqrt{k}} < \frac{2}{\sqrt{k} + \sqrt{k-1}} = 2\left(\sqrt{k} - \sqrt{k-1}\right).$$

Thus it follows that

$$\begin{aligned}20 &\leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{100}} \\ &< 2\sum_{k=1}^{100}\left(\sqrt{k} - \sqrt{k-1}\right) \\ &= 2\sqrt{100} = 20,\end{aligned}$$

which is impossible. We conclude that not all x_j's are distinct. ■

109. Let $f(x)$ be a polynomial with integer coefficients and of degree $n > 1$. Suppose $f(x) = 0$ has n real roots in the interval $(0, 1)$, not all equal. If a is the leading coefficient of $f(x)$, prove that

$$|a| \geq 2^n + 1.$$

Solution: Observe that $f(0) \neq 0$ and $f(1) \neq 0$. Hence $1 \leq |f(0)f(1)|$. Let $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ be the roots of $f(x) = 0$, which all lie in $(0, 1)$. Then

$$f(x) = a(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n).$$

Thus we have

$$1 \leq |f(0)f(1)| = |a|^2 |\alpha_1 \cdot \alpha_2 \cdots \alpha_n (1 - \alpha_1) \cdot (1 - \alpha_2) \cdots (1 - \alpha_n)|.$$

But we know that $x(1 - x) \leq 1/4$ for $x \in (0, 1)$ with equality only if $x = 1/2$. Since not all roots are equal, we conclude that

$$4^n < |a|^2.$$

Since a is an integer, it follows that $|a| \geq 2^n + 1$. ■

110. Show that the equation

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{w} + \frac{w}{x} = m,$$

has no solutions in positive reals for $m = 2, 3$.

Solution: If x, y, z, w are all positive, then the AM-GM inequality shows that

$$m = \frac{x}{y} + \frac{y}{z} + \frac{z}{w} + \frac{w}{x} \geq 4.$$

Hence the result follows. ■

111. Solve the system of equations

$$x = \frac{4z^2}{1 + 4z^2}, \quad y = \frac{4x^2}{1 + 4x^2}, \quad z = \frac{4x^2}{1 + 4x^2},$$

for real numbers x, y, z.

Solution: If any one of x, y, z is 0, then the remaining two are also 0. Suppose none of them is 0. Then x, y, z are all positive. Multiplying all the relations,

$$64xyz = \left(1 + 4x^2\right)\left(1 + 4y^2\right)\left(1 + 4z^2\right) \geq 4x \cdot 4y \cdot 4z,$$

where we have used the AM-GM inequality in the last step. This shows that equality holds in the inequality, implying $x = y = z = 1/2$. Thus we have two solutions:

$$(x, y, z) = (0, 0, 0), (1/2, 1/2, 1/2).$$

■

112. Suppose a, b are nonzero real numbers and that all the roots of the real polynomial

$$ax^n - ax^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 - n^2bx + b = 0$$

are real and positive. Prove that all the roots are in fact equal.

Solution: Let α_1,α_2,α_3,. . .,α_n be the roots of the given equation. Then

$$\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n = 1, \quad \frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \cdots + \frac{1}{\alpha_n} = n^2.$$

This shows that

$$n^2 = \Big(\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n\Big)\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \cdots + \frac{1}{\alpha_n}\right) \geq n^2,$$

by the AM-GM inequality. Hence equality holds in the inequality and this is precisely the case when $\alpha_1=\alpha_2=\alpha_3=\cdots=\alpha_n$. ■

113. Find all triples (a, b, c) of positive integers such that the product of any two leaves the remainder 1 when divided by the third number.

Solution: Observe that no two can be equal. Nor any one of them can be equal to 1. Thus we may assume that $1 < a < b < c$. Write $bc = 1 + ax$, $ca = 1 + by$ and $ab = 1 + cz$. From this we obtain

$$xyzabc = (bc - 1)(ca - 1)(ab - 1) = (abc)^2 - abc(a + b + c) + ab + bc + ca - 1.$$

This expression shows that abc divides $ab + bc + ca - 1$. Since abc and $ab + bc + ca - 1$ are positive integers, we can immediately get

$$abc < ab + bc + ca. \qquad (*)$$

Using $a < b < c$, we get $abc < ab + bc + ca < 3bc$ and hence it follows that $a < 3$. The condition $a > 1$ now implies that $a = 2$. This can be put back in the inequality (*) to get $bc < 2(b + c)$. Again using $b < c$, we get $bc < 4c$ or equivalently $b < 4$. Taking in to account $2 = a < b$, we can only have $b = 3$. Using $a = 2$ and $b = 3$, we see that the only possibilities for c are 4 and 5 of which 4 can be ruled out (why?). Thus the only solution (a, b, c) with $a < b < c$ is $(2, 3, 5)$. Permutations of this solution give all the other solutions. ■

114. Show that a triangle is equilateral if and only if

$$a\cos(\beta - \gamma) + b\cos(\gamma - \alpha) + c\cos(\alpha - \beta) = \frac{a^4 + b^4 + c^4}{abc},$$

where a, b, c are the sides and α, β, γ are the angles opposite to the sides a, b, c respectively.

Solution: If ABC is equilateral, then $\alpha = \beta = \gamma$ and $a = b = c$. Thus both the left hand side and the right hand side reduce to $3a$.

Suppose the above relation holds. Then the AM-GM inequality shows that

$$\begin{aligned} a^4 + b^4 + c^4 &\geq a^2b^2 + b^2c^2 + c^2a^2 \\ &\geq a^2bc + b^ca + c^ab \\ &= abc(a+b+c). \end{aligned}$$

Thus,

$$\begin{aligned} \frac{a^4 + b^4 + c^4}{abc} &\geq a + b + c \\ &\geq a\cos(\beta-\gamma) + b\cos(\gamma-\alpha) + c\cos(\alpha-\beta) \\ &= \frac{a^4 + b^4 + c^4}{abc}. \end{aligned}$$

Hence equality holds every where. This implies that

$$a^4 + b^4 + c^4 = abc(a+b+c),$$

which in turn gives $a = b = c$ ■

115. Find all positive solutions of the system

$$x_1 + \frac{1}{x_2} = 4, x_2 + \frac{1}{x_3} = 1, \ldots, x_{1999} + \frac{1}{x_{2000}} = 4, x_{2000} + \frac{1}{x_1} = 1.$$

Solution: Using the AM-GM inequality, we have

$$x_1 + \frac{1}{x_2} \geq 2\sqrt{\frac{x_1}{x_2}}, \cdots x_{2000} + \frac{1}{x_1} \geq 2\sqrt{\frac{x_{2000}}{x_1}}.$$

Hence it follows that

$$4^{1000} = \left(x_1 + \frac{1}{x_2}\right)\left(x_2 + \frac{1}{x_3}\right) \cdots \left(x_{2000} + \frac{1}{x_1}\right) \geq 4^{1000}.$$

We conclude that

$$x_1 = \frac{1}{x_2}, x_2 = \frac{1}{x_3}, \cdots, x_{2000} = \frac{1}{x_1}.$$

It follows that

$$x_1 = x_3 = x_5 = \cdots = x_{1999} = 2, \text{ and } x_2 = x_4 = x_6 = \cdots = x_{2000} = \frac{1}{2}.$$

■

116. Find all positive solutions of the system

$$\begin{aligned} x + y + z &= 1, \\ x^3 + y^3 + z^3 + xyz &= x^4 + y^4 + z^4 + 1. \end{aligned}$$

Solution: Using the first relation, we have

$$1 = x + y + z \geq 3\big(xyz\big)^{1/3},$$

so that $xyz \leq 1/27$. Using the second relation, we get

$$x^3(1-x) + y^3(1-y) + z^3(1-z) = 1 - xyz \geq \frac{26}{27}.$$

However, for any $t \in (0,1)$, we have

$$3t^3(1-t) = t \cdot t \cdot t \cdot (3-3t) \leq \left(\frac{t+t+t+3-3t}{4}\right)^4 = \frac{81}{256}.$$

Thus,

$$x^3(1-x) + y^3(1-y) + z^3(1-z) \leq \frac{81}{256},$$

which contradicts $x^3(1-x)+y^3(1-y)+z^3(1-z) \geq 26/27$. We conclude that there are no solutions to the given system in positive real numbers. ■

117. Let a, b be positive integers such that each equation

$$(a+b-x)^2 = a-b, \quad (ab+1-x)^2 = ab-1$$

has two distinct real roots. Suppose the bigger of these roots are the same. Show that the smaller roots are also the same.

Solution: Observe the equation $(a+b-x)^2 = a-b$ has two distinct roots if and only if $a > b$. Similarly, the second equation has distinct roots if and only if $ab > 1$. Since $a > 0$, we have $a^2 > ab > 1$ so that $a > 1$. Thus we obtain

$$a > b > \frac{1}{a}, \quad a > 1.$$

Now, the bigger roots of the two equations are

$$a + b + \sqrt{a-b}, \quad ab + 1 + \sqrt{ab-1}.$$

Equality of these two gives

$$\sqrt{a-b} = (a-1)(b-1) + \sqrt{ab-1}.$$

If $a > b \geq 1$, then

$$\sqrt{a-b} \geq \sqrt{ab-1}.$$

It follows that $(a+1)(b-1) \leq 0$, showing that $b \leq 1$. Hence $b = 1$. On the other-hand if $(1/a) < b \leq 1$, then

$$\sqrt{a-b} \leq \sqrt{ab-1},$$

and hence $(a+1)(b-1) \geq 0$. It follows that $b \geq 1$, and thus $b = 1$. Thus the bigger two roots are equal if and only if $b = 1$ and $a > 1$. In this case both the equations reduce to $(a+1-x)^2 = (a-1)$ and hence the smaller roots are also equal. ■

118. Suppose the polynomial

$$P(x) = x^n + nx^{n-1} + a_2x^{n-2} + \cdots + a_n$$

has real roots $\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n$. If

$$\alpha_1^{16} + \alpha_2^{16} + \cdots + \alpha_n^{16} = n.$$

Find $\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n$.

Solution: We have

$$\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n = -n.$$

If we use the Cauchy-Schwarz inequality, we get

$$n^2 = \left(\sum_{j=1}^{n} \alpha_j\right)^2 \leq n \sum_{j=1}^{n} \alpha_j^2.$$

Repeatedly applying this we obtain

$$\begin{aligned}
n^4 &\leq n^2 \cdot n \cdot \sum_{j=1}^{n} \alpha_j^4 = n^3 \sum_{j=1}^{n} \alpha_j^4, \\
n^8 &\leq n^6 \cdot n \cdot \sum_{j=1}^{n} \alpha_j^8 = n^7 \sum_{j=1}^{n} \alpha_j^8, \\
n^{16} &\leq n^{14} \cdot n \cdot \sum_{j=1}^{n} \alpha_j^{16} = n^{16}.
\end{aligned}$$

Thus equality holds in the Cauchy-Schwarz inequality. This implies that all the α_j's are equal. Hence $\alpha_j = -1$ for all j. ■

119. Find all the solutions of the following system of inequalities:

$$\begin{array}{lrcl}
(i) & \left(x_1^2 - x_3x_5\right)\left(x_2^2 - x_3x_5\right) & \leq & 0, \\
(ii) & \left(x_2^2 - x_4x_1\right)\left(x_3^2 - x_4x_1\right) & \leq & 0, \\
(iii) & \left(x_3^2 - x_5x_2\right)\left(x_4^2 - x_5x_2\right) & \leq & 0, \\
(iv) & \left(x_4^2 - x_1x_3\right)\left(x_5^2 - x_1x_3\right) & \leq & 0, \\
(v) & \left(x_5^2 - x_2x_4\right)\left(x_1^2 - x_2x_4\right) & \leq & 0.
\end{array}$$

Solution: It is easy to check that $x_1 = x_2 = x_3 = x_4 = x_5 = \lambda$ is a solution. Suppose not all the x_j's are equal. Thus among x_3, x_5, x_2, x_4, x_1, two consecutive numbers are distinct; say $x_3 \neq x_5$. We observe that whenever $(x_1, x_2, x_3, x_4, x_5)$ is a solution, $(1/x_1, 1/x_2, 1/x_3, 1/x_4, 1/x_5)$ is also a solution. Hence we may assume that $x_3 < x_5$.

Suppose $x_1 \le x_2$. Then $x_1^2 - x_3x_5 \le 0$ and $x_2^2 - x_3x_5 \ge 0$. This shows that $x_1 \le \sqrt{x_3x_5} < x_5$ and $x_2 \ge \sqrt{x_3x_5} > x_3$. Thus $x_1x_3 < x_5^2$ and $x_4^2 \le x_1x_3 < x_3x_5$. But $x_3^2 < x_2x_3 < x_2x_5$. Hence relation (iii) shows that $x_4^2 \ge x_2x_5 > x_3x_5$, because $x_2 > x_3$. This is impossible.

Suppose $x_1 > x_2$. Then (i) gives $x_1 \ge \sqrt{x_3x_5} > x_3$ and $x_2 \le \sqrt{x_3x_5} < x_5$. Using (ii) and (iv), we infer that

$$x_1x_4 \le \max\{x_2^2, x_3^3\} \le x_3x_5, \quad x_2x_4 \ge \min\{x_1^2, x_5^2\} \ge x_3x_5.$$

This shows that $x_2x_4 \ge x_1x_4$, contradicting $x_1 > x_2$. Thus all the x_j's must be equal. ■

120. Solve the following system of equations, when a is a real number such that $|a| > 1$:

$$\begin{aligned} x_1^2 &= ax_2 + 1, \\ x_2^2 &= ax_3 + 1, \\ &\vdots \\ x_{999}^2 &= ax_{1000} + 1, \\ x_{1000}^2 &= ax_1 + 1. \end{aligned}$$

Solution: We may assume that $a > 1$. For, if $a < -1$, we can take $a' = -a$ and $x_j' = -x_j$, $1 \le j \le 1000$, to get a similar system in which $a' > 1$. Since left sides are squares, we get $x_j' \ge -1/a > -1$, for $1 \le j \le 1000$. We may assume $x_1 \ge x_j$, $1 \le j \le 1000$. Thus $x_1 \ge x_2, x_3$. If $x_1 \ge 0$, then $x_{1000}^2 \ge 1$ and hence $x_{1000} \ge 1$. This implies that $x_{999} > 1$. Using induction, we get $x_j > 1$ for all j. Thus either $x_j > 1$ for all j or $x_j < 0$ for all j.

Suppose $x_j > 1$ for all j. Then $x_1 \ge x_2$ and hence $x_1^2 \ge x_2^2$. This implies that $x_2 \ge x_3$. Using induction, we have $x_1 \ge x_2 \ge x_3 \ge \cdots \ge x_{1000} \ge x_1$. Consequently all x_j's are equal. Thus we have to solve the equation $x^2 = ax + 1$. We obtain

$$x_j = \frac{1}{2}\left(a + \sqrt{a^2+4}\right), \text{ for } 1 \le j \le 1000.$$

Suppose $x_j < 0$ for all j. Then $x_1 \ge x_3$ implies that $x_1^2 \le x_3^2$ and hence $x_2 \le x_4$. This implies that $x_3 \ge x_5$. Again induction shows that $x_1 \ge x_3 \ge x_5 \ge \cdots \ge x_{999} \ge x_1$ and $x_2 \le x_4 \le \cdots \le x_{1000} \le x_2$. Thus we get $x_2 = x_4 = \cdots = x_{1000}$ and $x_1 = x_3 = \cdots = x_{999}$. In this case we have two equations: $x_1^2 = ax_2 + 1$, $x_2^2 = ax_1 + 1$. Hence

$$x_1^2 - x_2^2 = a(x_2 - x_1).$$

If $x_1 = x_2$, then all x_j's are equal and each is $\left(a - \sqrt{a^2+4}\right)/2$. Otherwise $x_1 + x_2 + a = 0$. In this case $x_1^2 + ax_1 + (a^2 - 1) = 0$. The discriminant is

$4 - 3a^2$. If $a > 2/\sqrt{3}$, then there is no solution to the system. If $a \le 2/\sqrt{3}$, then

$$x_1 = \frac{1}{2}\Big(- a - \sqrt{a^2+4}\Big) \text{ and } x_2 = \frac{1}{2}\Big(- a + \sqrt{a^2+4}\Big),$$

or the other way round. ■

121. Let a_1,a_2,a_3,. . .,a_n be n positive integers such that $\sum_{j=1}^{n} a_j = \prod_{j=1}^{n} a_j$. Let V_n denote this common value. Show that $V_n \ge n + s$, where s is the least positive integer such that $2^s - s \ge n$.

Solution: (By Rishi Raj) We begin with the observation that for $a, b \in \mathbb{N}$,

$$2^a - a \ge 2^b - b \Longleftrightarrow a \ge b.$$

This follows easily from the fact that the function $f(x) = 2^x - x$ is a non-decreasing function on $\mathbb{R}$. Thus we see that

$$V_n - n \ge s \Longleftrightarrow 2^{V_n - n} - V_n + n \ge 2^s - s.$$

The definition of s shows that it is sufficient to prove

$$2^{V_n - n} - \big(V_n - n\big) \ge n.$$

We also have $2^{m-1} \ge m$ for $m \ge 1$, which can be easily established by induction. Thus

$$2^{V_n - n} = 2^{\sum_{j=1}^{n}(a_j - 1)} = 2^{(a_1-1)}2^{(a_2-1)} \cdots 2^{(a_n-1)} \ge a_1 \cdot a_2 \cdots a_n = V_n.$$

This gives

$$2^{V_n - n} - \big(V_n - n\big) \ge n.$$

The desired inequality follows.

Alternate Solution:(By Aravind Narayanan)

Let k be the number of a_j which are strictly larger than 1. Let us take $a_1 > 1$, $a_2 > 1$, . . . , $a_k > 1$, and $a_j = 1$ for $k + 1 \le j \le n$. Then

$$V_n = a_1 + a_2 + \cdots + a_k + (n - k) = a_1 \cdot a_2 \cdot a_3 \cdots a_k.$$

Observe that $V_n \ge n$. Suppose $V_n = n + r$ for some r such that $1 \le r < s$. Then we have

$$\begin{aligned}
(n-k) + a_1 + a_2 + \cdots + a_k &= n + r \\
\Longrightarrow \quad & a_1 + a_2 + \cdots + a_k = r + k \\
\Longrightarrow \quad & r + k \ge \underbrace{2 + 2 + \cdots + 2}_{k} = 2k \\
\Longrightarrow \quad & r \ge k.
\end{aligned}$$

But we see that

$$\underbrace{1+1+\cdots+1}_{r-k}+a_1+a_2+\cdots+a_k=r-k+r+k=2r.$$

This gives

$$2r=\underbrace{1+1+\cdots+1}_{r-k}+a_1+a_2+\cdots+a_k\geq r\big(a_1a_2\cdots a_k\big)^{1/r}.$$

We have used the AM-GM inequality. Thus we obtain

$$a_1a_2\cdots a_k\leq 2^r.$$

Hence

$$n+r=V_n=a_1a_2\cdots a_k\leq 2^r.$$

It follows that $2^r-r\geq n$. But this contradicts the minimality of s. We conclude that $r\geq s$. This gives $V_n=n+r\geq n+s$. ■

122. Let $z_1,z_2,z_3,\ldots,z_n$ be n complex numbers such that $\sum_{j=1}^n|z_j|=1$. Prove that there exists a subset S of the set $\big\{z_1,z_2,z_3,\ldots,z_n\big\}$ such that

$$\left|\sum_{z\in S}z\right|\geq\frac{1}{4}.$$

Solution: Let us express $z_k=x_k+iy_k$ for $1\leq k\leq n$. Then we have

$$1=\sum_{j=1}^n|z_j|\leq\sum_{j=1}^n\Big(|x_j|+|y_j|\Big)=\sum_{x_j\geq 0}|x_j|+\sum_{x_j<0}|x_j|+\sum_{y_j\geq 0}|y_j|+\sum_{y_j<0}|y_j|.$$

By the pigeonhole principle, at least one of the sums is not smaller than 1/4. By symmetry, we may assume it to be the first sum. Thus we get

$$\frac{1}{4}\leq\sum_{x_j\geq 0}|x_j|=\left|\sum_{x_k\geq 0}x_k\right|.$$

It follows that

$$\left|\sum_{x_k\geq 0}z_k\right|\geq\left|\sum_{x_k\geq 0}x_k\right|\geq\frac{1}{4}.$$

We, in fact, show that the constant $1/4$ may be replaced by $1/\pi$. For a real number x, let $x^+=\max\big\{x,0\big\}$. We write for $1\leq k\leq n$,

$$z_k=r_k\big(\cos\theta_k+i\sin\theta_k\big),$$

where $r_k = |z_k|$, $0 \le \theta_k < 2\pi$. Define

$$f(\theta) = \sum_{k=1}^{n} r_k \Big(\cos(\theta - \theta_k) \Big)^+.$$

We see that

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta &= \frac{1}{2\pi} \sum_{k=1}^{n} r_k \int_0^{2\pi} \Big(\cos(\theta - \theta_k) \Big)^+ d\theta \\ &= \frac{1}{2\pi} \sum_{k=1}^{n} r_k \int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta \\ &= \frac{1}{\pi} \sum_{k=1}^{n} r_k. \end{aligned}$$

Hence there is a number θ_0 such that

$$f(\theta_0) \ge \frac{1}{\pi} \sum_{k=1}^{n} |z_k|.$$

We set

$$T = \Big\{ j \,\Big|\, 1 \le j \le n, \cos(\theta_0 - \theta_k) > 0 \Big\}.$$

With this set T, we obtain

$$\begin{aligned} \Big| \sum_{j \in T} z_j \Big| &= \Big| e^{-i\theta_0} \sum_{j \in T} z_j \Big| \\ &\ge \mathrm{Re}\Big(\sum_{j \in T} e^{-i\theta_0} z_j \Big) \\ &= \sum_{j \in T} r_j \cos(\theta_0 - \theta_j) \\ &= \sum_{j \in T} r_j \Big(\cos(\theta_0 - \theta_j) \Big)^+ \\ &= f(\theta_0) \ge \frac{1}{\pi} \sum_{j=1}^{n} |z_j|. \end{aligned}$$

Here $1/\pi$ is the best constant, but the proof needs more work.

■

123. Let $(a_1, a_2, a_3, \ldots, a_n)$ and $(b_1, b_2, b_3, \ldots, b_n)$ be two sequences of real numbers which are not proportional. Let $(x_1, x_2, x_3, \ldots, x_n)$ be another sequence of real numbers such that

$$\sum_{j=1}^{n} a_j x_j = 0, \quad \sum_{j=1}^{n} b_j x_j = 1.$$

Prove that

$$\sum_{j=1}^{n} x_j^2 \geq \frac{\sum_{j=1}^{n} a_j^2}{\left(\sum_{j=1}^{n} a_j^2\right)\left(\sum_{j=1}^{n} b_j^2\right) - \left(\sum_{j=1}^{n} a_j b_j\right)^2}.$$

When does equality hold?

Solution: Put

$$A = \sum_{j=1}^{n} a_j^2, B = \sum_{j=1}^{n} b_j^2, C = \sum_{j=1}^{n} a_j b_j, y_j = \frac{Ab_j - Ca_j}{AB - C^2}.$$

An easy computation shows that

$$\sum_{j=1}^{n} y_j x_j = \frac{A}{AB - C^2}.$$

The definition of y_j gives

$$\begin{aligned}
\sum_{j=1}^{n} a_j y_j &= \frac{A \sum a_j b_j - C \sum a_j^2}{AB - C^2} = 0, \\
\sum_{j=1}^{n} b_j y_j &= \frac{A \sum b_j^2 - C \sum a_j b_j}{AB - C^2} = 1.
\end{aligned}$$

Using these, we may now obtain

$$\sum_{j=1}^{n} y_j^2 = \sum y_j \frac{Ab_j - Ca_j}{AB - C^2} = \frac{A}{AB - C^2}.$$

It follows that

$$\begin{aligned}
\sum_{j=1}^{n} x_j^2 - \sum_{j=1}^{n} y_j^2 &= \sum_{j=1}^{n} x_j^2 - \sum_{j=1}^{n} y_j x_j \\
&= \sum_{j=1}^{n} x_j \left(x_j - y_j\right) \\
&= \sum_{j=1}^{n} x_j \left(x_j - y_j\right) - \sum_{j=1}^{n} y_j x_j + \sum_{j=1}^{n} y_j^2 \\
&= \sum_{j=1}^{n} x_j \left(x_j - y_j\right) - \sum_{j=1}^{n} y_j \left(x_j - y_j\right) \\
&= \sum_{j=1}^{n} \left(x_j - y_j\right)^2 \geq 0.
\end{aligned}$$

Hence we get

$$\sum_{j=1}^{n} x_j^2 \geq \sum_{j=1}^{n} y_j^2 = \frac{A}{AB - C^2},$$

which is the required inequality.

Corollary:

If $\left(a_1,a_2,a_3,\ldots,a_n\right)$ and $\left(b_1,b_2,b_3,\ldots,b_n\right)$, $n \geq 2$, are two sequences of real numbers such that $a_j b_k \neq a_k b_j$ for $j \neq k$, then

$$\frac{\sum_j a_j^2}{\left(\sum_j a_j^2\right)\left(\sum_j b_j^2\right) - \left(\sum_j a_j b_j\right)^2} \leq \binom{n}{2}^{-2} \sum_j \left(\sum_{k \neq j} \frac{a_k}{a_k b_j - a_j b_k}\right)^2.$$

Define

$$x_j = \binom{n}{2}^{-1} \sum_{k \neq j} \frac{a_k}{a_k b_j - a_j b_k},$$

for $1 \leq j \leq n$. Then we get

$$\sum_{j=1}^{n} a_j x_j = \binom{n}{2}^{-1} \sum_{j=1}^{n} \sum_{k \neq j} \frac{a_j a_k}{a_k b_j - a_j b_k}.$$

For any $l \neq m$, we may regroup the terms of the form

$$\frac{a_m a_l}{a_m b_l - a_l b_m} + \frac{a_l a_m}{a_l b_m - a_m b_l},$$

to get the sum equal to 0. Thus we see that

$$\sum_{j=1}^{n} a_j x_j = 0.$$

Similarly, we obtain

$$\sum_{j=1}^{n} b_j x_j = 1.$$

Now we can apply the inequality in the above problem. ■

124. Let $\left(a_1,a_2,a_3,\ldots,a_n\right)$ and $\left(b_1,b_2,b_3,\ldots,b_n\right)$ be two sequences of real numbers such that

$$b_1^2 - b_2^2 - \cdots - b_n^2 > 0 \text{ or } a_1^2 - a_2^2 - \cdots - a_n^2 > 0.$$

Prove that

$$\left(a_1^2 - a_2^2 - \cdots - a_n^2\right)\left(b_1^2 - b_2^2 - \cdots - b_n^2\right) \leq \left(a_1 b_1 - a_2 b_2 - \cdots - a_n b_n\right)^2,$$

and show that equality holds if and only if $a_j = \lambda b_j$, $1 \leq j \leq n$, for some real constant λ.

Solution: We may suppose that $b_1^2 - b_2^2 - \cdots - b_n^2 > 0$. Consider the function $f(x)$ defined on $\mathbb{R}$ by

$$\begin{aligned} f(x) &= \big(b_1^2 - b_2^2 - \cdots - b_n^2\big)x^2 - 2\big(a_1b_1 - a_2b_2 - \cdots - a_nb_n\big)x \\ &\qquad\qquad\qquad\qquad\qquad + \big(a_1^2 - a_2^2 - \cdots - a_n^2\big) \\ &= \big(b_1x - a_1\big)^2 - \big(b_2x - a_2\big)^2 - \cdots - \big(b_nx - a_n\big)^2. \end{aligned}$$

Now $b_1^2 - b_2^2 - \cdots - b_n^2 > 0$ implies that $b_1 \neq 0$. Taking $x = a_1/b_1$, we obtain

$$f\left(\frac{a_1}{b_1}\right) = -\left(b_2\frac{a_1}{b_1} - a_2\right)^2 - \cdots - \left(b_n\frac{a_1}{b_1} - a_m\right)^2 \leq 0.$$

However since the leading coefficient of the quadratic expression $f(x)$ is positive, $f(x) \to \infty$ as $x \to \pm\infty$. Since $f\big(a_1/b_1\big) \leq 0$, the equation $f(x) = 0$ has one root each in the intervals $(-\infty, a_1/b_1\big]$ and $\big[a_1/b_1, +\infty)$. Hence the discriminant of the quadratic function $f(x)$ must be non-negative. This gives

$$\big(a_1^2 - a_2^2 - \cdots - a_n^2\big)\big(b_1^2 - b_2^2 - \cdots - b_n^2\big) \leq \big(a_1b_1 - a_2b_2 - \cdots - a_nb_n\big)^2.$$

Equality holds if and only if a_1/b_1 is a double root of $f(x) = 0$. This forces $a_j = \lambda b_j$ for $1 \leq j \leq n$, where $\lambda = a_1/b_1$. ■

125. Let $x_1, x_2, x_3, \ldots, x_n$ be n positive real numbers. Prove that

$$\sum_{j=1}^{n} \frac{x_j}{2x_j + x_{j+1} + \cdots + x_{j+n-2}} \leq n,$$

where $x_{n+k} = x_k$.

Solution: The inequality is equivalent to

$$\sum_{j=1}^{n} \frac{x_{j+1} + x_{j+2} + \cdots + x_{j+n-2}}{2x_j + x_{j+1} + \cdots + x_{j+n-2}} \geq n - 2.$$

Using the Cauchy-Schwarz inequality,

$$\sum_{j=1}^{n} \frac{x_{j+1} + x_{j+2} + \cdots + x_{j+n-2}}{2x_j + x_{j+1} + \cdots + x_{j+n-2}} \geq$$

$$\frac{\Big(\sum_{\text{cyclic}} x_{j+1} + x_{j+2} + \cdots + x_{j+n-2}\Big)^2}{\sum_{\text{cyclic}} \Big(x_{j+1} + x_{j+2} + \cdots + x_{j+n-2}\Big)\Big(2x_j + x_{j+1} + \cdots + x_{j+n-2}\Big)}.$$

Let us put $S = \sum_{j=1}^{n} x_j$. Then

$$x_{j+1} + x_{j+2} + \cdots + x_{j+n-2} = S - x_{j-1} - x_j,$$

for $1 \le j \le n$. (Here the indices are taken modulo n.) Thus we have to prove that

$$\left(\sum_{j=1}^{n} S - x_{j-1} - x_j\right)^2 \ge \left(n-2\right) \sum_{j=1}^{n} \left(S - x_{j-1} - x_j\right)\left(S - x_{j-1} + x_j\right).$$

This is equivalent to

$$\begin{aligned} (n-2)S^2 &\ge \sum_{j=1}^{n} \left(S - x_{j-1}\right)^2 - x_j^2 \\ &= \sum_{j=1}^{n} \left(S^2 - 2Sx_{j-1} + x_{j-1}^2 - x_j^2\right). \end{aligned}$$

However using $\sum_{j=1}^{n} x_{j-1}^2 = \sum_{j=1}^{n} x_j^2$, the right side is

$$\sum S^2 - 2Sx_{j-1} = nS^2 - 2S \sum x_{j-1} = (n-2)S^2.$$

Thus the inequality follows. ■

126. Let x_1,x_2,x_3,. . .,x_n be $n \ge 2$ positive real numbers and k be a fixed integer such that $1 \le k \le n$. Show that

$$\sum_{\text{cyclic}} \frac{x_1 + 2x_2 + \cdots + 2x_{k-1} + x_k}{x_k + x_{k+1} + \cdots + x_n} \ge \frac{2n(k-1)}{n-k+1}.$$

Solution: Put $\lambda^{-1} = \sum_{j=1}^{n} x_j$ and $y_j = \lambda x_j$, for $1 \le j \le n$. Then y_j's are in $(0, 1)$ and $\sum_{j=1}^{n} y_j = 1$. The inequality takes the form

$$\sum_{\text{cyclic}} \frac{y_1 + 2y_2 + \cdots 2y_{k-1} + y_k}{1 - \left(y_1 + y_2 + \cdots + y_{k-1}\right)} \ge \frac{2n(k-1)}{n-k+1}.$$

Using the new symbols

$$\begin{aligned} \alpha_1 &= 1 - \left(y_1 + y_2 + \cdots + y_{k-1}\right), \\ \alpha_2 &= 1 - \left(y_2 + y_3 + \cdots + y_k\right), \\ \vdots \quad & \vdots \quad \vdots \quad \vdots \\ \alpha_n &= 1 - \left(y_n + y_1 + \cdots + y_{k-2}\right), \end{aligned}$$

we may now write $y_1 + 2y_2 + \cdots 2y_{k-1} + y_k = 2 - \alpha_1 - \alpha_2$. The inequality to be proved is

$$\sum_{\text{cyclic}} \frac{2 - \alpha_1 - \alpha_2}{\alpha_1} \geq \frac{2n(k-1)}{n-k+1},$$

where $\alpha_j \in (0,1)$ for $1 \leq j \leq n$. We observe that $\sum_{j=1}^{n} \alpha_j = (n-k+1)$. Let us put

$$z_j = \frac{1}{\alpha_j}, \quad \mu_j = 2 - \alpha_j - \alpha_{j+1}, \quad 1 \leq j \leq n.$$

Then

$$\begin{aligned}
\sum_{j=1}^{n} \mu_j &= \sum_{j=1}^{n} \bigl(2 - \alpha_j - \alpha_{j+1}\bigr) \\
&= 2n - 2(n-k+1) = 2(k-1).
\end{aligned}$$

Taking $\lambda_j = \mu_j / 2(k-1)$, we have $\sum_{j=1}^{n} \lambda_j = 1$ This entails us to use Jensen's inequality for the convex function $f(z) = 1/z$ in the interval $(0, \infty)$. Thus

$$\frac{1}{\sum_{j=1}^{n} \lambda_j z_j} \leq \sum_{j=1}^{n} \lambda_j \frac{1}{z_j}.$$

Using the definitions of z_j and λ_j, this reduces to

$$\frac{2(k-1)}{\sum \bigl(2 - \alpha_j - \alpha_{j+1}\bigr)/\alpha_j} \leq \frac{1}{2(k-1)} \sum_{j=1}^{n} \bigl(2 - \alpha_j - \alpha_{j+1}\bigr)\alpha_j.$$

This further simplifies to

$$\begin{aligned}
\sum_{j=1}^{n} \frac{\bigl(2 - \alpha_j - \alpha_{j+1}\bigr)}{\alpha_j} &\geq \frac{4(k-1)^2}{2\sum_{j=1}^{n} \alpha_j - \sum_{j=1}^{n} \alpha_j \bigl(\alpha_j + \alpha_{j+1}\bigr)} \\
&= \frac{8(k-1)^2}{4(n-k+1) - \sum_{j=1}^{n} \bigl(\alpha_j + \alpha_{j+1}\bigr)^2};
\end{aligned}$$

we have used

$$2\sum_{j=1}^{n} \alpha_j \bigl(\alpha_j + \alpha_{j+1}\bigr) = \sum_{j=1}^{n} \bigl(\alpha_j + \alpha_{j+1}\bigr)^2.$$

Thus we need to prove that

$$\frac{8(k-1)^2}{4(n-k+1) - \sum_{j=1}^{n} \bigl(\alpha_j + \alpha_{j+1}\bigr)^2} \geq \frac{2n(k-1)}{n-k+1}.$$

This in turn is equivalent to

$$\sum_{j=1}^{n} \bigl(\alpha_j + \alpha_{j+1}\bigr)^2 \geq \frac{4}{n}(n-k+1)^2.$$

However this is a consequence of the Cauchy-Schwarz inequality:

$$\begin{aligned}\sum_{j=1}^{n}\left(\alpha_j+\alpha_{j+1}\right)^2 &\geq \frac{1}{n}\left(\sum_{j=1}^{n}\left(\alpha_j+\alpha_{j+1}\right)\right)^2\\ &= \frac{1}{n}\Big(2(n-k+1)\Big)^2\\ &= \frac{4}{n}(n-k+1)^2.\end{aligned}$$

This completes the proof. ∎

127. Let z and ξ be two complex numbers such that $|z|\leq r$, $|\xi|\leq r$ and $z\neq\xi$. Show that for any natural number n, the inequality

$$\left|\frac{z^n-\xi^n}{z-\xi}\right|\leq\frac{1}{2}n(n-1)r^{n-2}|z-\xi|$$

holds.

Solution: We have

$$\begin{aligned}\frac{z^n-\xi^n}{z-\xi}-n\xi^{n-1} &= \sum_{k=0}^{n-1}z^k\xi^{n-1-k}-n\xi^{n-1}\\ &= \sum_{k=0}^{n-1}\xi^{n-1-k}\left(z^k-\xi^k\right)\\ &= \sum_{k=0}^{n-1}\left(z-\xi\right)\xi^{n-1-k}\left(\sum_{l=0}^{k-1}z^l\xi^{k-1-l}\right).\end{aligned}$$

This gives

$$\begin{aligned}\left|\frac{z^n-\xi^n}{z-\xi}\right| &\leq |z-\xi|\sum_{k=0}^{n-1}|\xi|^{n-1-k}\left(\sum_{l=0}^{k-1}|z|^l|\xi|^{k-1-l}\right)\\ &\leq |z-\xi|\sum_{k=0}^{n-1}r^{n-1-k}\left(\sum_{l=0}^{k-1}r^lr^{k-1-l}\right)\\ &= |z-\xi|\sum_{k=0}^{n-1}kr^{n-2}\\ &= \frac{1}{2}n(n-1)|z-\xi|r^{n-2}.\end{aligned}$$

∎

128. For any three vectors, $\boldsymbol{x} = \big(x_1,x_2,x_3,\ldots,x_n\big)$, $\boldsymbol{y} = \big(y_1,y_2,y_3,\ldots,y_n\big)$, and $\boldsymbol{z} = \big(z_1,z_2,z_3,\ldots,z_n\big)$ in $\mathbb{R}^n$, prove that

$$||\boldsymbol{x}|| + ||\boldsymbol{y}|| + ||\boldsymbol{z}|| - ||\boldsymbol{x}+\boldsymbol{y}|| - ||\boldsymbol{y}+\boldsymbol{z}|| - ||\boldsymbol{z}+\boldsymbol{x}|| + ||\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}|| \geq 0.$$

Solution: It is easy to prove the following identity:

$$\begin{aligned}&\Big\{||\boldsymbol{x}|| + ||\boldsymbol{y}|| + ||\boldsymbol{z}|| - ||\boldsymbol{x}+\boldsymbol{y}|| - ||\boldsymbol{y}+\boldsymbol{z}|| - ||\boldsymbol{z}+\boldsymbol{x}|| + ||\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}||\Big\}\\ &\qquad\qquad\Big\{||\boldsymbol{x}|| + ||\boldsymbol{y}|| + ||\boldsymbol{z}|| + ||\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}||\Big\}\\ &\quad = \prod\Big\{||\boldsymbol{y}|| + ||\boldsymbol{z}|| - ||\boldsymbol{y}+\boldsymbol{z}||\Big\}\Big\{||\boldsymbol{x}|| - ||\boldsymbol{y}+\boldsymbol{z}|| + ||\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}||\Big\},\end{aligned}$$

where the product is taken cyclically. But triangle inequality for vectors shows that the right side is non-negative. Hence the result follows. ■

129. Let $A_1A_2A_3\cdots A_{n+1}$ be a polygon with centre O, in which $A_1 = A_{n+1}$ is fixed and the remaining A_j's vary on the circle. Show that the area and the perimeter of the polygon are the largest when the polygon is regular.

Solution: Let $\angle A_jOA_{j+1} = \alpha_j$. Then $0 < \alpha_j < \pi$ for $1 \leq j \leq n$. Using the concavity of $\sin x$ on $(0,\pi)$, we have

$$\frac{1}{n}\sum_{j=1}^{n}\sin\big(\alpha_j/2\big) \leq \sin\bigg(\sum_{j=1}^{n}\big(\alpha_j/2n\big)\bigg) = \sin\big(\pi/n\big).$$

We observe that $A_jA_{j+1} = 2R\sin\big(\alpha_j/2\big)$, where R is the radius of the circle. This gives

$$\sum_{\text{cyclic}} A_jA_{j+1} \leq 2nR\sin\big(\pi/n\big).$$

Equality holds if and only if $\alpha_1 = \alpha_2 = \cdots = \alpha_n$; i.e., when the polygon is regular.

Similarly, the total area is

$$\begin{aligned}\frac{1}{2}R^2\sum_{j=1}^{n}\sin\big(\alpha_j\big) &= \frac{1}{2}nR^2\sum_{j=1}^{n}\frac{1}{n}\sin\big(\alpha_j\big)\\ &\leq \frac{1}{2}nR^2\sin\bigg(\sum_{j=1}^{n}\alpha_j/n\bigg)\\ &= \frac{1}{2}nR^2\sin\bigg(\frac{2\pi}{n}\bigg).\end{aligned}$$

Equality holds if and only if $\alpha_1 = \alpha_2 = \cdots = \alpha_n$. ■

130. A sequence $\langle a_n \rangle$ is said to be convex if $a_n - 2a_{n+1} + a_{n+2} \geq 0$ for all $n \geq 1$. Let $a_1, a_2, a_3, \ldots, a_{2n+1}$ be a convex sequence. Show that

$$\frac{a_1 + a_3 + a_5 + \cdots + a_{2n+1}}{n+1} \geq \frac{a_2 + a_4 + a_6 + \cdots + a_{2n}}{n},$$

and equality holds if and only if $a_1, a_2, a_3, \ldots, a_{2n+1}$ is an arithmetic progression.

Solution: We have

$$\begin{aligned} j\big(n-j+1\big)\big(a_{2j-1} - 2a_{2j} + a_{2j+1}\big) &\geq 0, \text{ for } 1 \leq j \leq n, \\ j\big(n-j\big)\big(a_{2j} - 2a_{2j+1} + a_{2j+2}\big) &\geq 0, \text{ for } 1 \leq j \leq n-1. \end{aligned}$$

Adding these and summing over j, the resulting sum is non-negative. If α_j denotes the coefficient of a_j in this sum, then

$$\begin{aligned} \alpha_{2k+1} &= k(n-k+1) - 2k(n-k) + (k+1)(n-k) \\ &= k(n-k+1) + (n-k)(1-k) \\ &= (n-k) + k = n, \\ \alpha_{2k} &= -2k(n-k+1) + k(n-k) + (k-1)(n-k+1) \\ &= k(n-k) - k(n-k+1) - (n-k+1) \\ &= -k - (n-k+1) = -(n+1). \end{aligned}$$

Thus it follows that

$$n\big(a_1 + a_3 + \cdots + a_{2n+1}\big) - \big(n+1\big)\big(a_2 + a_4 + \cdots + a_{2n}\big) \geq 0.$$

This simplifies to

$$\frac{a_1 + a_3 + a_5 + \cdots + a_{2n+1}}{n+1} \geq \frac{a_2 + a_4 + a_6 + \cdots + a_{2n}}{n}.$$

Equality holds if and only if each a_j is the average of a_{j-1} and a_{j+1}. Equivalently, the given sequence is an arithmetic progression. ■

131. Suppose $a_1, a_2, a_3, \ldots, a_n$ are n positive real numbers. For each k, define

$$x_k = a_{k+1} + a_{k+2} + \cdots + a_{k+n-1} - (n-2)a_k,$$

where $a_j = a_{j-n}$ for $j > n$. Suppose $x_k \geq 0$ for $1 \leq k \leq n$. Prove that

$$\prod_{k=1}^{n} a_k \geq \prod_{k=1}^{n} x_k.$$

Show that for $n = 3$ the inequality is still true without the non-negativity of x_k's, but for $n > 3$ these conditions are essential.

Solution: For each $k = 1, 2, 3, \ldots, n$,

$$(n-1)a_k = \sum_{j=1}^{n-1} x_{k+j},$$

where the indices are read modulo n. This gives

$$\begin{aligned}
\prod_{k=1}^{n} a_k &= \prod_{k=1}^{n} \left(\frac{1}{n-1} \sum_{j=1}^{n-1} x_{k+j} \right) \\
&\geq \prod_{k=1}^{n} \left(\prod_{j=1}^{n-1} x_{k+j} \right)^{1/(n-1)} \\
&= \prod_{k=1}^{n} x_k.
\end{aligned}$$

Equality holds if and only if $x_1 = x_2 = \cdots = x_n$. Equivalently a_j's are all equal.

Suppose $n = 3$. Then $x_1 = a_2 + a_3 - a_1$, $x_2 = a_3 + a_1 - a_2$, $x_3 = a_1 + a_2 - a_3$. Suppose only two of the x_j's are negative, say, $x_1 < 0$ and $x_2 < 0$. Then $x_1 + x_2 < 0$, forcing $a_3 < 0$ which is a contradiction. Thus either only one of the x_j's is negative or all three negative. But then $x_1 x_2 x_3$ is negative and hence $a_1 a_2 a_3 \geq x_1 x_2 x_3$. This proves the result for $n = 3$ without any restrictions on x_j.

For $n \geq 4$, we take $a_1 = a_2 = 1$ and $a_3 = a_4 = \cdots = a_n = \lambda > 0$. If the inequality holds for this sequence, then

$$\lambda^{n-2} \big(2 - \lambda\big)^{n-2} \big(-n + 3 + (n-2)\lambda \big)^2.$$

As $\lambda \to 0$, we see that

$$2^{n-2}(n-3)^2 \leq 0.$$

But this is impossible since $n > 3$. ■

132. Let a, c be positive reals and b be a complex number such that

$$f(z) = a|z|^2 + 2\,\mathrm{Re}(bz) + c \geq 0,$$

for all complex numbers z, where $\mathrm{Re}(z)$ denotes the real part of z. Prove that

$$|b|^2 \leq ac,$$

and

$$f(z) \leq \big(a + c\big)\big(1 + |z|^2\big).$$

Show that $|b|^2 = ac$ only if $f(z) = 0$ for some $z \in \mathbb{C}$.

Solution: Write $z = re^{i\theta}$ and $b = \rho e^{it}$. Then

$$f(z) = ar^2 + 2\rho r\cos(\theta + t) + c \geq 0.$$

In particular $ar^2 - 2\rho r + c \geq 0$. Since this holds for all r, the discriminant of the quadratic must be non-negative. We thus get $|b|^2 = \rho^2 \leq ac$. Equality holds if and only if $(ar - \rho)^2 = 0$. Equivalently $r = \rho/a$. Taking $z_0 = (\rho/a)e^{i\theta_0}$, where $\cos(\theta_0 + t) = -1$, we get $f(z_0) = 0$.

Since $\rho^2 \leq ac$, we have

$$\rho r \leq r\sqrt{ac} = \sqrt{a.cr^2} \leq \frac{1}{2}\left(a + cr^2\right).$$

Thus

$$\begin{aligned} f(z) &= ar^2 + 2\rho r\cos(\theta + t) + c \\ &\leq \left(ar^2 + c\right) + \left(a + cr^2\right) \\ &= (a + c)\left(1 + r^2\right). \end{aligned}$$

■

133. Suppose $x_1 \leq x_2 \leq x_3 \leq \ldots \leq x_n$ be n real numbers. Show that

$$\left(\sum_{j=1}^{n}\sum_{k=1}^{n}\left|x_j - x_k\right|\right)^2 \leq \frac{2\left(n^2 - 1\right)}{3}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(x_j - x_k\right)^2.$$

Prove also that equality holds if and only if the sequence $\langle x_j \rangle$ is in arithmetic progression.

Solution: We start with the observation that it is sufficient to consider the case where $\sum_{j=1}^{n} x_j = 0$. Otherwise we may replace each x_j by $x_j - a$, where $a = \left(\sum_{j=1}^{n} x_j\right)/n$, without affecting the inequality. Hence we assume that $\sum_{j=1}^{n} x_j = 0$. Consider the right hand side. We have

$$\begin{aligned} \sum_{j=1}^{n}\sum_{k=1}^{n}\left(x_j - x_k\right)^2 &= 2n\sum_{j=1}^{n} x_j^2 - 2\left(\sum_{j=1}^{n} x_j\right)^2 \\ &= 2n\sum_{j=1}^{n} x_j^2, \end{aligned}$$

as evident from expanding the square, rearranging it and the assumption

$\sum x_j = 0$. We also see that

$$\begin{aligned}
\left(\sum_{j=1}^{n}\sum_{k=1}^{n}\left|x_j - x_k\right|\right)^2 &= 4\left(\sum_{j<k}\left|x_j - x_k\right|\right)^2 \\
&= 4\left(\sum_{j=1}^{n}(2j-1-n)x_j\right)^2 \\
&\leq 4\left(\sum_{j=1}^{n}(2j-1-n)^2\right)\left(\sum_{j=1}^{n}x_j^2\right).
\end{aligned}$$

Here we have used the Cauchy-Schwarz inequality. Hence it is sufficient to show that

$$\sum_{j=1}^{n}(2j-1-n)^2 \leq \frac{n\left(n^2-1\right)}{3}.$$

However, we see that

$$\begin{aligned}
&\sum_{j=1}^{n}(2j-1-n)^2 \\
&\quad= 4\sum_{j=1}^{n}j^2 - 4(n+1)\sum_{j=1}^{n}j + n(n+1)^2 \\
&\quad= 4\frac{n(n+1)(2n+1)}{6} - 4(n+1)\frac{n(n+1)}{2} + n(n+1)^2 \\
&\quad= \frac{n\left(n^2-1\right)}{3}.
\end{aligned}$$

This proves the inequality. Moreover, equality holds if and only if equality holds in the Cauchy-Schwarz inequality. This is equivalent to the fact that x_j is proportional to $2j-1-n$ for each j, i.e., $x_j = k(2j-1-n)$, for $1 \leq j \leq n$. This is same as $x_j = k(1-n) + 2kj$ or $\langle x_j \rangle$ is an arithmetic progression. ■

134. Suppose $\langle a_n \rangle$ is an infinite sequence of real numbers with the properties:

(i) there is some real constant c such that $0 \leq a_n \leq c$, for all $n \geq 1$;

(ii) $\left|a_j - a_k\right| \geq \dfrac{1}{j+k}$ for all j, k with $j \neq k$.

Prove that $c \geq 1$.

Solution: Fix any integer $n \geq 2$ and consider the first n elements in the sequence: $a_1, a_2, \ldots, a_n$. Let σ be that permutation of $1, 2, 3, \ldots, n$ which orders these n elements as an increasing sequence:

$$0 \leq a_{\sigma(1)} \leq a_{\sigma(2)} \leq \cdots \leq a_{\sigma(n)} \leq c.$$

Then

$$
\begin{aligned}
c &\geq a_{\sigma(n)} - a_{\sigma(1)} \\
&= \sum_{j=2}^{n} \left(a_{\sigma(j)} - a_{\sigma(j-1)} \right) \\
&\geq \sum_{j=2}^{n} \frac{1}{\sigma(j) + \sigma(j-1)} \\
&\geq \frac{(n-1)^2}{\sum_{j=2}^{n} \left(\sigma(j) + \sigma(j-1) \right)},
\end{aligned}
$$

where the Cauchy-Schwarz inequality has been used in the last step. On the other hand

$$
\begin{aligned}
\sum_{j=2}^{n} \left(\sigma(j) + \sigma(j-1) \right) &= 2\left(\sum_{j=1}^{n} \sigma(j) \right) - \sigma(1) - \sigma(n) \\
&= n(n+1) - \sigma(1) - \sigma(n) \\
&\leq n^2 + n - 3,
\end{aligned}
$$

since $\sigma(1) + \sigma(n) \geq 3$. Using $n^2 + n - 3 \leq n^2 + 2n + 3 = (n-1)(n+3)$, we obtain

$$
c \geq \frac{(n-1)^2}{(n-1)(n+3)} = \frac{n-1}{n+3} = 1 - \frac{4}{n+3}.
$$

This holds for all values of n. It follows that $c \geq 1$. (If $c < 1$, then choose a large n such that $4/(n+3) < 1 - c$. This is possible since $1 - c > 0$ and $4/(n+3)$ can be made arbitrarily small by choosing a sufficiently large n.) ■

135. Let ABC be a right-angle triangle with medians m_a, m_b, m_c. Let $A'B'C'$ denote the triangle whose sides are m_a, m_b, m_c. If R and R' denote respectively the circum-radii of ABC and $A'B'C'$, prove that $R' \geq \dfrac{5}{6}R$.

Solution: Let AD denote the median from A on to BC, and G the centroid. Extend AD to M such that $GD = DM$. If m_a denotes the length of AD, then $GD = m_a/3$ and hence $GM = 2m_a/3$. Note that $BG = 2m_b/3$ and $CG = 2m_c/3$. We also observe that $BMCG$ is a parallelogram and hence $BM = CG = 2m_c/3$. Thus GBM is a triangle whose sides are $2m_a/3$, $2m_b/3$ and $2m_c/3$. Moreover $[GBM] = 2[GBD] = [ABC]/3$. Hence the area of a triangle whose sides are m_a, m_b, m_c is $\Delta' = (9/4) \times [ABC]/3 = 3[ABC]/4$.

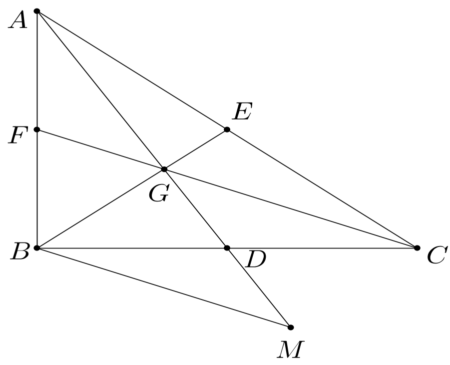

Fig. 6.19

We also have

$$m_b = b/2, \quad 4m_a^2 = a^2 + 4c^2, \quad 4m_c^2 = 4a^2 + c^2.$$

Hence,

$$m_a m_b m_c = \frac{b}{8}\sqrt{a^2+4c^2}\sqrt{c^2+4a^2} \geq \frac{b}{8}\big(a \cdot c + 2c \cdot 2a\big) = \frac{5abc}{8}.$$

We have used the Cauchy-Schwarz inequality here. This shows that

$$R' = \frac{m_a m_b m_c}{4\Delta'} \geq \frac{5abc}{8}\frac{4}{12[ABC]} = \frac{5}{6}R.$$

■

136. Let ABC be an equilateral triangle and D, E, F be arbitrary points on the segments BC, CA, AB respectively. Prove that

$$[DEF]\left\{\frac{1}{[BDF]} + \frac{1}{[CED]} + \frac{1}{[AFE]}\right\} \geq 3.$$

(Here $[XYZ]$ denotes the area of the triangle XYZ.)

Solution: Let us take $AB = BC = CA = 1$ unit and $AF = x$, $BD = y$, $CE = z$. Then using $[ABC] = (\sin A)/2$, we get

$$[AEF] = \frac{1}{2}AF \cdot AE \cdot \sin A = x(1-z)[ABC].$$

Similarly, we obtain

$$[BFD] = y(1-x)[ABC], \quad [CDE] = z(1-y)[ABC].$$

Thus, we have to prove

$$1 - x(1-z) - y(1-x) - z(1-y) \geq \frac{3}{\dfrac{1}{x(1-z)} + \dfrac{1}{y(1-x)} + \dfrac{1}{z(1-y)}}.$$

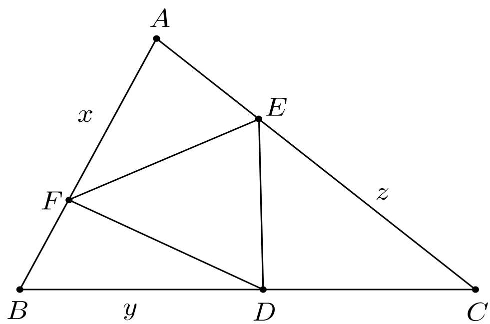

Fig. 6.20

Putting $u = z(1-y)$, $v = x(1-z)$, $w = y(1-x)$, we have to show that

$$\Big(1 - u - v - w\Big)\left(\frac{1}{u} + \frac{1}{v} + \frac{1}{w}\right) \geq 3.$$

Expanding, this may be written in an equivalent form:

$$\frac{1-v-w}{u} + \frac{1-w-u}{v} + \frac{1-u-v}{w} \geq 6.$$

However,

$$\begin{aligned}
\frac{1-v-w}{u} &= \frac{1 - x(1-z) - y(1-x)}{z(1-y)} \\
&= \frac{1 - x + xz - y + yx}{z(1-y)} \\
&= \frac{(1-y)(1-x) + zx}{z(1-y)} \\
&= \frac{1-x}{z} + \frac{x}{1-y}.
\end{aligned}$$

Similarly, we obtain

$$\frac{1-w-u}{v} = \frac{1-y}{x} + \frac{y}{1-z}, \qquad \frac{1-u-v}{w} = \frac{1-z}{y} + \frac{z}{1-x}.$$

Thus, we need to prove that

$$\frac{1-x}{z}+\frac{x}{1-y}+\frac{1-y}{x}+\frac{y}{1-z}+\frac{1-z}{y}+\frac{z}{1-x}\geq 6.$$

This follows from the AM-GM inequality: $a/b+b/a\geq 2$ for positive a,b. ■

137. Let the diagonals of a convex quadrilateral $ABCD$ meet in P. Prove that

$$\sqrt{[APB]}+\sqrt{[CPD]}\leq\sqrt{[ABCD]},$$

where as usual square-bracket denotes the area.

Solution: Let $BP=x$ and $PD=y$. The inequality to be proved is:

$$[APB]+[CPD]+2\sqrt{[APB]}\sqrt{[CPD]}\leq[ABCD].$$

This may be reduced to

$$2\sqrt{[APB]}\sqrt{[CPD]}\leq[BPC]+[DPA].$$

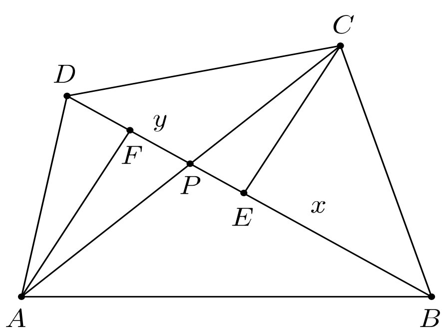

Fig. 6.21

But observe that

$$\frac{[APD]}{[APB]}=\frac{y}{x},\quad\frac{[CPB]}{[CPD]}=\frac{x}{y}.$$

Thus it is sufficient to prove that

$$2\sqrt{[APB]}\sqrt{[CPD]}\leq\frac{x}{y}[CPD]+\frac{y}{x}[APB].$$

This follows from the AM-GM inequality. Here equality holds if and only if

$$\frac{x}{y}[CPD]=\frac{y}{x}[APB].$$

If AE and CF are altitudes respectively from A, C on BD, we have

$$[CPD] = \frac{1}{2}y \cdot CF, \quad [APB] = \frac{1}{2}x \cdot AE.$$

Thus, equality holds if and only if

$$\frac{y}{x} = \frac{CF}{AE} = \frac{CP}{PA}.$$

This is possible only if CPD and APB are similar. Equivalently AB is parallel to CD. ■

138. Let AD be the median from A on to BC of a triangle ABC; let r, r_1, r_2 denote the in-radii of triangles ABC, ABD, ADC respectively. Prove that

$$\frac{1}{r_1} + \frac{1}{r_2} \geq 2\left(\frac{1}{r} + \frac{2}{BC}\right).$$

Solution:

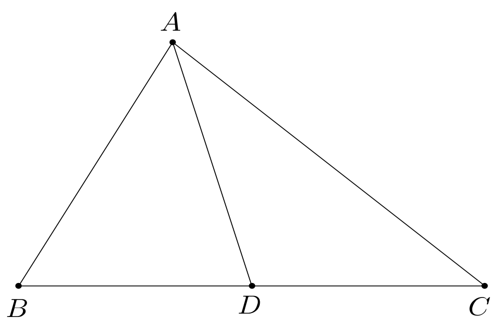

Fig. 6.22

Let p, p_1, p_2 be the perimeters of ABC, ABD, ADC respectively. Then $p_1 + p_2 = p + 2AD$. Hence

$$\frac{2[ABD]}{r_1} + \frac{2[ADC]}{r_2} = \frac{2[ABC]}{r} + 2AD.$$

Since AD is a median, $[ABD] = [ADC] = [ABC]/2$. Thus we get

$$\frac{[ABC]}{r_1} + \frac{[ABC]}{r_2} = \frac{2[ABC]}{r} + 2AD.$$

Using $AD \geq h_a = 2[ABC]/BC$, we obtain

$$\frac{1}{r_1} + \frac{1}{r_2} \geq \frac{2}{r} + \frac{4}{BC}.$$

■

139. Let a, b, c be positive reals such that $a+b+c=1$. Prove that

$$a\big(1+b-c\big)^{1/3} + b\big(1+c-a\big)^{1/3} + c\big(1+a-b\big)^{1/3} \leq 1.$$

Solution: We observe that $1+b-c = a+b+c+b-c = a+2b > 0$. Hence the AM-GM inequality gives

$$\big(1+b-c\big)^{1/3} \leq \frac{1+1+1+b-c}{3} = 1 + \frac{b-c}{3}.$$

Thus,

$$a\big(1+b-c\big)^{1/3} \leq a + \frac{a(b-c)}{3}.$$

Similarly, we obtain

$$\begin{aligned} b\big(1+c-a\big)^{1/3} &\leq b + \frac{b(c-a)}{3} \\ c\big(1+a-b\big)^{1/3} &\leq c + \frac{c(a-b)}{3}. \end{aligned}$$

Adding, we get the desired result. ■

140. Show that in a triangle ABC,

$$\Big(a^2m_a^2 + b^2m_b^2 + c^2m_c^2\Big)\Big(a^2+b^2+c^2\Big) \geq 16m_a^2m_b^2m_c^2,$$

where m_a, m_b, m_c denote the medians on to the sides BC, CA, AB from A, B, C respectively.

Solution: Using

$$m_a^2 = \frac{2b^2+2c^2-a^2}{4},$$

and similar expressions for m_b^2, m_c^2, we get

$$\begin{aligned} \sum_{\text{cyclic}} a^2m_a^2 &= \frac{1}{4}\sum_{\text{cyclic}} a^2\big(2b^2+2c^2-a^2\big) \\ &= \frac{1}{2}\bigg(\sum_{\text{cyclic}} a^2\bigg)^2 - \frac{3}{4}\sum_{\text{cyclic}} a^4. \end{aligned}$$

Similarly,

$$\begin{aligned}
16m_a^2m_b^2m_c^2 &= \frac{1}{4}\prod_{\text{cyclic}}\left(2\left(a^2+b^2+c^2\right)-3a^2\right)\\
&= \frac{1}{4}\Big\{-4\left(a^2+b^2+c^2\right)^3-27a^2b^2c^2\\
&\qquad +18\left(a^2+b^2+c^2\right)\left(a^2b^2+b^2c^2+c^2a^2\right)\Big\}.
\end{aligned}$$

Thus, the inequality to be proved is

$$\begin{aligned}
&\left(2\left(a^2+b^2+c^2\right)^2-3\left(a^4+b^4+c^4\right)\right)\left(a^2+b^2+c^2\right)\\
&\quad\geq -4\left(a^2+b^2+c^2\right)^3-27a^2b^2c^2+18\left(a^2+b^2+c^2\right)\left(a^2b^2+b^2c^2+c^2a^2\right).
\end{aligned}$$

Simplification reduces this to

$$\sum_{\text{cyclic}} a^2\left(a^2-b^2\right)\left(a^2-c^2\right)\geq 0.$$

This follows from Schur's inequality. ■

141. Let $x_1,x_2,x_3,\ldots,x_n$ be n positive reals which add up to 1. Find the minimum value of

$$\sum_{j=1}^{n}\frac{x_j}{1+\sum_{k\neq j}x_k}\quad .$$

Solution: Let S denote the sum. Then

$$S=\sum_{j=1}^{n}\frac{x_j}{2-x_j}=-n+\sum_{j=1}^{n}\frac{2}{2-x_j}.$$

Using the Cauchy-Schwarz inequality, we get

$$2n^2=\left(\sum_{j=1}^{n}\sqrt{2}\right)^2\leq\left(\frac{2}{2-x_j}\right)\left(\sum_{j=1}^{n}(2-x_j)\right).$$

Hence,

$$\sum_{j=1}^{n}\frac{2}{2-x_j}\geq\frac{2n^2}{2n-1}.$$

Thus,

$$S\geq -n+\frac{2n^2}{2n-1}=\frac{n}{2n-1}.$$

Equality holds if and only if $x_1=x_2=\cdots=x_n=1/n$. Thus the least value of S is $n/(2n-1)$. ■

142. Find all possible values of

$$\frac{a}{a+b+d}+\frac{b}{a+b+c}+\frac{c}{b+c+d}+\frac{d}{a+c+d},$$

when a, b, c, d vary over positive reals.

Solution: Let us write

$$S_1=\frac{a}{a+b+d}+\frac{c}{b+c+d}, \quad S_2=\frac{b}{a+b+c}+\frac{d}{a+c+d}.$$

We may assume $a+b+c+d=1$. Let us introduce $a+c=x$, $b+d=y$. Then

$$\begin{aligned} S_1=\frac{a}{1-c}+\frac{c}{1-a} &= \frac{a+c-(a^2+c^2)}{1-(a+c)+ac} \\ &= \frac{2ac+x-x^2}{ac+1-x}. \end{aligned}$$

We observe that $S_1 \geq x$ and $S_1 = x$ whenever $a=0$ or $c=0$. Similarly, it is easy to see that

$$S_1 \leq \frac{2x}{2-x}.$$

(In fact this is equivalent to $(a+c)^2 \geq 4ac$.) Thus the set of values of S_1 is $\big(x, 2x/(2-x)\big]$. Similarly, the set of values of S_2 is $\big(y, 2y/(2-y)\big]$. Combining, we get that the set of values of S_1+S_2 is

$$\big(x+y, 2x/(2-x)+2y/(2-y)\big].$$

But $x+y=1$ and

$$\frac{2x}{2-x}+\frac{2y}{2-y}=\frac{4-4xy}{2+xy} \leq 2.$$

Equality holds only if $xy=0$. Since $x \neq 0$ and $y \neq 0$, it follows that $S_1+S_2<2$. Thus, the set of values of the given sum is the interval $(1,2)$. ■

143. Let $\langle F_n \rangle$ be the Fibonacci sequence defined by

$$F_1=F_2=1, \quad F_{n+2}=F_{n+1}+F_n, \text{ for } n \geq 1.$$

Prove that

$$\sum_{j=1}^{n} \frac{F_j}{2^j}<2,$$

for all $n \geq 1$.

Solution: Let S_n denote the sum above. Then

$$\begin{aligned}
S_n &= \frac{F_1}{2} + \frac{F_2}{2^2} + \frac{F_1+F_2}{2^3} + \frac{F_2+F_3}{2^4} + \cdots + \frac{F_{n-2}+F_{n-1}}{2^n} \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{4}\sum_{j=1}^{n-2}\frac{F_j}{2^j} + \frac{1}{2}\sum_{j=2}^{n-1}\frac{F_j}{2^j} \\
&= \frac{3}{4} + \frac{1}{4}S_n + \frac{1}{2}S_n - \frac{1}{4}\left(\frac{F_{n-1}}{2^{n-1}} + \frac{F_n}{2^n}\right) - \frac{1}{2}\frac{F_n}{2^n} - \frac{1}{4} \\
&= \frac{1}{2} + \frac{3}{4}S_n - \frac{F_{n-1}}{2^{n+1}} - \frac{3F_n}{2^{n+2}}.
\end{aligned}$$

It follows that $S_n < 2$. ■

144. Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with real coefficients such that $|P(0)| = P(1)$. Suppose all the roots of $P(x) = 0$ are real and lie in the interval $(0,1)$. Prove that the product of the roots does not exceed $1/2^n$.

Solution: Let $\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n$ be the roots of $P(x) = 0$. Then we have

$$\alpha_1 \cdot \alpha_2 \cdots \alpha_n = (1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_n).$$

Introduce

$$\beta_j = \frac{1-\alpha_j}{\alpha_j}, \quad 1 \le j \le n.$$

Then $\alpha_j = 1/(1+\beta_j)$ for $1 \le j \le n$. Thus

$$\begin{aligned}
\alpha_1 \cdot \alpha_2 \cdots \alpha_n &= \frac{1}{(1+\beta_1)(1+\beta_2)\cdots(1+\beta_n)} \\
&\le \frac{1}{2^n\sqrt{\beta_1 \cdot \beta_2 \cdots \beta_n}}.
\end{aligned}$$

Note that $\beta_1 \cdot \beta_2 \cdots \beta_n = 1$. Hence it follows that

$$\alpha_1 \cdot \alpha_2 \cdots \alpha_n \le \frac{1}{2^n}.$$

■

145. If x, y are real numbers such that

$$2x + y + \sqrt{8x^2 + 4xy + 32y^2} = 3 + 3\sqrt{2},$$

prove that $x^2y \le 1$.

Solution: Using the AM-GM inequality, we have

$$2x+y=x+x+y\geq 3\big(x^2y\big)^{1/3}.$$

Similarly,

$$\begin{aligned}8x^2+4xy+32y^2 &= 4\big(2x^2\big)+4xy+32y^2\\ &\geq 9\Big(\big(2x^2\big)^4\big(xy\big)^4 32y^2\Big)^{1/9}\\ &= 18\big(x^2y\big)^{2/3}.\end{aligned}$$

Thus,

$$\begin{aligned}3+3\sqrt{2} &= 2x+y+\sqrt{8x^2+4xy+32y^2}\\ &\geq 3\big(x^2y\big)^{1/3}+3\sqrt{2}\big(x^2y\big)^{1/3}\\ &= \big(3+3\sqrt{2}\big)\big(x^2y\big)^{1/3}.\end{aligned}$$

It follows that $x^2y\leq 1$. ■

146. If α, β, γ are the angles of a triangle whose circum-radius is R and in-radius r, prove that

$$\cos^2\left(\frac{\alpha-\beta}{2}\right)\geq\frac{2r}{R}.$$

Solution: We have

$$\begin{aligned}r &= 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\\ &= 2R\sin\frac{\alpha}{2}\left(\cos\left(\frac{\beta-\gamma}{2}\right)-\cos\left(\frac{\beta+\gamma}{2}\right)\right)\\ &= 2R\sin\frac{\alpha}{2}\cos\Big((\beta-\gamma)/2\Big)-2R\sin^2\frac{\alpha}{2}.\end{aligned}$$

This shows that $\sin(\alpha/2)$ is a solution of the quadratic equation

$$2Rx^2-2R\cos\Big((\beta-\gamma)/2\Big)x+r=0.$$

Hence the discriminant of the quadratic expression is non-negative. We thus get,

$$4R^2\cos^2\left(\frac{\beta-\gamma}{2}\right)\geq 8Rr.$$

This reduces to

$$\cos^2\left(\frac{\alpha-\beta}{2}\right)\geq\frac{2r}{R}.$$

■

147. Let I be the in-centre of a triangle ABC. Suppose the internal bisectors of angles A, B, C meet the opposite sides at A', B' and C'. Prove that

$$\frac{1}{4} < \frac{AI \cdot BI \cdot CI}{AA' \cdot BB' \cdot CC'} \leq \frac{8}{27}.$$

Solution: Observe that

$$\frac{A'B}{AB} = \frac{A'I}{AI} = \frac{A'C}{AC}.$$

Hence

$$\frac{A'I}{AI} = \frac{A'B + A'C}{AB + AC} = \frac{a}{c+b}.$$

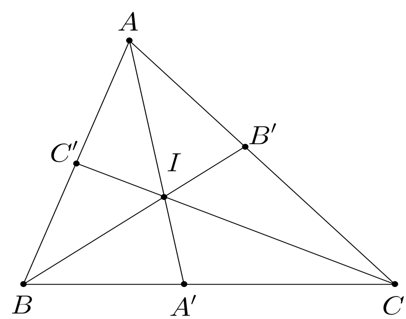

Fig. 6.23

Thus, we get

$$\frac{AI}{AA'} = \frac{AI}{AI + IA'} = \frac{c+b}{a+b+c}.$$

Similarly,

$$\frac{BI}{BB'} = \frac{c+a}{a+b+c}, \quad \frac{CI}{CC'} = \frac{a+b}{a+b+c}.$$

The inequality to be proved is, therefore,

$$\frac{1}{4} < \frac{(a+b)(b+c)(c+a)}{(a+b+c)^3} \leq \frac{8}{27}.$$

Using the AM-GM inequality, we have

$$\frac{(a+b)(b+c)(c+a)}{(a+b+c)^3} \leq \frac{1}{27}\left(\frac{a+b}{a+b+c} + \frac{b+c}{a+b+c} + \frac{c+a}{a+b+c}\right)^3 = \frac{8}{27}.$$

For getting the other inequality, take $x = s - a$, $y = s - b$, $z = s - c$. Then x, y, z are positive real numbers and $x + y = c$, $y + z = a$, $z + x = b$. The left side in equality reduces to

$$(y + z + 2x)(z + x + 2y)(x + y + 2z) > 2(x + y + z)^3.$$

This can easily be verified after expanding the left side of the inequality. ■

148. Determine the maximum value of

$$\sum_{j<k} x_j x_k \left(x_j + x_k\right),$$

over all n-tuples $(x_1,x_2,x_3,\ldots,x_n)$ of reals such that $x_j \geq 0$ for $1 \leq j \leq n$.

Solution: Observe that

$$\begin{aligned}\sum_{j<k} x_j x_k \left(x_j + x_k\right) &= \frac{1}{2}\sum_{j\neq k} x_j x_k \left(x_j + x_k\right)\\ &= \sum_{j\neq k} x_j^2 x_k.\end{aligned}$$

We may assume that $x_1 \geq x_2 \geq \cdots \geq x_n$. Introducing

$$y_1 = x_1, y_2 = x_2, \cdots, y_{n-2} = x_{n-2}, y_{n-1} = x_{n-1} + x_n,$$

we see that $y_j \geq 0$ and $\sum_{j=1}^{n-1} y_j = 1$. Moreover

$$\begin{aligned}\sum_{j\neq k} y_j^2 y_k - \sum_{j\neq k} x_j^2 x_k &= \left(y_{n-1}^2 - x_{n-1}^2 - x_n^2\right) \sum_{j=1}^{n-1} x_j\\ &= 2x_{n-1}x_n \sum_{j=1}^{n-1} x_j \geq 0.\end{aligned}$$

Thus, it follows that

$$\sum_{j\neq k} x_j^2 x_k \leq \sum_{j\neq k} y_j^2 y_k.$$

This shows that the number of variables can be reduced by 1. Continuing by induction on the number of variables, we see that the required maximum is the maximum of $\alpha\beta(\alpha + \beta)$ under the condition $\alpha + \beta = 1$. This maximum is 1/4. Thus the required maximum is 1/4. ■

149. If α, β, γ are the angles of a triangle, prove that

$$3 \sum_{\text{cyclic}} \cos\alpha \geq 2 \sum_{\text{cyclic}} \sin\alpha \sin\beta.$$

Solution: (Amit Diwadkar) We know that $\sum_{\text{cyclic}} \big(\cos\alpha - \cos\beta\big)^2 \geq 0$. Expanding this we get

$$\sum_{\text{cyclic}} \cos^2\alpha \geq \sum_{\text{cyclic}} \cos\alpha\cos\beta\,.$$

This may be written in the form

$$3\sum_{\text{cyclic}} \cos\alpha\cos\beta \leq \left(\sum_{\text{cyclic}} \cos\alpha\right)^2 \leq \frac{3}{2}\sum_{\text{cyclic}} \cos\alpha,$$

because $\sum_{\text{cyclic}} \cos\alpha \leq 3/2$. Thus we obtain

$$2\sum_{\text{cyclic}} \cos\alpha\cos\beta \leq \sum_{\text{cyclic}} \cos\alpha.$$

Using $2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$, this reduces to

$$\sum_{\text{cyclic}} \cos(\alpha-\beta) \leq 2\sum_{\text{cyclic}} \cos\alpha.$$

Adding $\sum_{\text{cyclic}} \cos\alpha$ both sides, this further reduces to

$$\sum_{\text{cyclic}} \cos(\alpha-\beta) - \sum_{\text{cyclic}} \cos(\alpha+\beta) \leq 3\sum_{\text{cyclic}} \cos\alpha.$$

After expanding, we get

$$2\sum_{\text{cyclic}} \sin\alpha\sin\beta \leq 3\sum_{\text{cyclic}} \cos\alpha.$$

■

150. Let x_1,x_2,x_3,. . .,x_N be positive real numbers. Prove that

$$\sum_{j=1}^{N} \big(x_1x_2\cdots x_j\big)^{1/j} < 3\left(\sum_{j=1}^{N} x_j\right).$$

Solution: Let c_1,c_2,c_3,. . .,c_N be N positive real numbers. Then for $1 \leq j \leq N$, we have

$$\Big(x_1x_2\cdots x_j\Big)^{1/j} \leq \frac{c_1x_1 + c_2x_2 + \cdots + c_jx_j}{j\big(c_1c_2\cdots c_j\big)^{1/j}}.$$

This follows from the AM-GM inequality. We choose c_j to meet our requirement. Taking

$$c_j = \frac{(j+1)^j}{j^{j-1}},$$

we see that

$$\begin{aligned}\Big(c_1 c_2 \cdots c_j\Big) &= \left(\frac{2}{1}\cdot\frac{3^2}{2}\cdot\frac{4^3}{3^2}\cdots\frac{(j+1)^j}{j^{j-1}}\right)^{1/j}\\ &= j+1.\end{aligned}$$

This leads to

$$\Big(x_1 x_2 \cdots x_j\Big)^{1/j} \le \frac{c_1x_1 + c_2x_2 + \cdots + c_jx_j}{j\big(j+1\big)}.$$

Summing over j, we obtain

$$\begin{aligned}\sum_{j=1}^{N}\big(x_1x_2\cdots x_j\big)^{1/j} &\le \sum_{j=1}^{N}\left(\sum_{k=1}^{j} c_k x_k\right)\frac{1}{j\big(j+1\big)}\\ &= \sum_{k=1}^{N} c_k x_k\left(\sum_{j=k}^{N}\frac{1}{j\big(j+1\big)}\right)\\ &= \sum_{k=1}^{N} c_k x_k \sum_{j=k}^{N}\left(\frac{1}{j}-\frac{1}{j+1}\right)\\ &= \sum_{k=1}^{N} c_k x_k\left(\frac{1}{k}-\frac{1}{N}\right)\\ &< \sum_{k=1}^{N}\frac{c_k x_k}{k}.\end{aligned}$$

But we observe that

$$\frac{c_k}{k} = \left(\frac{1+k}{k}\right)^k = \left(1+\frac{1}{k}\right)^k < 3.$$

We conclude that

$$\sum_{j=1}^{N}\big(x_1x_2\cdots x_j\big)^{1/j} < 3\left(\sum_{j=1}^{N} x_j\right).$$

Here the best constant is e, not 3. This is known as Carleman's inequality. ■

151. Let $a_1 \le a_2 \le a_3 \le \cdots \le a_n$ be n real numbers with the property $\sum_{j=1}^{n} a_j = 0$. Prove that

$$na_1 a_n \sum_{j=1}^{n} a_j^2 \le 0.$$

Solution: Consider $(a_1 - a_j)(a_j - a_n)$, for $1 \le j \le n$. This product is non-negative since $a_1 \le a_j \le a_n$. Hence we get

$$a_1 a_j - a_1 a_n - a_j^2 + a_j a_n \ge 0.$$

Summing this over j, we obtain

$$a_1\left(\sum_{j=1}^{n} a_j\right) - na_1 a_n - \sum_{j=1}^{n} a_j^2 + a_n\left(\sum_{j=1}^{n} a_j\right) \ge 0.$$

This reduces to

$$na_1 a_n + \sum_{j=1}^{n} a_j^2 \le 0.$$

Alternate Solution:

Put $a_j = a_1 + r_j$ for $1 \le j \le n$. We get $0 = r_1 \le r_2 \le r_3 \le \cdots \le r_n$ and

$$na_1 + \sum_{j=1}^{n} r_j = 0.$$

Thus

$$\begin{aligned}
\sum_{j=1}^{n} a_j^2 &= \sum_{j=1}^{n} \left(a_1 + r_j\right)^2 \\
&= \sum_{j=1}^{n} \left(a_1^2 + 2a_1 r_j + r_j^2\right) \\
&= na_1^2 + 2a_1\left(\sum_{j=1}^{n} r_j\right) + \sum_{j=1}^{n} r_j^2 \\
&= -na_1^2 + \sum_{j=1}^{n} r_j^2.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
na_1 + a_n + \sum_{j=1}^{n} a_j^2 &= na_1\left(a_1 + r_n\right) - na_1^2 + \sum_{j=1}^{n} r_j^2 \\
&= -r_n\left(\sum_{j=1}^{n} r_j\right) + \sum_{j=1}^{n} r_j^2
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} r_j\left(r_j - r_n\right) \\
&\leq 0,
\end{aligned}
$$

since $r_j - r_n \leq 0$, for $1 \leq j \leq n$. ■

152. Suppose a, b, c are positive real numbers. Prove that

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{1+abc}.$$

Solution: This is a tricky problem. We add $3/(1+abc)$ to both sides to get an equivalent inequality

$$\left\{\frac{1}{a(1+b)} + \frac{1}{1+abc}\right\} + \left\{\frac{1}{b(1+c)} + \frac{1}{1+abc}\right\} + \left\{\frac{1}{c(1+a)} + \frac{1}{1+abc}\right\} \geq \frac{6}{1+abc}.$$

Observe that

$$\frac{1}{a(1+b)} + \frac{1}{1+abc} = \frac{1}{1+abc}\left\{\frac{1+a}{a(1+b)} + \frac{b(1+c)}{1+b}\right\}.$$

Thus we get

$$
\begin{aligned}
&\left\{\frac{1}{a(1+b)} + \frac{1}{1+abc}\right\} + \left\{\frac{1}{b(1+c)} + \frac{1}{1+abc}\right\} + \left\{\frac{1}{c(1+a)} + \frac{1}{1+abc}\right\} = \\
&\quad \frac{1}{1+abc}\left\{\frac{1+a}{a(1+b)} + \frac{b(1+c)}{1+b} + \frac{1+b}{b(1+c)} + \frac{c(1+a)}{1+c} + \frac{1+c}{c(1+a)} + \frac{a(1+b)}{1+a}\right\} \\
&= \frac{1}{1+abc}\left\{\left(\frac{1+a}{a(1+b)} + \frac{a(1+b)}{1+a}\right) + \left(\frac{b(1+c)}{1+b} + \frac{1+b}{b(1+c)}\right) \right. \\
&\qquad \left. + \left(\frac{c(1+a)}{1+c} + \frac{1+c}{c(1+a)}\right)\right\} \\
&\geq \frac{6}{1+abc}.
\end{aligned}
$$

Alternate Solution:

We may write the inequality in the form

$$\sum_{\text{cyclic}} \frac{1}{a(1+b)} + \sum_{\text{cyclic}} \frac{bc}{(1+b)} \geq 3.$$

This may be rearranged to

$$\sum_{\text{cyclic}} \left(\frac{1}{a(1+b)} + \frac{ab}{(1+a)}\right) \geq 3.$$

Equivalently, the inequality to be proved is

$$\sum_{\text{cyclic}} \left(\frac{1}{a} \cdot \frac{1}{1+b} + b \cdot \frac{1}{1+(1/a)} \right) \geq 3.$$

If $1/a \geq b$, then $(1+a)/a \geq 1+b$ and hence

$$\frac{1}{1+b} \geq \frac{1}{1+(1/a)}.$$

Similarly, $1/a \leq b$ implies that

$$\frac{1}{1+b} \leq \frac{1}{1+(1/a)}.$$

Thus, the pairs

$$\left(\frac{1}{a}, b\right), \quad \left(\frac{1}{1+b}, \frac{1}{1+(1/a)}\right)$$

are oppositely ordered. The rearrangement inequality gives

$$\begin{aligned} \frac{1}{a} \cdot \frac{1}{1+b} + b \cdot \frac{1}{1+(1/a)} &\geq \frac{1}{a} \cdot \frac{1}{1+(1/a)} + b \cdot \frac{1}{1+b} \\ &= \frac{1}{1+a} + \frac{b}{1+b}. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{\text{cyclic}} \left(\frac{1}{a} \cdot \frac{1}{1+b} + b \cdot \frac{1}{1+(1/a)} \right) &\geq \sum_{\text{cyclic}} \frac{1}{1+a} + \sum_{\text{cyclic}} \frac{b}{1+b} \\ &= \sum_{\text{cyclic}} \left(\frac{1}{1+a} + \frac{a}{1+a} \right) \\ &= 3, \end{aligned}$$

which is what we need to prove. ■

153. Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 = 2$. Prove that

$$x + y + z \leq 2 + xyz.$$

Find conditions under which equality holds.

Solution: Let us put $a = xy$, $b = yz$, $c = zx$, $s = x + y + z$ and $t = xyz$. Since $x^2 + y^2 + z^2 = 2$, we have

$$\begin{aligned} x^2 + (y-z)^2 &= 2 - 2b, \\ y^2 + (z-x)^2 &= 2 - 2c, \\ z^2 + (x-y)^2 &= 2 - 2a. \end{aligned}$$

This shows that $1-a$, $1-b$, $1-c$ are non-negative reals. Thus we obtain

$$(1-a)(1-b)(1-c) \geq 0.$$

We also observe that

$$\begin{aligned} s^2 = (x+y+z)^2 &= x^2+y^2+z^2+2\bigl(xy+yz+zx\bigr) \\ &= 2+2(a+b+c). \end{aligned}$$

Hence we obtain $a+b+c = \bigl(s^2-2\bigr)/2$. Moreover, we also have

$$ab+bc+ca = xyz(x+y+z) = ts, \quad abc = t^2.$$

We have to show that $s-t \leq 2$. We observe that

$$\begin{aligned} (1-a)(1-b)(1-c) &= 1-(a+b+c)+ab+bc+ca-abc \\ &= 1-\frac{1}{2}\Bigl(s^2-2\Bigr)+ts-t^2 \\ &= \frac{4-(s-t)^2-t^2}{2} \\ &= 2-\frac{1}{2}\Bigl(s-t\Bigr)^2-\frac{t^2}{2} \\ &\leq 2-\frac{1}{2}\Bigl(s-t\Bigr)^2. \end{aligned}$$

Since $(1-a)(1-b)(1-c) \geq 0$, we conclude that $(s-t)^2 \leq 4$ or $\bigl|s-t\bigr| \leq 2$. Equality holds if and only if $a=1$ or $b=1$ or $c=1$ and $t=0$. But $t=0$ holds only if one of the x, y, z is zero. If, for example, $x=0$, then $yz=1$ giving $y=z=1$. Thus equality holds if and only if one of the x, y, z is zero and the other two are equal to 1 each. ■

154. Let $0 \leq x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n$ be such that $\sum_{j=1}^{n} x_j = 1$, where $n \geq 2$ is an integer. If $x_n \leq 2/3$, prove that there exists a k such that $1 \leq k \leq n$ and

$$\frac{1}{3} \leq \sum_{j=1}^{k} x_j \leq \frac{2}{3}.$$

Solution: We consider two cases.

Case 1. Suppose $x_n > 1/3$. Here we have

$$\frac{1}{3} \leq 1-x_n < \frac{2}{3}.$$

But, we know that $1-x_n = \sum_{j=1}^{n-1} x_j$. Hence it follows that

$$\frac{1}{3} \leq \sum_{j=1}^{n-1} x_j < \frac{2}{3}.$$

Thus, the given inequality holds with $k = n - 1$.

Case 2. Suppose $x_n \leq 1/3$. We take $s_k = \sum_{j=1}^{k} x_j$. If we have

$$\frac{1}{3} \leq s_k < \frac{2}{3},$$

for some k, then we are through. Otherwise, choose the smallest k such that $s_k \geq 2/3$. Note that this is possible since $s_0 = 0$ and $s_n = 1$. The choice of this least k shows that $s_{k-1} < 1/3$; for otherwise we would have $\frac{1}{3} \leq s_{k-1} \leq \frac{2}{3}$ and we would have our result. But then

$$x_k = s_k - s_{k-1} > \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \geq x_n,$$

a contradiction.

Thus we conclude that the result holds for some k. ■

155. Let x, y, z be non-negative real numbers such that $xy+yz+zx+xyz = 4$. Prove that

$$x + y + z \geq xy + yz + zx.$$

Solution: We solve for z to get

$$z = \frac{4 - xy}{y + x + xy}.$$

This shows that $xy \leq 4$. We write the given inequality in the form

$$x + y - xy \geq z\big(x + y - 1\big) = \frac{(4 - xy)(x + y - 1)}{y + x + xy}.$$

We may reduce this to

$$\big(x + y - 2\big)^2 \geq xy(x - 1)(y - 1).$$

If $(x-1)(y-1) \leq 0$, then the inequality is obvious. Suppose $(x-1)(y-1) \geq 0$. Then we have,

$$\begin{aligned} \big(x + y - 2\big)^2 &= \big(x - 1 + y - 1\big)^2 \\ &\geq 4(x - 1)(y - 1) \\ &\geq xy(x - 1)(y - 1), \end{aligned}$$

since $0 \leq xy \leq 4$.

A Generalisation:

Here is a generalisation due to M. S. Klamkin(CRUX-2000). If $0 < a \leq 1$ and

if $xy+yz+zx+xyz=3a+a^3$, then $x+y+z \geq xy+yz+zx$. The symmetry consideration shows that it is sufficient to assume $x \geq y \geq z$. This implies that

$$3x+x^3 \geq 3a+a^3 \geq 3z+z^3.$$

It follows that $x \geq a \geq z$. Put $\lambda = 3a+a^3$. Then the condition $xy+yz+zx+xyz=3a+a^3$ gives

$$y = \frac{\lambda - xz}{x+z+xz}.$$

Substituting this in the required inequality, it is sufficient to prove that

$$x^2\left(1+z-z^2\right)+x\left(z^2+z-\lambda\right)+z^2-\lambda z+\lambda \geq 0.$$

Considering the left side as a quadratic expression in x, the inequality is equivalent to the non-positivity of its discriminant. Observe that the discriminant is given by

$$D = \left(z^2+z-\lambda\right)^2 - 4\left(z^2-\lambda z+\lambda\right)\left(1+z-z^2\right).$$

Now $D \leq 0$ is equivalent to

$$\lambda(4-\lambda)+z(1-z)\Big(2\lambda(1-2z)+z(5z+3)\Big) \geq 0.$$

Since $\lambda \leq 4$ and $z \leq 1$, the inequality is true if $z \leq 1/2$. If $z > 1/2$, then

$$2\lambda(1-2z)+z(5z+3) \geq 8(1-2z)+z(5z+3) = (1-z)(8-5z) \geq 0.$$

More generally it can be proved that if positive reals x, y, z satisfy $xy+yz+zx+xyz=\alpha$, then $x+y+z \geq xy+yz+zx$ if and only if $0 < \alpha \leq 4$. (CRUX-2000.) Note that $xy < \alpha$ and

$$z = \frac{\alpha - xy}{x+y+xy}.$$

Thus we have to determine positive α such that

$$x+y+\frac{\alpha - xy}{x+y+xy} \geq xy + \frac{(x+y)(\alpha - xy)}{x+y+xy}.$$

This simplifies to determine all positive α such that

$$f(x,y) = (x+y)^2 - (xy)^2 + (\alpha - xy)(1-x-y) \geq 0,$$

whenever $x, y > 0$ and $xy < \alpha$.
Taking $x = \alpha/2$ and making y approach zero, we see that

$$\frac{\alpha^2}{4} + \alpha\left(1 - \frac{\alpha}{2}\right) \geq 0.$$

This implies that $\alpha \leq 4$.

Conversely, suppose $\alpha \leq 4$ and let x, y be positive reals such that $xy < \alpha$. If $(x-1)(y-1) \geq 0$, then

$$f(x,y) = (x-y)^2 + (4-\alpha)xy + (\alpha - xy)(x-1)(y-1) \geq 0.$$

If $(x-1)(y-1) < 0$, we may write

$$f(x,y) = (x+y-2)^2 + (4-\alpha)xy + (\alpha - 4 - xy)(x-1)(y-1) \geq 0.$$

This proves the result.

■

156. Let x, y, z be non-negative real numbers such that $x+y+z=1$. Prove that

$$x^2y + y^2z + z^2x \leq \frac{4}{27}.$$

Solution: We may assume that z is the least number. Suppose $0 \leq z \leq y \leq x$. Then

$$x^2y + y^2z + z^2x \leq x^2y + xyz + xyz + z^2y = (x+z)^2y.$$

Here equality holds if and only if $z^2y = 0$ and $y^2z = xyz = z^2x$. If $y = 0$, we have $z = 0$ and hence the given inequality is trivially true. Hence we may assume that $y \neq 0$. Thus equality holds if and only if $z = 0$. It may be observed now that

$$(x+z)^2y = 4\left(\frac{x+z}{2}\right)\left(\frac{x+z}{2}\right)y \leq 4\left(\frac{x+z+y}{3}\right)^3 = \frac{4}{27}.$$

We have used the AM-GM inequality in the last step. Here equality holds if and only if $x+z=2y$. Since $z=0$, we must have $x=2y$. Using $x+y+z=1$, we see that equality holds if and only if $y=1/3$ and $x=2/3$, and of course $z=0$. The case $0 \leq z \leq x \leq y$ can be dealt in the same way. Thus equality holds if and only if

$$(x,y,z) = \big(2/3, 1/3, 0\big), \big(0, 2/3, 1/3\big), \big(1/3, 0, 2/3\big).$$

■

157. Let x, y, z be real numbers and let p, q, r be real numbers in the interval $(0, 1/2)$ such that $p+q+r=1$. Prove that

$$pqr\big(x+y+z\big)^2 \geq xyr\big(1-2r\big) + yzp\big(1-2p\big) + zxq\big(1-2q\big).$$

When does equality hold?

Solution: Putting $x = pu$, $y = qv$, $z = rw$, we get an equivalent inequality:

$$pqr\big(pu + qv + rw\big)^2 \geq pqr\Big\{uv\big(1 - 2r\big) + vw\big(1 - 2p\big) + wu\big(1 - 2q\big)\Big\}.$$

This further reduces to

$$\big(pu + qv + rw\big)^2 \geq uv\big(1 - 2r\big) + vw\big(1 - 2p\big) + wu\big(1 - 2q\big).$$

Taking $1 - 2p = a$, $1 - 2q = b$ and $1 - 2r = c$, we see that a, b, c are positive real numbers such that $a + b + c = 1$. The above inequality is equivalent to

$$\Big(a(v + w) + b(w + u) + c(u + v)\Big)^2 \geq 4\Big(avw + bwu + cuv\Big).$$

Suppose $u \leq v \leq w$. Then

$$\begin{aligned} v\big(u + w - v\big) &= vu + vw - v^2 \\ &= uw + (w - v)(v - u) \geq uw. \end{aligned}$$

Hence

$$\begin{aligned} &\Big(a(v + w) + b(w + u) + c(u + v)\Big)^2 \\ &= \Big\{(a + b + c)v + aw + b(w + u - v) + cu\Big\}^2 \\ &= \Big\{v + aw + b(w + u - v) + cu\Big\}^2 \\ &\geq 4v\big(aw + b(w + u - v) + cu\big) \\ &= 4\big(avw + bv(w + u - v) + cuv\big) \\ &\geq 4\big(avw + bwu + cuv\big). \end{aligned}$$

We observe that equality holds if and only if $v = w$ or $v = u$ and $v = aw + b(w + u - v) + cu$. If $v = w$, then we get $v = av + bu + cu = av + (1 - a)u$ and hence $v = u$ since $a \neq 1$. Similarly, $v = u$ implies that $v = w$. We conclude that equality holds if and only if $u = v = w$. This is equivalent to $x/p = y/q = z/r$. ■

158. Let x_1,x_2,x_3,. . .,x_n be n real numbers in the interval $[0, 1]$. Prove that

$$\left(\sum_{j=1}^{n} x_j\right) - \left(\sum_{j=1}^{n} x_j x_{j+1}\right) \leq \left[\frac{n}{2}\right],$$

where $x_{n+1} = x_1$.

Solution: Observe that

$$\begin{aligned}\sum_{j=1}^{n} x_j - \sum_{j=1}^{n} x_j x_{j+1} &= \sum_{j=1}^{n} x_j\left(1 - x_{j+1}\right)\\ &\le \frac{\sum_{j=1}^{n} x_j^2 + \sum_{j=1}^{n}\left(1 - x_{j+1}\right)^2}{2}\\ &= \frac{2\sum_{j=1}^{n} x_j^2 - 2\sum_{j=1}^{n} x_j + n}{2}\\ &= \frac{n}{2} + \sum_{j=1}^{n} x_j^2 - \sum_{j=1}^{n} x_j\\ &\le \frac{n}{2},\end{aligned}$$

since $x_j^2 \le x_j$ for all j, in view of the given condition that x_j's are from the interval $[0,1]$, for $1 \le j \le n$. The quantity $\sum_{j=1}^{n}\left(x_j - x_j x_{j+1}\right)$ is maximum only when $\sum_{j=1}^{n} x_j^2 - \sum_{j=1}^{n} x_j = 0$. Writing this in the form $\sum_{j=1}^{n} x_j\left(1 - x_j\right) = 0$, it follows that $x_j = 0$ or 1 for each j. Moreover, the condition for equality in the AM-GM inequality shows that equality holds in the above inequality only if

$$x_1 = 1 - x_2, x_2 = 1 - x_3, \ldots, x_{n-1} = 1 - x_n, x_n = 1 - x_1.$$

Thus, it follows that the quantity $\sum x_j - \sum x_j x_{j+1}$ is maximum if and only if each x_j is either 0 or 1 and alternate x_j's are equal.

Case 1: Suppose $n = 2k$. This implies that $x_1 = x_3 = \cdots = x_{2k-1}$ and $x_2 = x_4 = \cdots = x_{2k}$. Since each x_j is either 0 or 1, it follows that

$$x_1 = x_3 = \cdots = x_{2k-1} = 0 \quad \text{and} \quad x_2 = x_4 = \cdots = x_{2k} = 1,$$

or

$$x_1 = x_3 = \cdots = x_{2k-1} = 1 \quad \text{and} \quad x_2 = x_4 = \cdots = x_{2k} = 0.$$

Thus, we obtain

$$\sum_{j=1}^{n} x_j - \sum_{j=1}^{n} x_j x_{j+1} = k = [n/2].$$

Case 2: Suppose $n = 2k+1$. In this case, we have

$$x_1 = x_3 = \cdots = x_{2k+1} = 1 - x_1 \quad \text{and} \quad x_2 = x_4 = \cdots = x_{2k}.$$

Thus, it follows that

$$x_1 = x_3 = \cdots = x_{2k+1} = \frac{1}{2}.$$

But then $x_2 = x_4 = \cdots = x_{2k} = 1/2$ and

$$\sum_{j=1}^{n} x_j - \sum_{j=1}^{n} x_j x_{j+1} = \frac{n}{2} - \frac{n}{4} = \frac{n}{4} < [n/2].$$

■

159. Suppose x, y, z are positive real numbers such that $xyz \geq 1$. Prove that the inequality

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \geq 0$$

holds.

Solution: (by a Moldovian student)
We first show that

$$\sum_{\text{cyclic}} \frac{x^5 - x^2}{x^5 + y^2 + z^2} \geq \sum_{\text{cyclic}} \frac{x^5 - x^2}{x^3\left(x^2 + y^2 + z^2\right)},$$

where the sum is taken cyclically over x, y, z. In fact

$$\sum_{\text{cyclic}} \frac{x^5 - x^2}{x^5 + y^2 + z^2} - \sum_{\text{cyclic}} \frac{x^5 - x^2}{x^3\left(x^2 + y^2 + z^2\right)} = \frac{x^2\left(x^3 - 1\right)^2\left(y^2 + z^2\right)}{x^3\left(x^2 + y^2 + z^2\right)\left(x^5 + y^2 + z^2\right)},$$

which is non-negative. Thus it is sufficient to prove that

$$\sum_{\text{cyclic}} \frac{x^5 - x^2}{x^3\left(x^2 + y^2 + z^2\right)} \geq 0.$$

However, we have

$$\begin{aligned}
\sum_{\text{cyclic}} \frac{x^5 - x^2}{x^3\left(x^2 + y^2 + z^2\right)} &= \frac{1}{\left(x^2 + y^2 + z^2\right)} \sum_{\text{cyclic}} \left(x^2 - \frac{1}{x}\right) \\
&\geq \frac{1}{\left(x^2 + y^2 + z^2\right)} \sum_{\text{cyclic}} \left(x^2 - yz\right) \\
&= \frac{1}{2\left(x^2 + y^2 + z^2\right)} \sum_{\text{cyclic}} (x - y)^2 \\
&\geq 0.
\end{aligned}$$

■

160. Consider two sequences of positive real numbers, $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq b_3 \leq \cdots \leq b_n$, such that

$$\sum_{j=1}^{n} a_j \geq \sum_{j=1}^{n} b_j.$$

Suppose there exists a k, $1 \leq k \leq n$, such that $b_j \leq a_j$ for $1 \leq j \leq k$ and $b_j \geq a_j$ for $j > k$. Prove that

$$\prod_{j=1}^{n} a_j \geq \prod_{j=1}^{n} b_j.$$

Solution: We define

$$a'_j = a_k, \quad b'_j = \frac{b_j a_k}{a_j}, \quad 1 \leq j \leq n.$$

Then, for $1 \leq j \leq n$, we have

$$a'_j - b'_j = a_k \left(1 - \frac{b_j}{a_j}\right).$$

This gives

$$a'_j - b'_j - \left(a_j - b_j\right) = \left(a_j - b_j\right)\left(a_k - a_j\right)/a_j \geq 0,$$

for $1 \leq j \leq n$; this follows from $a_j - b_j \geq 0$, $a_k \geq a_j$, for $1 \leq j \leq k$, and $a_j - b_j \leq 0$, $a_k \leq a_j$, for $k < j \leq n$. Summing over all j, we obtain

$$\sum_{j=1}^{n} a'_j - \sum_{j=1}^{n} b'_j - \left(\sum_{j=1}^{n} a_j - \sum_{j=1}^{n} b_j\right) \geq 0.$$

Since we are given that $\sum_{j=1}^{n} a_j \geq \sum_{j=1}^{n} b_j$, we conclude that

$$n a_k = \sum_{j=1}^{n} a'_j \geq \sum_{j=1}^{n} b'_j.$$

Using the AM-GM inequality, we obtain

$$\left(b'_1 b'_2 \cdots b'_n\right)^{1/n} \leq \frac{\sum b'_j}{n} \leq a_k.$$

This implies that

$$\left(\frac{b_1 b_2 b_3 \cdots b_n a_k^n}{a_1 a_2 a_3 \cdots a_n}\right)^{1/n} \leq a_k.$$

Simplification gives

$$b_1 b_2 b_3 \cdots b_n \leq a_1 a_2 a_3 \cdots a_n.$$

■

161. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \leq \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c}.$$

Solution: We introduce elementary symmetric functions σ_1, σ_2, σ_3 in a, b, c: $\sigma_1 = a+b+c$; $\sigma_2 = ab+bc+ca$; $\sigma_3 = abc$. Note that $\sigma_1 \geq 3$, $\sigma_2 \geq 3$ and $\sigma_3 = 1$. The left side of the inequality is:

$$\frac{\sum_{\text{cyclic}}(1+b+c)(1+c+a)}{\prod_{\text{cyclic}}(1+a+b)}.$$

Now

$$\begin{aligned}
&\sum_{\text{cyclic}}(1+b+c)(1+c+a)\\
&\qquad = \sum_{\text{cyclic}}\left(1+a+b+c+c+ab+bc+ca+c^2\right)\\
&\qquad = 3+4\sigma_1+3\sigma_2+\sum_{\text{cyclic}}c^2\\
&\qquad = 3+4\sigma_1+\sigma_2+\sigma_1^2,
\end{aligned}$$

and

$$\prod_{\text{cyclic}}(1+a+b) = 2\sigma_1+\sigma_2+\sigma_1^2+\sigma_2\sigma_1.$$

Thus, the left side may be written in the form

$$\frac{3+4\sigma_1+\sigma_2+\sigma_1^2}{2\sigma_1+\sigma_2+\sigma_1^2+\sigma_2\sigma_1}.$$

Similarly, the right side of the inequality reduces to

$$\frac{12+4\sigma_1+\sigma_2}{9+4\sigma_1+2\sigma_2}.$$

The required inequality is, thus,

$$\frac{3+4\sigma_1+\sigma_2+\sigma_1^2}{2\sigma_1+\sigma_2+\sigma_1^2+\sigma_2\sigma_1} \leq \frac{12+4\sigma_1+\sigma_2}{9+4\sigma_1+2\sigma_2}.$$

Equivalently, one needs to prove

$$3\sigma_1^2\sigma_2+\sigma_1\sigma_2^2+6\sigma_1\sigma_2-5\sigma_1^2-\sigma_2^2-24\sigma_1-3\sigma_2-27 \geq 0.$$

Consider the quadratic expression:

$$Q(x) = \left(3\sigma_2-5\right)x^2+\left(\sigma_2^2+6\sigma_2-24\right)x-\left(\sigma_2^2+3\sigma_2+27\right).$$

For $x=0$, this is negative. For $x=3$, the expression reduces to

$$2\sigma_2^2+42\sigma_2-144,$$

which is non-negative because of $\sigma_2 \geq 3$. Note that $Q(x) = 0$ has a positive and a negative root. Since the leading coefficient is $3\sigma_2 - 5 \geq 9 - 5 = 4 > 0$, it follows that $Q(x) \geq 0$, for all $x \geq 3$. Since $\sigma_1 \geq 3$, it follows that $Q(\sigma_1) \geq 0$. This proves the required inequality.

Alternate Solution:

We have to prove that

$$\frac{\sum_{\text{cyclic}}(1+b+c)(1+c+a)}{\prod_{\text{cyclic}}(1+a+b)} \leq \frac{\sum_{\text{cyclic}}(2+b)(2+c)}{\prod_{\text{cyclic}}(2+a)}.$$

As in the earlier solution, we introduce $\sigma_1 = a+b+c$, $\sigma_2 = ab+bc+ca$, $\sigma_3 = abc = 1$. Then we have

$$\begin{aligned} \sum_{\text{cyclic}}(1+b+c)(1+c+a) &= 3 + 4\sigma_1 + \sigma_2 + \sigma_1^2, \\ \prod_{\text{cyclic}}(1+a+b) &= 2\sigma_1 + \sigma_2 + \sigma_1^2 + \sigma_2\sigma_1, \\ \sum_{\text{cyclic}}(2+b)(2+c) &= 12 + 4\sigma_1 + \sigma_2, \\ \prod_{\text{cyclic}}(2+a) &= 9 + 4\sigma_1 + 2\sigma_2. \end{aligned}$$

The equivalent inequality is

$$27 + 24\sigma_1 + 3\sigma_2 + 5\sigma_1^2 - 6\sigma_1\sigma_2 + \sigma_2^2 - 3\sigma_1^2\sigma_2 - \sigma_1\sigma_2^2 \leq 0.$$

We observe that

$$\begin{aligned} \sigma_1 = a+b+c &\geq 3\big(abc\big)^{1/3} = 3, \\ \sigma_2 = ab+bc+ca &\geq 3\big(a^2b^2c^2\big)^{1/3} = 3, \\ 3\sigma_2 = 3(ab+bc+ca) &\leq (a+b+c)^2 = \sigma_1^2. \end{aligned}$$

We may write the inequality in the form

$$27 + 24\sigma_1 + 5\sigma_1^2 + \sigma_2\big(3 - 6\sigma_1 + \sigma_2 - 3\sigma_1^2 - \sigma_1\sigma_2\big) \leq 0.$$

Observe that

$$\begin{aligned} 3 - 6\sigma_1 + \sigma_2 - 3\sigma_1^2 - \sigma_1\sigma_2 &\leq 3 - 6\sigma_1 + \sigma_2 - 3\sigma_1^2 - 3\sigma_1 \\ &= 3 - 9\sigma_1 + \sigma_2 - 3\sigma_1^2 \\ &\leq 0, \end{aligned}$$

since $\sigma_1 \geq 3$ and $\sigma_2 \leq \sigma_1^2/3$. Thus, using $\sigma_2 \geq 3$, we have

$$\begin{aligned} &27 + 24\sigma_1 + 5\sigma_1^2 + \sigma_2\big(3 - 6\sigma_1 + \sigma_2 - 3\sigma_1^2 - \sigma_1\sigma_2\big) \\ &\qquad \leq 36 + 6\sigma_1 - 4\sigma_1^2 - 3\sigma_1\sigma_2 + 3\sigma_2 \\ &\qquad = 36 + 6\sigma_1 - 4\sigma_1^2 - 3\sigma_2\big(1 - \sigma_1\big). \end{aligned}$$

Since $1-\sigma_1<0$, we get

$$\begin{aligned}36+6\sigma_1-4\sigma_1^2-3\sigma_2\left(1-\sigma_1\right)&\\ &\leq 36+6\sigma_1-4\sigma_1^2-9\left(1-\sigma_1\right)\ (\text{since } \sigma_2\geq 3)\\ &=45-3\sigma_1-4\sigma_1^2\\ &=-\left(\sigma_1-3\right)\left(4\sigma_1+15\right)\leq 0,\end{aligned}$$

since $\sigma_1 \geq 3$. ■

162. Let $n\geq 4$ and let $a_1,a_2,a_3,\ldots,a_n$ be real numbers such that

$$a_1+a_2+\cdots+a_n\geq n,\quad a_1^2+a_2^2+\cdots+a_n^2\geq n^2.$$

Prove that

$$\max\left\{a_1,a_2,a_3,\ldots,a_n\right\}\geq 2.$$

Solution: We assume $a_1\leq a_2\leq a_3\leq\cdots\leq a_n$. Put

$$\alpha=\left(\sum_{j=1}^{n}a_j\right)/n$$

and $b_j=a_j-\alpha$, for $1\leq j\leq n$. Then we observe that

$$b_1\leq b_2\leq b_3\leq\cdots\leq b_n,\quad \sum_{j=1} nb_j=0.$$

Consider $\left(b_1-b_j\right)\left(b_j-b_n\right)$. This product is non-negative since $b_1-b_j\leq 0$ and $b_j-b_n\leq 0$. Summing these products over j, we observe that

$$nb_1b_n+\sum_{j=1}^{n}b_j^2\leq 0.$$

Substituting back $b_j=a_j-\alpha$ and simplifying, we obtain

$$na_1a_n-n\alpha\left(a_1+a_n\right)+\sum_{j=1}^{n}a_j^2\leq 0;$$

we have used $\sum_{j=1}^{n}a_j\geq n$. Using $\sum_{j=1}^{n}a_j^2\geq n^2$, this further simplifies to

$$n\leq\alpha\left(a_1+a_n\right)-a_1a_n.$$

Suppose $a_n<2$. If $|a_j|\leq 2$ for all $j\leq n-1$, then

$$n^2\leq\sum_{j=1}^{n}a_j^2<4n,$$

and hence $n < 4$. Thus we may assume that $|a_j| > 2$ for some $j \neq n$ (and of course $a_n < 2$).

Suppose k is such that $a_1 \le a_2 \le a_3 \le \cdots \le a_k < -2$ and $-2 \le a_{k+1} \le a_{k+2} \le \cdots \le a_n < 2$. Since $a_1 + a_n \le 0$ and $\alpha \ge 1$, we have

$$n \le a_1 + a_n - a_1 a_n < a_1 + 2 - 2a_1 = -a_1 + 2.$$

Thus, we obtain $a_1 < -n + 2$. We further have

$$\begin{aligned} n &\le a_1 + a_2 + a_3 + \cdots + a_k + a_{k+1} + \cdots + a_n \\ &< a_1 + a_2 + a_3 + \cdots + a_k + 2(n-k), \end{aligned}$$

and hence $a_1 + a_2 + a_3 + \cdots + a_k > 2k - n$. This estimate leads to

$$\begin{aligned} 2k - n &< a_1 + a_2 + a_3 + \cdots + a_k \\ &< -n + 2 + a_2 + a_3 + \cdots + a_k \\ &< -n + 2 - 2k + 2. \end{aligned}$$

It follows that $k < 1$. But this contradicts $|a_j| > 2$ for at least one $j \neq n$. Thus, $a_n < 2$ forces $n < 4$. We conclude that for $n \ge 4$, the largest number among a_j's must exceed 2. ■

163. Let $x_1 \le x_2 \le x_3 \le \cdots \le x_{n+1}$ be $n+1$ positive integers. Prove that

$$\sum_{j=1}^{n+1} \frac{\sqrt{x_{j+1} - x_j}}{x_{j+1}} < \sum_{j=1}^{n^2} \frac{1}{j}.$$

Solution: Since x_j are positive integers, $x_{j+1} - x_j \ge 1$ for $1 \le j \le n$. This forces $\sqrt{x_{j+1} - x_j} \le x_{j+1} - x_j$, $1 \le j \le n$. Thus

$$\begin{aligned} \sum_{j=1}^{n+1} \frac{\sqrt{x_{j+1} - x_j}}{x_{j+1}} &\le \sum_{j=1}^{n+1} \frac{x_{j+1} - x_j}{x_{j+1}} \\ &= \frac{x_2 - x_1}{x_2} + \frac{x_3 - x_2}{x_3} + \cdots + \frac{x_{n+1} - x_n}{x_n} \\ &< 1 + \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x_2}\right) \\ &\quad + \left(\frac{1}{x_2 + 1} + \cdots + \frac{1}{x_3}\right) \\ &\quad + \cdots + \left(\frac{1}{x_n + 1} + \cdots + \frac{1}{x_{n+1}}\right) \\ &< 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x_{n+1}}. \end{aligned}$$

We prove the inequality by induction on n. For $n = 1$, we have

$$\frac{\sqrt{x_2 - x_1}}{x_2} < \frac{\sqrt{x_2}}{x_2} = \frac{1}{\sqrt{x_2}} < 1.$$

Suppose

$$\sum_{j=1}^{n} \frac{\sqrt{x_{j+1} - x_j}}{x_{j+1}} < \sum_{j=1}^{(n-1)^2} \frac{1}{j}.$$

If $x_{n+1} \leq n^2$, then the above estimate proves the inequality:

$$\begin{aligned} \sum_{j=1}^{n+1} \frac{\sqrt{x_{j+1} - x_j}}{x_{j+1}} &< 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x_{n+1}} \\ &\leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n^2}. \end{aligned}$$

Suppose $x_{n+1} > n^2$. Then we have

$$\frac{\sqrt{x_{n+1} - x_n}}{x_{n+1}} < \frac{1}{\sqrt{x_{n+1}}} < \frac{1}{n}.$$

On the other hand,

$$\frac{1}{(n-1)^2 + 1} + \frac{1}{(n-1)^2 + 2} + \cdots + \frac{1}{n^2} > \frac{2n - 1}{n^2} \geq \frac{1}{n}.$$

Thus, we get

$$\frac{\sqrt{x_{n+1} - x_n}}{x_{n+1}} < \frac{1}{(n-1)^2 + 1} + \frac{1}{(n-1)^2 + 2} + \cdots + \frac{1}{n^2}.$$

Using the induction hypothesis, we now obtain

$$\begin{aligned} \sum_{j=1}^{n+1} \frac{\sqrt{x_{j+1} - x_j}}{x_{j+1}} &= \left(\sum_{j=1}^{n} \frac{\sqrt{x_{j+1} - x_j}}{x_{j+1}} \right) + \frac{\sqrt{x_{n+1} - x_n}}{x_{n+1}} \\ &< \sum_{j=1}^{(n-1)^2} \frac{1}{j} + \sum_{j=(n-1)^2+1}^{n^2} \frac{1}{j} \\ &= \sum_{j=1}^{n^2} \frac{1}{j}. \end{aligned}$$

This completes the proof by induction. ■

164. Let a, b, c be three positive real numbers which satisfy $abc = 1$ and $a^3 > 36$. Prove that

$$\frac{2}{3}a^2 < a^2 + b^2 + c^2 - ab - bc - ca.$$

Solution: Consider the quadratic expression

$$a(b+c)^2 - a^2(b+c) + \frac{1}{3}a^3 - 3,$$

in $b+c$. Its discriminant is

$$D = a^4 - 4a\Big(\frac{1}{3}a^3 - 3\Big) = \frac{a}{3}\Big(36 - a^3\Big) < 0,$$

and hence

$$a(b+c)^2 - a^2(b+c) + \frac{1}{3}a^3 - 3 > 0,$$

for all values for $b+c$. Using $abc = 1$, this may be written in the form

$$a\Big\{(b+c)^2 - a(b+c) + \frac{1}{3}a^2 - 3bc\Big\} > 0.$$

Since $a > 0$, this reduces to

$$b^2 + c^2 - a(b+c) + \frac{1}{3}a^2 - bc > 0.$$

It follows that

$$\frac{2}{3}a^2 < a^2 + b^2 + c^2 - ab - bc - ca.$$

■

165. Let $z_1, z_2, z_3, \ldots, z_n$ be n complex numbers and consider n positive real numbers $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ which have the property that $\sum 1/\lambda_j = 1$. Prove that

$$\left|\sum_{j=1}^{n} z_j\right|^2 \le \sum_{j=1}^{n} \lambda_j |z_j|^2.$$

Solution: Suppose $\beta_1, \beta_2, \ldots, \beta_n$ be any n real numbers. Then the Cauchy-Schwarz inequality gives

$$\begin{aligned} \sum_{j=1}^{n} \lambda_j \beta_j^2 &= \Big(\sum_{j=1}^{n} \frac{1}{\lambda_j}\Big)\Big(\sum_{j=1}^{n} \lambda_j \beta_j^2\Big) \\ &\ge \Big(\sum_{j=1}^{n} \beta_j\Big)^2. \end{aligned}$$

We may assume $\sum_{j=1}^{n} |z_j| \neq 0$. Taking

$$\beta_j = \frac{|z_j|}{\sum_{j=1}^{n} |z_j|},$$

we obtain

$$\sum_{j=1}^{n} \lambda_j |z_j|^2 - \left(\sum_{j=1}^{n} |z_j|\right)^2 \geq 0.$$

However, triangle inequality gives $\left|\sum_{j=1}^{n} z_j\right| \leq \sum_{j=1}^{n} |z_j|$. Thus we obtain

$$\left|\sum_{j=1}^{n} z_j\right|^2 \leq \sum_{j=1}^{n} \lambda_j |z_j|^2.$$

Remark: For $n = 2$, this is known as Bohr's inequality. ■

166. Let a, b, c be three distinct real numbers. Prove that

$$\begin{aligned} 3\min\{a,b,c\} &< \sum a - \left(\sum a^2 - \sum ab\right)^{1/2} \\ &< \sum a + \left(\sum a^2 - \sum ab\right)^{1/2} < 3\max\{a,b,c\}, \end{aligned}$$

where the sum is cyclically over a, b, c.

Solution: Consider the monic polynomial whose roots are a, b, c:

$$\begin{aligned} p(x) &= (x-a)(x-b)(x-c) \\ &= x^3 - \Big(\sum_{\text{cyclic}} a\Big)x^2 + \Big(\sum_{\text{cyclic}} ab\Big)x - abc. \end{aligned}$$

We know that the roots of $p'(x) = 0$ lie between $\min\{a,b,c\}$ and $\max\{a,b,c\}$. But $p'(x) = 3x^2 - 2\left(\sum a\right)x + \left(\sum ab\right)$ whose zeros are

$$\frac{\sum\limits_{\text{cyclic}} a + \sqrt{\sum\limits_{\text{cyclic}} a^2 - \sum\limits_{\text{cyclic}} ab}}{3}, \quad \frac{\sum\limits_{\text{cyclic}} a - \sqrt{\sum\limits_{\text{cyclic}} a^2 - \sum\limits_{\text{cyclic}} ab}}{3}.$$

Since a, b, c are distinct, $\sum_{\text{cyclic}} a^2 - \sum_{\text{cyclic}} ab > 0$, and hence

$$\begin{aligned} 3\min\{a,b,c\} &< \sum_{\text{cyclic}} a - \Big(\sum_{\text{cyclic}} a^2 - \sum_{\text{cyclic}} ab\Big)^{1/2} \\ &< \sum_{\text{cyclic}} a + \Big(\sum_{\text{cyclic}} a^2 - \sum_{\text{cyclic}} ab\Big)^{1/2} < 3\max\{a,b,c\}. \end{aligned}$$

■

167. Suppose a, b, c are real numbers such that $a^3 + b^3 + c^3 = 0$. Prove that

$$\left(\sum a^2\right)^3 \le \left(\sum (b-c)^2\right)\left(\sum a^4\right),$$

where the sum is cyclically over a, b, c.

Solution: We have

$$\begin{aligned}(a-b)(b-c)(c-a) &= ab^2 - a^2b + bc^2 - b^2c + ca^2 - c^2a \\ &= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}.\end{aligned}$$

Hence it follows that

$$(a-b)^2(b-c)^2(c-a)^2 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}^2.$$

But for any two matrices A and B, we have $\det(AB) = \det(A)\det(B)$ and $\det(A) = \det(A^t)$. This implies

$$\begin{aligned}(a-b)^2(b-c)^2(c-a)^2 &= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}^2 \\ &= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \\ &= \left|\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}\begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}\right|\end{aligned}$$

Introducing $s_k = a^k + b^k + c^k$ for $k \ge 1$, the above relation simplifies to

$$\begin{aligned}&(a-b)^2(b-c)^2(c-a)^2 \\ &\quad= \begin{vmatrix} 3 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix} \\ &\quad= s_4\left(3s_2 - s_1^2\right) - s_3\left(3s_3 - s_1s_2\right) + s_2\left(s_3s_1 - s_2^2\right).\end{aligned}$$

Since $s_3 = 0$, we get

$$(a-b)^2(b-c)^2(c-a)^2 = s_4\left(3s_2 - s_1^2\right) - s_2^3.$$

This shows that

$$s_2^3 \le s_4\left(3s_2 - s_1^2\right).$$

However,

$$3s_2 - s_1^2 = 3\left(a^2+b^2+c^2\right) - (a+b+c)^2 = \sum_{\text{cyclic}} (b-c)^2.$$

Thus, we obtain

$$\left(\sum_{\text{cyclic}} a^2\right)^3 \le \left(\sum_{\text{cyclic}} (b-c)^2\right)\left(\sum_{\text{cyclic}} a^4\right).$$

■

168. Show that for all complex numbers z with real part of $z > 1$, the following inequality holds:

$$|z^{n+1} - 1| > |z^n||z-1|, \text{ for all } n \ge 1.$$

Solution: We may easily verify this for $n = 1$ and 2. We assume $n \ge 3$. Put $\operatorname{Re}(z) = r\cos\theta$, where $\operatorname{Re}(z)$ denotes the real part of z. Then $r > 1$ and $0 \le \theta \le \pi/2$. We have

$$\begin{aligned}\left|z^{n+1} - z^n\right|^2 &= r^{2n+2} + r^{2n} - 2r^{2n+1}\cos\theta,\\ \left|z^{n+1} - 1\right|^2 &= r^{2n+2} + 1 - 2r^{n+1}\cos(n+1)\theta.\end{aligned}$$

Thus, we have to show that

$$r^{2n+2} + r^{2n} - 2r^{2n+1}\cos\theta < r^{2n+2} + 1 - 2r^{n+1}\cos(n+1)\theta.$$

This is equivalent to the relation

$$r^{2n} - 1 < 2r^{2n+1}\cos\theta - 2r^{n+1}\cos(n+1)\theta.$$

Here we consider two cases.

Case 1. Suppose $r = 1 + d$, where

$$d \ge \frac{2}{n(n-2)}, \quad n \ge 3.$$

Then

$$r^2 - 1 = 2d + d^2 \le (n-1)^2 d^2 = (1 + nd - r)^2 < (r^n - r)^2.$$

Thus we get in this case $r^{2n} - 2r^{n+1} + 1 > 0$ and hence

$$\begin{aligned}2r^{2n+1}\cos\theta - 2r^{n+1}\cos(n+1)\theta &\ge 2r^{2n+1}\cos\theta - 2r^{n+1}\\ &\ge 2r^{2n} - 2r^{n+1}\\ &> r^{2n} - 1.\end{aligned}$$

Case 2. Suppose

$$d < \frac{2}{n(n-2)}.$$

Since $r\cos\theta > 1$, we have

$$1 - \frac{2}{n(n-2)} < 1 - d < \frac{1}{r} < \cos\theta < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}.$$

This implies that

$$\theta^4 - 12\theta^2 + \frac{48}{n(n-2)} > 0.$$

Since $\theta^2 < 6$, we get

$$\theta^2 \le 6 - 6\sqrt{D},$$

where $D = 1 - \dfrac{4}{3n(n-2)}$. If $n \ge 4$, it is easy to check that

$$8n^4 + 10n^3 - 87n^2 - 50n - 16 > 0,$$

and hence

$$1 - \frac{6}{(n+2)^2} < \sqrt{D}.$$

This gives

$$\theta^2 \le 6 - 6\sqrt{D} < \frac{36}{(n+2)^2} < \frac{4\pi^2}{(n+2)^2}.$$

Thus we obtain $(n+2)\theta < 2\pi$. Hence $3\pi/2 < (n+1)\theta \le 2\pi$ and this implies that $\cos(n+1)\theta < \cos(2\pi - \theta) = \cos\theta$. Obviously, this holds for $n = 3$ also, since $0 \le \theta \le \pi/2$. Thus we get

$$2r^{2n+1}\cos\theta - 2r^{n+1}\cos(n+1)\theta \ge 2r^{2n} - 2r^n > r^{2n} - 1.$$

■

169. Suppose a, b, c are positive real numbers and let

$$x = a + b - c, \quad y = b + c - a, \quad z = c + a - b.$$

Prove that

$$abc(xy + yz + zx) \ge xyz(ab + bc + ca).$$

Solution:

Since the expressions are symmetric in a, b, c, we may assume that $a \ge b \ge c > 0$. Note that $x > 0$ and $z > 0$. If $y > 0$, then the inequality is equivalent to

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \ge \frac{1}{a} + \frac{1}{b} + \frac{1}{c}. \tag{169.1}$$

But using the AM-HM inequality, we see that

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{b+c-a}+\frac{1}{c+a-b}\geq\frac{2}{c}.$$

Similarly, we can prove that

$$\frac{1}{x}+\frac{1}{y}\geq\frac{2}{b},\quad \frac{1}{z}+\frac{1}{x}\geq\frac{2}{a}.$$

Adding these we get (169.1).

If $y=0$, then the inequality reduces to $abczx\geq 0$ which is true. If $y<0$, then the inequality is equivalent to

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\leq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}. \tag{169.2}$$

Since $z\geq c$ and $x\geq a$, we have

$$\frac{1}{z}\leq\frac{1}{c},\quad \frac{1}{x}\leq\frac{1}{a},$$

and

$$\frac{1}{y}<0<\frac{1}{b}.$$

Adding these we get (169.2). ■

170. Let a,b,c be positive real numbers. Prove that

$$\sum_{\text{cyclic}}\frac{a^3}{b^2-bc+c^2}\geq\frac{3\sum_{\text{cyclic}}ab}{\sum_{\text{cyclic}}a}.$$

Solution: The left hand side is equal to

$$\sum_{\text{cyclic}}\frac{a^3(b+c)}{b^3+c^3},$$

and this may be written as

$$\frac{\sum_{\text{cyclic}}a^3\left(a^3+b^3\right)\left(a^3+c^3\right)(b+c)}{\prod_{\text{cyclic}}\left(b^3+c^3\right)}.$$

However, we have

$$\begin{aligned}
&a^3(b+c)(a^3+b^3)(a^3+c^3)\\
&= a^3(b+c)\left\{a^3\Big(\sum_{\text{cyclic}} a^3\Big)+b^3c^3\right\}\\
&= (b+c)\left\{a^6\Big(\sum_{\text{cyclic}} a^3\Big)+a^3b^3c^3\right\}\\
&= \Big(\sum_{\text{cyclic}} a\Big)\Big(\sum_{\text{cyclic}} a^3\Big)a^6+\Big(\sum_{\text{cyclic}} a\Big)a^3b^3c^3-a^7\Big(\sum_{\text{cyclic}} a^3\Big)-a\Big(a^3b^3c^3\Big).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\sum_{\text{cyclic}} a^3(a^3+b^3)(a^3+c^3)(b+c) &= \Big(\sum_{\text{cyclic}} a\Big)\Big(\sum_{\text{cyclic}} a^3\Big)\Big(\sum_{\text{cyclic}} a^6\Big)\\
&\quad -\Big(\sum_{\text{cyclic}} a^3\Big)\Big(\sum_{\text{cyclic}} a^7\Big)+2\Big(\sum_{\text{cyclic}} a\Big)a^3b^3c^3.
\end{aligned}$$

On the other hand, we also have

$$\prod_{\text{cyclic}}(b^3+c^3)=2a^3b^3c^3+\Big(\sum_{\text{cyclic}} a^3\Big)\Big(\sum_{\text{cyclic}} a^6\Big)-\sum_{\text{cyclic}} a^9.$$

Thus, the inequality to be proved is

$$\begin{aligned}
&\Big(\sum_{\text{cyclic}} a\Big)\Big\{\Big(\sum_{\text{cyclic}} a\Big)\Big(\sum_{\text{cyclic}} a^3\Big)\Big(\sum_{\text{cyclic}} a^6\Big)\\
&\qquad -\Big(\sum_{\text{cyclic}} a^3\Big)\Big(\sum_{\text{cyclic}} a^7\Big)+2\Big(\sum_{\text{cyclic}} a\Big)a^3b^3c^3\Big\}\\
&\qquad \geq 3\Big(\sum_{\text{cyclic}} ab\Big)\left\{2a^3b^3c^3+\Big(\sum_{\text{cyclic}} a^3\Big)\Big(\sum_{\text{cyclic}} a^6\Big)-\sum_{\text{cyclic}} a^9\right\}.
\end{aligned}$$

This may be written in the form

$$\begin{aligned}
&\Big\{\Big(\sum_{\text{cyclic}} a\Big)^2-3\sum_{\text{cyclic}} ab\Big\}\Big\{2a^3b^3c^3+\Big(\sum_{\text{cyclic}} a^3\Big)\Big(\sum_{\text{cyclic}} a^6\Big)-\sum_{\text{cyclic}} a^9\Big\}\\
&\qquad +\Big(\sum_{\text{cyclic}} a\Big)\Big\{\Big(\sum_{\text{cyclic}} a\Big)\Big(\sum_{\text{cyclic}} a^9\Big)-\Big(\sum_{\text{cyclic}} a^3\Big)\Big(\sum_{\text{cyclic}} a^7\Big)\Big\}\geq 0.
\end{aligned}$$

We observe the following:

$$\Big(\sum_{\text{cyclic}} a\Big)^2 - 3\sum_{\text{cyclic}} ab = \frac{1}{2}\sum_{\text{cyclic}} (a-b)^2 \geq 0,$$

$$2a^3b^3c^3 + \Big(\sum_{\text{cyclic}} a^3\Big)\Big(\sum_{\text{cyclic}} a^6\Big) - \sum_{\text{cyclic}} a^9 > 0,$$

and

$$\begin{aligned}
&\Big(\sum_{\text{cyclic}} a\Big)\Big(\sum_{\text{cyclic}} a^9\Big) - \Big(\sum_{\text{cyclic}} a^3\Big)\Big(\sum_{\text{cyclic}} a^7\Big) \\
&= \sum_{\text{cyclic}} ab^7\left(b^2-a^2\right) + \sum_{\text{cyclic}} ba^7\left(a^2-b^2\right) \\
&= \sum_{\text{cyclic}} ab\left(b^6-a^6\right)\left(b^2-a^2\right) \\
&= \sum_{\text{cyclic}} ab\left(b^2-a^2\right)^2\left(b^4+b^2a^2+a^4\right) \\
&\geq 0.
\end{aligned}$$

This proves the given inequality. Equality holds if and only if $a = b = c$. ■

171. Let $a_1, a_2, \ldots, a_n < 1$ be non-negative real numbers satisfying

$$a = \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{\sqrt{3}}{3}.$$

Prove that

$$\frac{a_1}{1-a_1^2} + \frac{a_2}{1-a_2^2} + \cdots + \frac{a_n}{1-a_n^2} \geq \frac{na}{1-a^2}.$$

Solution: We first prove that it is sufficient to consider the case where all the a_j's are positive. Suppose, for example, $a_n = 0$. Then

$$a = \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_{n-1}^2}{n}}.$$

If we set

$$b = \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_{n-1}^2}{n-1}},$$

then the inequality for $(n-1)$ numbers $a_1, a_2, \ldots, a_{n-1}$ is

$$\sum_{j=1}^{n-1} \frac{a_j}{1-a_j^2} \geq \frac{(n-1)b}{1-b^2}.$$

Note that

$$a \geq \frac{1}{\sqrt{3}} \Longrightarrow b = a \cdot \sqrt{\frac{n}{n-1}} > a \geq \frac{1}{\sqrt{3}}.$$

Thus, we need to prove

$$\frac{(n-1)b}{1-b^2} \geq \frac{na}{1-a^2}.$$

Using $b = a \cdot \sqrt{\dfrac{n}{n-1}}$, this reduces to

$$\frac{\sqrt{n(n-1)}}{1 - \dfrac{na^2}{(n-1)}} \geq \frac{n}{1-a^2}.$$

Simplification leads to

$$1 - a^2 \leq \frac{n^2 + \sqrt{n(n-1)^3}}{n^3 - (n-1)^3}.$$

Since $a^2 \geq 1/3$ implies $1 - a^2 \leq 2/3$, it is sufficient to prove

$$\frac{2}{3} \leq \frac{n^2 + \sqrt{n(n-1)^3}}{n^3 - (n-1)^3}.$$

A further simplification reduces this to

$$n^2 - 2n + \frac{2}{3} \leq \sqrt{n(n-1)^3}.$$

Finally we observe

$$n^2 - 2n + \frac{2}{3} < (n-1)^2 < \sqrt{n(n-1)^3},$$

proving our claim.

The above argument shows that we can discard all those a_j's which are equal to zero without affecting the inequality. We henceforth assume $a_j \neq 0$ for $1 \leq j \leq n$.

We may write the inequality in the form

$$\sum_{j=1}^{n} \frac{a_j^2}{na^2} \frac{1}{a_j\left(1 - a_j^2\right)} \geq \frac{1}{a\left(1 - a^2\right)}.$$

Setting $w_j = a_j^2/na^2$ for $1 \leq j \leq n$, we see that $w_j > 0$, for $1 \leq j \leq n$ and $\sum_{j=1}^{n} w_j = 1$. Consider the function

$$f(x) = \frac{1}{x\left(1 - x^2\right)}, \text{ for } 0 < x < 1.$$

It is easy to check that $f''(x) \geq 0$, for $0 < x < 1$. Thus $f(x)$ is a convex function on the interval $(0,1)$. Now Jensen's inequality gives

$$\sum_{j=1}^{n} w_j f(a_j) \geq f\left(\sum_{j=1}^{n} w_j a_j\right).$$

This takes the form

$$\sum_{j=1}^{n} w_j \frac{1}{a_j\left(1-a_j^2\right)} \geq \frac{1}{\sum_{j=1}^{n} w_j a_j - \left(\sum_{j=1}^{n} w_j a_j\right)^3}.$$

Thus it is sufficient to prove that

$$\sum_{j=1}^{n} w_j a_j - \left(\sum_{j=1}^{n} w_j a_j\right)^3 \leq a(1-a^2).$$

Taking $\lambda = \sum_{j=1}^{n} a_j^3$, this reduces to

$$\lambda^3 - n^2 a^4 \lambda + n^3 a^7\left(1-a^2\right) \geq 0.$$

But using the factorisation

$$\lambda^3 - n^2 a^4 \lambda + n^3 a^7\left(1-a^2\right) = \left(\lambda - na^3\right)\left(\lambda^2 + na^3\lambda + n^2a^4\left(1-a^2\right)\right),$$

and the observation that for $a^2 < 1$ and $\lambda \geq 0$, the expression $\lambda^2 + na^3\lambda + n^2a^4\left(1-a^2\right)$ remains positive, it suffices to prove that $\lambda - na^3 \geq 0$. Thus we need to prove that

$$\sum_{j=1}^{n} a_j^3 \geq n\left(\sum_{j=1}^{n} \frac{a_j^2}{n}\right)^{3/2}.$$

This reduces to the form

$$n\left(\sum_{j=1}^{n} a_j^3\right)^2 \geq \left(\sum_{j=1}^{n} a_j^2\right)^3.$$

However, this follows from Hölder's inequality with exponents $p = 3/2$ and $q = 3$:

$$\sum_{j=1}^{n} a_j^2 \leq \left(\sum_{j=1}^{n} a_j^3\right)^{2/3} n^{1/3}.$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$. ∎

172. Suppose a, b, c are non-negative real numbers such that $a^2+b^2+c^2+abc = 4$. Prove that

$$0 \leq ab + bc + ca - abc \leq 2.$$

Solution: The given condition on a, b, c shows that at least one of a, b, c cannot exceed 1; say $a \leq 1$. Then

$$ab + bc + ca - abc = a(b + c) + bc(1 - a) \geq 0.$$

Here equality holds if and only if $a(b + c) = bc(1 - a) = 0$. If $a = 1$, then $b + c = 0$ forcing $b = c = 0$ which is impossible in view of the condition $a^2 + b^2 + c^2 + abc = 4$. Thus $1 - a \neq 0$, and only one of b, c can be equal to zero. Suppose, for example, $b = 0$. Then c cannot be zero and $ca = 0$. This implies that $a = 0$ and hence $c^2 = 4$. But then $c = 2$. Thus the equality holds on the left side if and only if $(a, b, c) = (0, 0, 2)$, $(0, 2, 0)$, $(2, 0, 0)$.

To prove the right side inequality, we show that the given condition forces the existence of a triangle with angles α, β, γ such that $a = 2\sin(\alpha/2)$, $b = 2\sin(\beta/2)$, $c = 2\sin(\gamma/2)$. In fact, if α, β, γ are the angles of a triangle, then

$$\begin{aligned}
\sum_{\text{cyclic}} \sin^2(\alpha/2) &= \frac{1}{2} \sum_{\text{cyclic}} 2\sin^2(\alpha/2) \\
&= \frac{1}{2} \sum_{\text{cyclic}} (1 - \cos\alpha) \\
&= \frac{3}{2} - \frac{1}{2} \sum_{\text{cyclic}} \cos\alpha \\
&= \frac{3}{2} - \frac{1}{2}\Big(1 + 4 \prod_{\text{cyclic}} \sin(\alpha/2)\Big) \\
&= 1 - 2 \prod_{\text{cyclic}} \sin(\alpha/2).
\end{aligned}$$

Thus we obtain

$$\sum_{\text{cyclic}} \sin^2(\alpha/2) + 2 \prod_{\text{cyclic}} \sin(\alpha/2) = 1.$$

Conversely, suppose there are non-negative real numbers x, y, z such that $x^2 + y^2 + z^2 + 2xyz = 1$. We show that there is a triangle with angles α, β, γ such that $x = \sin(\alpha/2)$, $y = \sin(\beta/2)$, $z = \sin(\gamma/2)$. In fact solving for x, we obtain

$$x = -yz + \sqrt{\left(1 - y^2\right)\left(1 - z^2\right)}.$$

Putting $y = \sin v$, $z = \sin w$, $0 < v, w < 90°$, we get

$$x = -\sin v \sin w + \cos v \cos w = \cos(v + w).$$

Let $\beta = 2v$, $\gamma = 2w$ and $\alpha = \pi - \beta - \gamma$. We observe that $1 > y^2 + z^2 = \sin^2(\beta/2) + \sin^2(\gamma/2)$ and hence $\cos^2(\beta/2) > \sin^2(\gamma/2)$. Since $0 < \beta/2, \gamma/2 < 90°$, we must have $\cos(\beta/2) > \sin(\gamma/2) = \cos(\pi/2 - \gamma/2)$. Thus it follows that $\beta/2 < \pi/2 - \gamma/2$. Hence $\beta + \gamma < \pi$. The definition of α shows that α, β, γ are

the angles of a triangle and $x = \cos(v + w) = \sin(\alpha/2)$, $y = \sin v = \sin(\beta/2)$, $z = \sin w = \sin(\gamma/2)$.

Coming back to the problem, since $a^2 + b^2 + c^2 + abc = 4$, there is a triangle with angles α, β, γ such that

$$a = 2\sin(\alpha/2), \quad b = 2\sin(\beta/2), \quad c = 2\sin(\gamma/2).$$

Then

$$\begin{aligned} ab &= 2\Big(\sin\alpha\sin\beta\tan(\alpha/2)\tan(\beta/2)\Big)^{1/2} \\ &\le \sin\alpha\tan(\beta/2) + \sin\beta\tan(\alpha/2) \\ &= \sin\alpha\cot\left(\frac{\alpha+\gamma}{2}\right) + \sin\beta\cot\left(\frac{\beta+\gamma}{2}\right). \end{aligned}$$

Similarly

$$\begin{aligned} bc &\le \sin\beta\cot\left(\frac{\beta+\alpha}{2}\right) + \sin\gamma\cot\left(\frac{\gamma+\alpha}{2}\right) \\ ca &\le \sin\gamma\cot\left(\frac{\gamma+\beta}{2}\right) + \sin\alpha\cot\left(\frac{\alpha+\beta}{2}\right). \end{aligned}$$

Adding these we obtain

$$\begin{aligned} ab + bc + ca &\le \sum_{\text{cyclic}} (\sin\alpha + \sin\beta)\cot\left(\frac{\alpha+\beta}{2}\right) \\ &= 2\sum_{\text{cyclic}} \cos\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) \\ &= 2\sum_{\text{cyclic}} \cos\alpha \\ &= 6 - 4\Big(\sum_{\text{cyclic}} \sin^2(\alpha/2)\Big) \\ &= 6 - \left(a^2 + b^2 + c^2\right) \\ &= 2 + abc. \end{aligned}$$

■

173. Suppose a, b, c are complex numbers such that $|a| = |b| = |c|$. Prove that

$$\left|\frac{ab}{a^2 - b^2}\right| + \left|\frac{bc}{b^2 - c^2}\right| + \left|\frac{ca}{c^2 - a^2}\right| \ge \sqrt{3}.$$

Solution: We may assume that a^2, b^2, c^2 are all distinct and $abc \neq 0$. Put $a = re^{i\alpha}$, $b = re^{i\beta}$, and $c = re^{i\gamma}$. Then the inequality reduces to

$$|\operatorname{cosec}(\alpha-\beta)| + |\operatorname{cosec}(\beta-\gamma)| + |\operatorname{cosec}(\gamma-\alpha)| \geq 2\sqrt{3}.$$

Now the AM-GM inequality gives

$$\sum_{\text{cyclic}} |\operatorname{cosec}(\alpha-\beta)| \geq 3\Big|\prod_{\text{cyclic}} \operatorname{cosec}(\alpha-\beta)\Big|^{1/3}.$$

Putting $A = \alpha-\beta$, $B = \beta-\gamma$, $C = \pi-(\alpha-\gamma) = \pi-(A+B)$, This takes the form

$$\sum_{\text{cyclic}} |\operatorname{cosec}(\alpha-\beta)| \geq 3\Big|\prod_{\text{cyclic}} \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C\Big|^{1/3},$$

and it is sufficient to prove that

$$\Big|\prod_{\text{cyclic}} \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C\Big| \geq \frac{8}{3\sqrt{3}}.$$

The function $f(t) = \log|\sin t|$ is a concave function and Jensen's inequality gives

$$f(A) + f(B) + f(C) \leq 3f\left(\frac{A+B+C}{3}\right) = 3\log\Big(\sin(\pi/3)\Big).$$

This implies that

$$\log\Big|\sin A \sin B \sin C\Big| \leq 3\log\left(\frac{\sqrt{3}}{2}\right).$$

Hence

$$\Big|\prod_{\text{cyclic}} \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C\Big| \geq \frac{8}{3\sqrt{3}},$$

is true. ■

174. Suppose x, y, z are non-negative real numbers such that $x^2+y^2+z^2 = 1$. Prove that

(a) $1 \leq \sum_{\text{cyclic}} \dfrac{x}{1-yz} \leq \dfrac{3\sqrt{3}}{2}$;

(b) $1 \leq \sum_{\text{cyclic}} \dfrac{x}{1+yz} \leq \sqrt{2}$.

Solution: (a):
Observe that $x/(1-yz) \geq x$, $y/(1-zx) \geq y$, and $z/(1-xy) \geq z$. Thus, we get

$$\sum_{\text{cyclic}} \frac{x}{1-yz} \geq x+y+z \geq x^2+y^2+z^2 = 1.$$

Equality holds if and only if $(x,y,z) = (1,0,0)$, $(0,1,0)$ or $(0,0,1)$.

On the other hand,

$$1-yz \geq 1-\left(\frac{y^2+z^2}{2}\right) = 1-\frac{(1-x^2)}{2} = \frac{1+x^2}{2},$$

which gives

$$\frac{x}{1-yz} \leq \frac{2x}{1+x^2}.$$

However, the inequality

$$\frac{2x}{1+x^2} \leq \frac{3\sqrt{3}}{8}\left(1+x^2\right)$$

holds for all real x. In fact

$$\begin{aligned}\left(1+x^2\right)^2 - \frac{16\sqrt{3}}{9}x &= \frac{1}{9}\left(9x^4+18x^2+9-16\sqrt{3}\,x\right)\\ &= \frac{1}{9}\left(\sqrt{3}\,x-1\right)^2\left(3x^2+2\sqrt{3}\,x+9\right) \geq 0.\end{aligned}$$

Equality holds if and only if $x = 1/\sqrt{3}$. It follows that

$$\sum_{\text{cyclic}} \frac{x}{1-yz} \leq \sum_{\text{cyclic}} \frac{2x}{1+x^2} \leq \frac{3\sqrt{3}}{8}\left(3+\sum_{\text{cyclic}} x^2\right) = \frac{3\sqrt{3}}{2}.$$

Equality holds if and only if $x=y=z=1/\sqrt{3}$.

(b):
Using $x^3-3x+2 = (x-1)^2(x+2) \geq 0$, the inequality

$$\begin{aligned}x+xyz &\leq x+\frac{1}{2}x\left(y^2+z^2\right)\\ &= x+\frac{1}{2}x\left(1-x^2\right)\\ &= \frac{1}{2}\left(3x-x^3\right) \leq 1\end{aligned}$$

is obtained. This implies

$$\sum_{\text{cyclic}} \frac{x}{1+yz} = \sum_{\text{cyclic}} \frac{x^2}{x+xyz} \geq \sum_{\text{cyclic}} x^2 = 1,$$

which proves the first inequality in (b).

To prove the right side inequality in (b), we may assume that $x \le z$ and $y \le z$. Thus

$$\sum_{\text{cyclic}} \frac{x}{1+yz} \le \frac{x+y+z}{1+xy},$$

and it is sufficient to prove that

$$x+y+z \le \sqrt{2}(1+xy),$$

under the condition that $x \le z$, $y \le z$ and $x^2+y^2+z^2=1$. This is equivalent to the inequality

$$\sqrt{1-x^2-y^2} \le \sqrt{2}(1+xy)-x-y.$$

Since both sides are non-negative, this is further equivalent to

$$1-x^2-y^2 \le \left(\sqrt{2}(1+xy)-x-y\right)^2.$$

Setting $x+y=\alpha$, $xy=\beta$, the inequality to be proved is

$$1-\alpha^2+2\beta \le \left(\sqrt{2}(1+\beta)-\alpha\right)^2.$$

This simplifies to

$$\left(\sqrt{2}\alpha-\beta-1\right)^2+\beta^2 \ge 0.$$

Since the left side is the sum of squares of two real numbers, the result follows. We also observe that the case of equality occurs only when $\beta=0$ and $\sqrt{2}\alpha-\beta-1=0$. This corresponds to $xy=0$ and $x+y=1/\sqrt{2}$. We conclude that

$$(x,y,z)=\left(0,1/\sqrt{2},1/\sqrt{2}\right) \quad \text{or} \quad \left(1/\sqrt{2},0,1/\sqrt{2}\right).$$

By symmetry, we get one more case of equality:

$$(x,y,z)=\left(1/\sqrt{2},1/\sqrt{2},0\right).$$

■

175. Let x,y,z be non-negative real numbers satisfying $x+y+z=1$. Prove that

$$xy^2+yz^2+zx^2 \ge xy+yz+zx-\frac{2}{9}.$$

Solution: Introduce new variables a, b, c by

$$x = a + \frac{1}{3}, \quad y = b + \frac{1}{3}, \quad z = c + \frac{1}{3}.$$

Then $a + b + c = 0$, $3a + 1 \geq 0$, $3b + 1 \geq 0$ and $3c + 1 \geq 0$. The inequality reduces to

$$\sum_{\text{cyclic}} ab \leq 3 \sum_{\text{cyclic}} ab^2 + \sum_{\text{cyclic}} a^2,$$

We may assume a to be the largest among a, b, c. Then $a > 0$. (If $a = 0$, we see that $b = c = 0$ and the inequality is, in fact, an equality.) Substituting $b = -a - c$, and after some simplification, an equivalent inequality is obtained:

$$a^3 + (3c + 1)a^2 + ca + (c^2 - c^3) \geq 0.$$

If $c \geq 0$, this is true in view of the condition $c < 1$ since $z \leq 1$. Hence we may assume that $c < 0$. For a fixed c, consider the function

$$f(a) = a^3 + (3c + 1)a^2 + ca + (c^2 - c^3).$$

Its derivative is $f'(a) = 3a^2 + 2(3c + 1) + c$. Setting the derivative equal to zero, two values of a are easily computed:

$$a = \frac{-(3c + 1) \pm \sqrt{9c^2 + 3c + 1}}{3}.$$

Since $a \geq 0$, the positive square root has to be chosen. Using $3c + 1 \geq 0$, it may be deduced that $f(a)$ has an extremum at

$$a_c = \frac{-(3c + 1) + \sqrt{9c^2 + 3c + 1}}{3}.$$

The second derivative $f''(a) = 2(3a + 3c + 1) > 0$, since $3c + 1 \geq 0$ and $a > 0$. Thus the function $f(a)$ has minimum at $a = a_c$. We show that $f(a_c) \geq 0$. An involved computation gives

$$f(a_c) = \frac{1}{27}\left\{27c^3 + 54c^2 + 9c + 2 - 2(9c^2 + 3c + 1)\sqrt{9c^2 + 3c + 1}\right\}.$$

Thus it is sufficient to prove that

$$27c^3 + 54c^2 + 9c + 2 - 2(9c^2 + 3c + 1)\sqrt{9c^2 + 3c + 1} \geq 0,$$

for $-(1/3) \leq c \leq 0$. Using $27c^3 + 54c^2 + 9c + 2 = 27c^2 + (9c^2 + 3c + 1)(3c + 2)$ and rationalisation process, the inequality reduces to

$$\frac{(9c^2 + 3c + 1)(-27c^2)}{3c + 2 + 2\sqrt{9c^2 + 3c + 1}} + 27c^2 \geq 0.$$

This may be seen to be equivalent to

$$2\sqrt{9c^2+3c+1} \geq 9c^2-1.$$

But observe that

$$9c^2-1=\big(3c+1\big)\big(3c-1\big)\leq 0,$$

whereas the left side is non-negative. ■

176. Let a, b, c, d be four positive real numbers such that $a+b+c+d=2$. Prove that

$$\sum_{\text{cyclic}} \frac{a^2}{\big(a^2+1\big)^2} \leq \frac{16}{25}.$$

Solution: From the symmetry, it may be assumed that $a \leq b \leq c \leq d$. Suppose $a \geq 1/8$. Then

$$\big(48a-4\big)\big(a^2+1\big)^2-125a^2=\big(2a-1\big)^2\big(12a^3+11a^2+32a-4\big)\geq 0.$$

Thus it follows that

$$\frac{a^2}{\big(a^2+1\big)^2} \leq \frac{48a-4}{125}.$$

Summing over a, b, c, d, we get

$$\sum_{\text{cyclic}} \frac{a^2}{\big(a^2+1\big)^2} \leq \frac{48}{125}(a+b+c+d)-\frac{16}{25}=\frac{16}{25}.$$

Suppose $a<1/8$. In this case, we observe that for $x \geq 0$,

$$\begin{aligned}&\big(540x+108\big)\big(x^2+1\big)^2-2197x^2\\ &\qquad\qquad\qquad\big(3x-2\big)^2\big(60x^3+92x^2+216x+27\big)\geq 0,\end{aligned}$$

which gives

$$\frac{x^2}{\big(x^2+1\big)^2} \leq \frac{540x+108}{2197}.$$

Thus it follows that

$$\begin{aligned}\frac{b^2}{\big(b^2+1\big)^2}+\frac{c^2}{\big(c^2+1\big)^2}+\frac{d^2}{\big(d^2+1\big)^2} &\leq \frac{540(b+c+d)+324}{2197}\\ &= \frac{108}{169}-\frac{540a}{2197},\end{aligned}$$

and

$$\frac{a^2}{\big(a^2+1\big)^2}<a^2<\frac{a}{8}<\frac{540a}{2197}.$$

We thus obtain

$$\sum \frac{a^2}{\left(a^2+1\right)^2} < \frac{108}{169} < \frac{16}{25}.$$

■

177. Prove that

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1$$

for all positive real numbers a, b and c.

Solution: We show that

$$\frac{a}{\sqrt{a^2+8bc}} \geq \frac{a^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}. \tag{177.1}$$

This is equivalent to the inequality

$$(a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}})^2 \geq a^{\frac{2}{3}}(a^2+8bc). \tag{177.2}$$

However we observe that

$$\begin{aligned}
\left(a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}\right)^2 - \left(a^{\frac{4}{3}}\right)^2 &= \left(b^{\frac{4}{3}}+c^{\frac{4}{3}}\right)\left(a^{\frac{4}{3}}+a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}\right) \\
&\geq 2b^{\frac{2}{3}}c^{\frac{2}{3}} \cdot 4a^{\frac{2}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}} \\
&= 8a^{\frac{2}{3}}bc,
\end{aligned}$$

using the AM-GM inequality. This in turn gives the inequality (177.2) and hence (177.1). Thus we obtain

$$\begin{aligned}
&\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \\
&\qquad \geq \frac{a^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}} + \frac{b^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}} + \frac{c^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}} = 1.
\end{aligned}$$

Alternate Solution.

Introducing

$$\frac{bc}{a^2} = x, \quad \frac{ca}{b^2} = y, \quad \frac{ab}{c^2} = z,$$

the inequality to be proved is

$$\frac{1}{\sqrt{1+8x}} + \frac{1}{\sqrt{1+8y}} + \frac{1}{\sqrt{1+8z}} \geq 1, \tag{177.3}$$

under the restriction $xyz = 1$. Let us put

$$\frac{1}{\sqrt{1+8x}} = p, \frac{1}{\sqrt{1+8y}} = q, \frac{1}{\sqrt{1+8z}} = r,$$

so that (177.3) reduces to

$$p + q + r \geq 1, \tag{177.4}$$

and $xyz = 1$ transforms to

$$(1-p^2)(1-q^2)(1-r^2) = 512p^2q^2r^2. \tag{177.5}$$

Suppose (177.4) is not true under the condition (177.5). Then we have $q+r < 1-p$ and $q+r+2p < 1+p$. Thus we obtain

$$1-p^2 > (q+r)(2p+q+r) \geq 2(qr)^{\frac{1}{2}} \cdot 4(p^2qr)^{\frac{1}{4}} = 8p^{\frac{1}{2}}(qr)^{\frac{3}{4}},$$

by an application of the AM-GM inequality. Similarly, we obtain

$$1-q^2 > 8q^{\frac{1}{2}}(rp)^{\frac{3}{4}}, 1-r^2 > 8r^{\frac{1}{2}}(pq)^{\frac{3}{4}}.$$

It follows that,

$$(1-p^2)(1-q^2)(1-r^2) > 8^3p^2q^2r^2,$$

contradicting (177.5). We conclude that (177.4) holds and hence the required inequality is true.

A generalisation.

We prove a more general inequality(due to Oleg Mushkarov, Bulgarian leader for IMO-2001). We show that

$$\frac{a}{\sqrt{a^2+\lambda bc}} + \frac{b}{\sqrt{b^2+\lambda ca}} + \frac{c}{\sqrt{c^2+\lambda ab}} \geq \frac{3}{\sqrt{1+\lambda}}, \tag{177.6}$$

for all positive real numbers a, b, c and for all $\lambda \geq 8$. As in the first solution, introduce the new variables x, y, z, and the inequality to be proved is

$$\frac{1}{\sqrt{1+\lambda x}} + \frac{1}{\sqrt{1+\lambda y}} + \frac{1}{\sqrt{1+\lambda z}} \geq \frac{3}{\sqrt{1+\lambda}}, \tag{177.7}$$

where $xyz = 1$ and $\lambda \geq 8$. Put

$$1+\lambda x = u^2, \quad 1+\lambda y = v^2, \quad 1+\lambda z = w^2,$$

so that (177.7) reduces to

$$uv + vw + wu \geq \frac{3uvw}{\sqrt{1+\lambda}}, \tag{177.8}$$

under the restriction $\lambda \geq 8$ and

$$(u^2-1)(v^2-1)(w^2-1) = \lambda^3. \tag{177.9}$$

We consider two possibilities:

Case 1. Suppose $uvw \leq (1+\lambda)^{3/2}$. Then, an application of the AM-GM inequality yields

$$uv + vw + wu \geq 3(uvw)^{\frac{2}{3}} \geq \frac{3uvw}{\sqrt{1+\lambda}}.$$

Case 2. Suppose $uvw \geq (1+\lambda)^{3/2}$. We write (177.9) in the form

$$\begin{aligned}\lambda^3 &= u^2v^2w^2 - (u^2v^2 + v^2w^2 + w^2u^2) + (u^2 + v^2 + w^2) - 1 \\ &= (uvw + u + v + w)^2 - (uv + vw + wu + 1)^2.\end{aligned}$$

Thus the inequality (177.8) reduces to

$$(uvw + u + v + w)^2 - \left(\frac{3uvw}{\sqrt{1+\lambda}} + 1\right)^2 \geq \lambda^3.$$

Since $u + v + w \geq 3(uvw)^{1/3}$, we see that by setting $uvw = X^3$, it is sufficient to prove that

$$(X^3 + 3X)^2 - \left(\frac{3X^3}{\sqrt{1+\lambda}} + 1\right)^2 \geq \lambda^3, \tag{177.10}$$

under the restriction $X \geq \sqrt{1+\lambda}$. We write (177.10) in the form

$$\left[X^3\left(1 + \frac{3}{\sqrt{1+\lambda}}\right) + 3X + 1\right]\left[X^3\left(1 - \frac{3}{\sqrt{1+\lambda}}\right) + 3X - 1\right] \geq \lambda^3.$$

Since $\lambda \geq 8$, we observe that $1 - \dfrac{3}{\sqrt{1+\lambda}} \geq 0$. Using $X \geq \sqrt{1+\lambda}$, we obtain

$$\begin{aligned}&X^3\left(1 - \frac{3}{\sqrt{1+\lambda}}\right) + 3X - 1 \\ &\qquad \geq (1+\lambda)^{\frac{3}{2}}\left(1 - \frac{3}{\sqrt{1+\lambda}}\right) + 3\sqrt{1+\lambda} - 1 \\ &\qquad = \sqrt{1+\lambda}(\lambda + 4) - (3\lambda + 4).\end{aligned}$$

Similarly, we show that

$$X^3\left(1 + \frac{3}{\sqrt{1+\lambda}}\right) + 3X + 1 \geq \sqrt{1+\lambda}(\lambda + 4) + (3\lambda + 4).$$

Combining these, we get

$$(X^3 + 3X)^2 - \left(\frac{3X^3}{\sqrt{1+\lambda}} + 1\right)^2 \geq (1+\lambda)(\lambda+4)^2 - (3\lambda+4)^2 = \lambda^3.$$

This proves (177.10) and hence the required inequality (177.6). ■

178. Prove that in a triangle ABC, the inequality

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\le\frac{9R^2}{4[ABC]},$$

holds.

Solution: We know that

$$r=(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{c}{2}.$$

Hence we get

$$\begin{aligned}
&\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\\
&\qquad= r\left(\frac{1}{s-a}+\frac{1}{s-a}+\frac{1}{s-a}\right)\\
&\qquad= \frac{(s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)}{[ABC]}.
\end{aligned}$$

Thus, we need to prove that

$$(s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)\le\frac{9R^2}{4}.$$

Using

$$(s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)=r(4R+r),$$

the inequality to be proved is

$$4r(4R+r)\le 9R^2.$$

This may be written as

$$(9R+2r)(R-2r)\ge 0,$$

which follows from $R\ge 2r$.

■

179. Prove that in a triangle with angles α,β,γ, the inequality

$$\sum_{\text{cyclic}}\sin\alpha\le\sqrt{\frac{15}{4}+\sum\cos(\alpha-\beta)}$$

holds.

Solution: Since both $\sum_{\text{cyclic}} \sin\alpha$ and $\sqrt{\frac{15}{4} + \sum_{\text{cyclic}} \cos(\alpha-\beta)}$ are positive, the inequality is equivalent to

$$\Big(\sum_{\text{cyclic}} \sin\alpha\Big)^2 \le \frac{15}{4} + \sum_{\text{cyclic}} \cos(\alpha-\beta).$$

Expansion and some cancellation leads to

$$3 - \sum_{\text{cyclic}} \cos^2\alpha \le \frac{15}{4} + \sum_{\text{cyclic}} \big(\cos\alpha\cos\beta - \sin\alpha\sin\beta\big).$$

This further reduces to

$$\sum_{\text{cyclic}} \big(\cos\alpha - 1/2\big)^2 \ge 0.$$

Equality holds if and only if $\cos\alpha = \cos\beta = \cos\gamma = 1/2$, which corresponds to the case of an equilateral triangle. ∎

180. If x, y are real numbers such that $x^3 + y^4 \le x^2 + y^3$, prove that $x^3 + y^3 \le 2$.

Solution: It may be seen that

$$3x^2 - 2x^3 - 1 = -(x-1)^2(2x+1),$$

and

$$4y^3 - 3y^4 - 1 = -\big(y-1\big)^2\big(3y^2 + 2y + 1\big).$$

Thus, we obtain

$$\begin{aligned} &3\big(x^2 + y^3 - x^3 - y^4\big) + x^3 + y^3 - 2 \\ &\qquad\qquad = -(x-1)^2(2x+1) - \big(y-1\big)^2\big(3y^2+2y+1\big). \end{aligned}$$

Tt follows that

$$\begin{aligned} 2 - x^3 - y^3 &= (x-1)^2(2x+1) + \big(y-1\big)^2\big(3y^2+2y+1\big) \\ &\quad +3\big(x^2 + y^3 - x^3 - y^4\big) \\ &\ge 0 \end{aligned}$$

∎

181. Let a, b, c be three positive real numbers. Prove that

$$\sum \frac{ab}{c(c+a)} \ge \sum \frac{a}{c+a},$$

where the sum is taken cyclically over a, b, c.

Solution: We introduce the variables x, y, z by

$$\frac{a}{b} = x, \quad \frac{b}{c} = y, \quad \frac{c}{a} = z.$$

Then $xyz = 1$ and the inequality takes the form

$$\frac{x-1}{y+1} + \frac{y-1}{z+1} + \frac{z-1}{x+1} \geq 0.$$

Expanding, this may be written in the form

$$x^2 + y^2 + z^2 - x - y - z + xy^2 + yz^2 + zx^2 - 3 \geq 0.$$

However, we have

$$\begin{aligned} x^2 + y^2 + z^2 &\geq \frac{1}{3}(x+y+z)^2 \\ &\geq (x+y+z)(xyz)^{1/3} = x + y + z, \end{aligned}$$

and

$$xy^2 + yz^2 + zx^2 \geq 3(x^3y^3z^3)^{1/3} = 3.$$

Hence the result follows. ■

182. Prove that for any real x, and real numbers a, b,

$$\big(\sin x + a\cos x\big)\big(\sin x + b\cos x\big) \leq 1 + \left(\frac{a+b}{2}\right)^2.$$

Solution: If $\cos x = 0$, then the inequality is clear. Suppose $\cos x \neq 0$. We may write the inequality in the form

$$\big(\tan x + a\big)\big(\tan x + b\big) \leq \sec^2 x + \left(\frac{a+b}{2}\right)^2 \sec^2 x.$$

Taking $t = \tan x$, this may be written as

$$t^2 + t(a+b) + ab \leq \left(1 + t^2\right)\left(1 + \left(\frac{a+b}{2}\right)^2\right).$$

This reduces to

$$\left(\frac{t(a+b)}{2} - 1\right)^2 + \left(\frac{a-b}{2}\right)^2 \geq 0.$$

Hence the inequality follows. ■

183. Let x, y be two real numbers, where y is non-negative and $y(y+1) \leq (x+1)^2$. Prove that $y(y-1) \leq x^2$.

Solution: If $0 \le y \le 1$, then $y(y-1) \le 0 \le x^2$. Suppose $y > 1$. If $x \le y - (1/2)$, then

$$\begin{aligned} y(y-1) = y(y+1) - 2y &\le (x+1)^2 - 2y \\ &= x^2 + 2x + 1 - 2y \\ &\le x^2. \end{aligned}$$

If $x \ge y - (1/2)$, we have

$$x^2 \ge y^2 - y + \frac{1}{4} > y(y-1).$$

Thus, $y(y-1) \le x^2$ holds in all cases. ■

184. Let x, y, z be positive real numbers. Prove that

$$\left(\frac{xy + yz + zx}{3}\right)^{1/2} \le \left(\frac{(x+y)(y+z)(z+x)}{8}\right)^{1/3}.$$

Solution: Put $x + y = c$, $y + z = a$, $z + x = b$. Then a, b, c are the sides of a triangle, and $x = s - a$, $y = s - b$, $z = s - c$, where $s = (a + b + c)/2$. The inequality may be written in the form

$$\left(\frac{1}{3}\Big((s-a)(s-b) + (s-b)(s-c) + (s-c)(s-a)\Big)\right)^{1/2} \le \left(\frac{abc}{8}\right)^{1/3}.$$

But, we also know

$$(s-a)(s-b) + (s-b)(s-c) + (s-c)(s-a) = r(4R + r), \quad abc = 4Rrs,$$

where R, r are the circum-radius and in-radius of the triangle whose sides are a, b, c. The inequality reduces to

$$4r(4R + r)^3 \le 27R^2s^2.$$

Using $3(ab + bc + ca) \le (a + b + c)^2$, we obtain

$$(ab + bc + ca) - s^2 \le \frac{1}{3}s^2.$$

Hence, we get

$$\begin{aligned} r(4R + r) &= (s-a)(s-b) + (s-b)(s-c) + (s-c)(s-a) \\ &= (ab + bc + ca) - s^2 \\ &\le \frac{1}{3}s^2. \end{aligned}$$

Thus, it suffices to prove that

$$4(4R + r)^2 \le 81R^2.$$

This equivalent to

$$(17R + 2r)(R - 2r) \geq 0,$$

which follows from $R \geq 2r$. ■

185. Let a, b, c be positive real numbers such that $abc = 1$. Show that

$$\sum \frac{a^9 + b^9}{a^6 + a^3b^3 + b^6} \geq 2,$$

where the sum is cyclical.

Solution: We observe that

$$\frac{a^9 + b^9}{a^6 + a^3b^3 + b^6} \geq \frac{1}{3}(a^3 + b^3).$$

This follows easily by cross-multiplication and rearrangement of terms. Hence

$$\begin{aligned}\sum_{\text{cyclic}} \frac{a^9 + b^9}{a^6 + a^3b^3 + b^6} &\geq \frac{2}{3}(a^3 + b^3 + c^3) \\ &\geq 2(abc) = 2.\end{aligned}$$

■

186. Let a, b, c be the sides of a triangle and set $x = 2(s - a)$, $y = 2(s - b)$, $z = 2(s - c)$, where s denotes the semi-perimeter. Prove that

$$abc(ab + bc + ca) \geq xyz(xy + yz + zx).$$

Solution: Note that

$$a = \frac{y + z}{2}, \quad b = \frac{z + x}{2}, \quad c = \frac{x + y}{2}.$$

Thus,

$$\begin{aligned}abc &= \left(\frac{y + z}{2}\right)\left(\frac{z + x}{2}\right)\left(\frac{x + y}{2}\right) \\ &\geq \sqrt{yz}\sqrt{zx}\sqrt{xy} = xyz.\end{aligned}$$

Moreover,

$$ab = \frac{1}{4}(xy + yz + zx + z^2),$$

so that

$$\begin{aligned}
\sum_{\text{cyclic}} ab &= \frac{1}{4}\Big(\sum_{\text{cyclic}} x^2 + 3\sum_{\text{cyclic}} xy\Big) \\
&\geq \frac{1}{4}\Big(\sum_{\text{cyclic}} xy + 3\sum_{\text{cyclic}} xy\Big) \\
&= \sum_{\text{cyclic}} xy.
\end{aligned}$$

Thus, it follows that

$$abc\sum_{\text{cyclic}} ab \geq xyz\sum_{\text{cyclic}} xy.$$

■

187. Let $a_1,a_2,a_3,\ldots,a_n$ $(n > 2)$ be positive real numbers and let s denote their sum. Let $0 < \beta \leq 1$ be a real number. Prove that

$$\sum_{k=1}^{n}\left(\frac{s-a_k}{a_k}\right)^{\beta} \geq (n-1)^{2\beta}\sum_{k=1}^{n}\left(\frac{a_k}{s-a_k}\right)^{\beta}.$$

When does equality hold?

Solution: Define

$$\begin{aligned}
A &= \sum_{k=1}^{n}\left(\frac{s-a_k}{a_k}\right)^{\beta} \\
&= (n-1)^{\beta}\sum_{k=1}^{n}\frac{1}{a_k^{\beta}}\left(\frac{1}{n-1}\sum_{j\neq k} a_j\right)^{\beta}.
\end{aligned}$$

The function $f(x) = x^{\beta}$ is concave for $x > 0$. Hence Jensen's inequality gives

$$\left(\frac{1}{n-1}\sum_{j\neq k} a_j\right)^{\beta} \geq \frac{1}{n-1}\sum_{j\neq k} a_j^{\beta},$$

with equality if and only if $a_1 = a_2 = \cdots = a_{k-1} = a_{k+1} = \cdots = a_n$. Thus, we have

$$\begin{aligned}
A &\geq (n-1)^{\beta}\sum_{k=1}^{n} a_k^{-\beta}\frac{1}{n-1}\Big(\sum_{j\neq k} a_j^{\beta}\Big) \\
&= (n-1)^{\beta}\sum_{j=1}^{n} a_j^{\beta}\frac{1}{n-1}\Big(\sum_{k\neq j} a_k^{-\beta}\Big).
\end{aligned}$$

The function $g(x) = x^{-\beta}$ is convex for $x > 0$. Hence we get

$$\begin{aligned} A &\geq (n-1)^{\beta} \sum_{j=1}^{n} a_j^{\beta} \left(\sum_{k \neq j} \frac{1}{n-1} a_k \right)^{-\beta} \\ &= (n-1)^{2\beta} \sum_{j=1}^{n} a_j^{\beta} \left(\sum_{k \neq j} a_k \right)^{-\beta} \\ &= (n-1)^{2\beta} \sum_{j=1}^{n} a_j^{\beta} (s - a_j)^{-\beta} \\ &= (n-1)^{2\beta} \sum_{j=1}^{n} \left(\frac{a_j}{s - a_j} \right)^{\beta}, \end{aligned}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$. ■

188. A point D on the segment BC of a triangle ABC is such that the in-radii of ABD and ACD are equal, say r_1. Similarly define r_2 and r_3. Prove that

(i) $2r_1 + 2\dfrac{\sqrt{s(s-a)}}{a} r = h_a$.

(ii) $2\bigl(r_1 + r_2 + r_3\bigr) + s \geq h_a + h_b + h_c$.

Solution: Let O_1 and O_2 be the in-centres of ABD and ACD respectively; and let s_1 and s_2 be their semi-perimeters. Then we have

$$[ABC] = [ABD] + [ACD] = r_1 s_1 + r_1 s_2 = r_1 \bigl(s_1 + s_2\bigr).$$

But

$$\begin{aligned} s_1 + s_2 &= \frac{1}{2}(AB + AD + BD + AD + DC + AC) \\ &= \frac{1}{2}(a + b + c) + AD = s + AD. \end{aligned}$$

Thus $[ABC] = r_1 \bigl(s + AD\bigr)$.

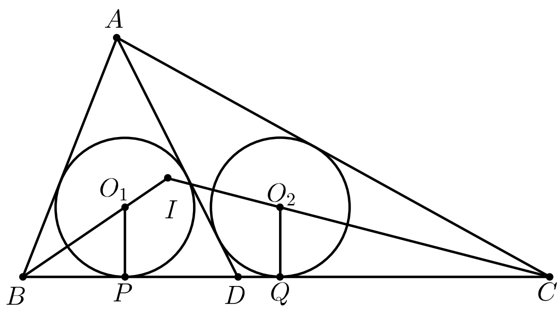

Fig. 6.24

Let P and Q denote the respective points of contact of the in-circles of ABD and ACD with BC. Then O_1PQO_2 is a rectangle. Hence

$$\begin{aligned} O_1O_2 = PQ = PD + DQ = s_1 - AB + s_2 - AC \\ = (s_1 + s_2) - (b + c) = s + AD - b - c. \end{aligned}$$

Let I be the in-centre of ABC. Then, IO_1O_2 is similar to IBC so that

$$\frac{O_1O_2}{BC} = \frac{r - r_1}{r}.$$

This gives $(s + AD - b - c)r = a(r - r_1)$. But $rs = [ABC] = r_1(s + AD)$. We get a quadratic equation:

$$ar_1^2 - 2rsr_1 + r^2 s = 0.$$

Solving, we get

$$r_1 = r\left(\frac{s \pm \sqrt{s(s-a)}}{a}\right).$$

Since $s + \sqrt{s(s-a)} > a$, we must take the negative sign. Thus, we obtain

$$\begin{aligned} r_1 &= r\left(\frac{s - \sqrt{s(s-a)}}{a}\right) \\ &= \frac{rs}{a} - \frac{r\sqrt{s(s-a)}}{a} \\ &= \frac{1}{2}h_a - \frac{r\sqrt{s(s-a)}}{a}. \end{aligned}$$

This reduces to

$$h_a = 2r_1 + \frac{2r\sqrt{s(s-a)}}{a},$$

which proves the first part.

From the first part we have

$$h_a + h_b + h_c = 2(r_1 + r_2 + r_3) + 2r\left\{ \frac{\sqrt{s(s-a)}}{a} + \frac{\sqrt{s(s-b)}}{b} + \frac{\sqrt{s(s-c)}}{c} \right\}.$$

Thus, we need to prove that

$$2\left\{ \frac{\sqrt{(s-a)}}{a} + \frac{\sqrt{(s-b)}}{b} + \frac{\sqrt{(s-c)}}{c} \right\} \leq \frac{\sqrt{s}}{r}.$$

But, note that $a = s - b + s - c$. Hence

$$\frac{\sqrt{(s-a)}}{a} = \frac{\sqrt{(s-a)}}{s-b+s-c} \leq \frac{\sqrt{(s-a)}}{2\sqrt{(s-b)}\sqrt{(s-c)}}.$$

Thus, we obtain

$$\begin{aligned}
&2\left\{ \frac{\sqrt{(s-a)}}{a} + \frac{\sqrt{(s-b)}}{b} + \frac{\sqrt{(s-c)}}{c} \right\} \\
&\quad \leq \quad \frac{\sqrt{(s-a)}}{\sqrt{(s-b)(s-c)}} + \frac{\sqrt{(s-b)}}{\sqrt{(s-c)(s-a)}} + \frac{\sqrt{(s-c)}}{\sqrt{(s-a)(s-b)}} \\
&\quad = \quad \frac{s}{\sqrt{(s-a)(s-b)(s-c)}} \\
&\quad = \quad \frac{s}{r\sqrt{s}} \\
&\quad = \quad \frac{\sqrt{s}}{r}.
\end{aligned}$$

■

189. For $n \geq 4$, let $a_1, a_2, a_3, \ldots, a_n$ be n positive real numbers such that $\sum_{j=1}^{n} a_j^2 = 1$. Show that

$$\frac{a_1}{a_2^2+1} + \frac{a_2}{a_3^2+1} + \cdots + \frac{a_n}{a_1^2+1} \geq \frac{4}{5}\Big(a_1\sqrt{a_1} + a_2\sqrt{a_2} + \cdots + a_n\sqrt{a_n}\Big)^2.$$

Solution: Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\sum_{j=1}^{n} \frac{a_j^3}{a_j^2 a_{j+1}^2 + a_j^2} &= \sum_{j=1}^{n} \frac{\left(a_j\sqrt{a_j}\right)^2}{\left(\sqrt{a_j^2 a_{j+1}^2 + a_j^2}\,\right)^2} \\
&\geq \frac{\left(\sum_{j=1}^{n} a_j\sqrt{a_j}\right)^2}{\sum_{j=1}^{n}\left(a_j^2 a_{j+1}^2 + a_j^2\right)},
\end{aligned}$$

where $a_{n+1} = a_1$. But

$$\sum_{j=1}^{n} \left(a_j^2 a_{j+1}^2 + a_j^2\right) = 1 + \sum_{j=1}^{n} a_j^2 a_{j+1}^2.$$

Thus, it is sufficient to prove that

$$\sum_{j=1}^{n} a_j^2 a_{j+1}^2 \leq \frac{1}{4},$$

under the restriction $\sum_{j=1}^{n} a_j^2 = 1$.

Let $x_1, x_2, x_3, \ldots, x_n$ be n positive real numbers such that $\sum_{j=1}^{n} x_j = 1$, where $n \geq 4$. If n is even, then

$$x_1x_2 + x_2x_3 + \cdots + x_nx_1 \leq \left(x_1 + x_3 + \cdots\right)\left(x_2 + x_4 \cdots\right) = y(1-y),$$

where $y = \sum_j x_{2j-1}$. But, $y(1-y) \leq 1/4$ since $0 \leq y \leq 1$. This takes care of even values of n. If n is odd, then $n \geq 5$, and we may assume $x_1 \geq x_2$ (since $x_j \geq x_{j+1}$ for some j). Thus

$$x_1x_2 + x_2x_3 + \cdots + x_nx_1 \leq x_1(x_2 + x_3) + (x_2 + x_3)x_4 + \cdots + x_nx_1 \leq \frac{1}{4},$$

using the previous argument to $n-1$ numbers $x_1, x_2 + x_3, x_4, \ldots x_n$. This completes the proof. ■

190. Does there exist an infinite sequence $\langle x_n \rangle$ of positive real numbers such that

$$x_{n+2} = \sqrt{x_{n+1}} - \sqrt{x_n},$$

for all $n \geq 2$.

Solution: The answer is **NO**. Suppose such a sequence $\langle x_n \rangle$ exists. Then $\sqrt{x_{n+1}} > \sqrt{x_n}$ and hence $\langle x_n \rangle$ is strictly increasing. Moreover,

$$\sqrt{x_{n+1}} - \sqrt{x_n} = x_{n+2} > x_{n+1} = \sqrt{x_n} - \sqrt{x_{n-1}},$$

so that

$$\sqrt{x_{n+1}} - \sqrt{x_n} > \sqrt{x_n} - \sqrt{x_{n-1}} > \cdots > \sqrt{x_2} - \sqrt{x_1}.$$

Thus, it follows that

$$\begin{aligned} \sqrt{x_{n+1}} - \sqrt{x_1} &= \sum_{j=1}^{n} \left(\sqrt{x_{j+1}} - \sqrt{x_j}\right) \\ &> n\left(\sqrt{x_2} - \sqrt{x_1}\right). \end{aligned}$$

This gives

$$\sqrt{x_{n+1}} > n\left(\sqrt{x_2} - \sqrt{x_1}\right) + \sqrt{x_1},$$

for all $n \geq 2$. Choose k large such that

$$k\left(\sqrt{x_2} - \sqrt{x_1}\right) + \sqrt{x_1} > \frac{1}{2}.$$

Then $\sqrt{x_{n+1}} > 1/2$ for all $n \geq k$. Thus if $n \geq k$, then

$$\begin{aligned} x_{n+3} &> x_{n+3} - x_{n+2} \\ &= \left(\sqrt{x_{n+3}} - \sqrt{x_{n+2}}\right)\left(\sqrt{x_{n+3}} + \sqrt{x_{n+2}}\right) \\ &> \left(\sqrt{x_{n+3}} - \sqrt{x_{n+2}}\right) \\ &= x_{n+4}. \end{aligned}$$

This contradicts the earlier observation that $\langle x_n \rangle$ is strictly increasing. ■

191. Let $a_1, a_2, a_3, \ldots, a_n$ be n positive real numbers and consider a permutation $b_1, b_2, b_3, \ldots, b_n$ of it. Prove that

$$\sum_{j=1}^{n} \frac{a_j^2}{b_j} \geq \sum_{j=1}^{n} a_j.$$

Solution: Using the Cauchy-Schwarz inequality, we get

$$\left(\sum_{j=1}^{n} a_j\right)^2 \leq \left(\sum_{j=1}^{n} \frac{a_j^2}{b_j}\right)\left(\sum_{j=1}^{n} b_j\right).$$

But $\sum_{j=1}^{n} b_j = \sum_{j=1}^{n} a_j$. Hence we get

$$\sum_{j=1}^{n} \frac{a_j^2}{b_j} \geq \sum_{j=1}^{n} a_j.$$

■

192. Let $a_1, a_2, a_3, \ldots, a_n$ and $b_1, b_2, b_3, \ldots, b_n$ be two sequences of positive real numbers such that $\sum_{j=1}^{n} a_j = \sum_{j=1}^{n} b_j = 1$. Prove that

$$\sum_{j=1}^{n} \frac{a_j^2}{a_j + b_j} \geq \frac{1}{2}.$$

Solution: The Cauchy-Schwarz inequality gives

$$\left(\sum_{j=1}^{n} a_j\right)^2 \leq \left(\sum_{j=1}^{n} \frac{a_j^2}{a_j + b_j}\right)\left(\sum_{j=1}^{n} a_j + b_j\right).$$

Using $\sum_{j=1}^{n} a_j = \sum_{j=1}^{n} b_j = 1$, we get the desired inequality. ■

193. Let x, y, z be positive real numbers. Prove that

$$\frac{y^2-x^2}{z+x}+\frac{z^2-y^2}{x+y}+\frac{x^2-z^2}{y+z}\geq 0.$$

Solution: Introducing $a=y+z$, $b=z+x$, $c=x+y$, we see that a,b,c are the sides of a triangle, and $x=s-a$, $y=s-b$, $z=s-c$, where s is the semi-perimeter of the triangle. We get

$$y-x=a-b,\quad \frac{y^2-x^2}{z+x}=\frac{(a-b)c}{b}.$$

We hence obtain

$$\frac{y^2-x^2}{z+x}+\frac{z^2-y^2}{x+y}+\frac{x^2-z^2}{y+z}=\frac{ac}{b}+\frac{ba}{c}+\frac{cb}{a}-(a+b+c).$$

Thus, we have to prove

$$a^2b^2+b^2c^2+c^2a^2\geq abc(a+b+c).$$

This follows from Muirhead's inequality, since $(2,1,1)\prec(2,2,0)$. ■

194. Find the greatest real value of k such that for every triple (a,b,c) of positive real numbers, the inequality

$$\left(a^2-bc\right)^2>k\left(b^2-ca\right)\left(c^2-ab\right)$$

holds.

Solution: We show that the largest value of k is 4. In fact, we show that

$$\left(a^2-bc\right)^2>4\left(b^2-ca\right)\left(c^2-ab\right),$$

for every positive triple (a,b,c), and for any $l>4$, it is possible to choose positive a,b,c such that

$$\left(a^2-bc\right)^2<l\left(b^2-ca\right)\left(c^2-ab\right).$$

Let us take $\lambda=a-\sqrt{bc}$. Then

$$\begin{aligned}
&\left(a^2-bc\right)^2-4\left(b^2-ca\right)\left(c^2-ab\right)\\
&= a^4-6bca^2+4\left(b^3+c^3\right)a-3b^2c^2\\
&= \left(\lambda+\sqrt{bc}\right)^4-6bc\left(\lambda+\sqrt{bc}\right)^2+4\left(b^3+c^3\right)\left(\lambda+\sqrt{bc}\right)-3b^2c^2\\
&= \lambda^4+4\sqrt{bc}\lambda^3+4\left(b\sqrt{b}-c\sqrt{c}\right)^2\lambda+4\sqrt{bc}\left(b\sqrt{b}-c\sqrt{c}\right)^2\\
&> 0.
\end{aligned}$$

On the other hand, suppose $l > 4$. Choose $\epsilon > 0$ such that $5\epsilon < l - 4$. Take $a = 1 + \epsilon$, $b = c = 1$. Then

$$a^2 - bc = (1+\epsilon)^2 - 1 = 2\epsilon + \epsilon^2 > 0.$$

Hence

$$\begin{aligned}\left(a^2 - bc\right)^2 = \left(2\epsilon + \epsilon^2\right)^2 &= \epsilon^2\left(\epsilon^2 + 4\epsilon + 4\right) \\ &< \epsilon^2\left(5\epsilon + 4\right) \\ &< l\epsilon^2 = l\left(-\epsilon\right)\left(-\epsilon\right) \\ &= l\left(b^2 - ca\right)\left(c^2 - ab\right).\end{aligned}$$

Thus, $k = 4$ is the largest constant for which the given inequality holds for all choices of the positive reals a, b, c. ■

195. Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that

$$\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+cd}{1+c} + \frac{1+da}{1+d} \geq 4.$$

Solution: Using $cd = 1/ab$ and $da = 1/bc$, the inequality is equivalent to

$$\frac{1+ab}{1+a} + \frac{1+ab}{ab+abc} + \frac{1+bc}{1+b} + \frac{1+bc}{bc+bcd} \geq 4.$$

The AM-GM inequality gives

$$\begin{aligned}\frac{1+ab}{1+a} + \frac{1+ab}{ab+abc} &\geq \frac{4(1+ab)}{1+a+ab+abc}, \\ \frac{1+bc}{1+b} + \frac{1+bc}{bc+bcd} &\geq \frac{4(1+bc)}{1+b+bc+bcd} = \frac{4a(1+bc)}{a+ab+abc+1}.\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{1+ab}{1+a} + \frac{1+ab}{ab+abc} + \frac{1+bc}{1+b} + \frac{1+bc}{bc+bcd} \\ &\qquad\qquad \geq \frac{4(1+ab)}{1+a+ab+abc} + \frac{4a(1+bc)}{a+ab+abc+1} = 4.\end{aligned}$$

■

196. Let a, b, c be the sides of a triangle such that $a + b + c = 1$, and let $n \geq 2$ be a natural number. Prove that

$$\left(a^n + b^n\right)^{1/n} + \left(b^n + c^n\right)^{1/n} + \left(c^n + a^n\right)^{1/n} < 1 + \frac{2^{1/n}}{2}.$$

Solution: Taking $x = s - a$, $y = s - b$, $z = s - c$, we see that x, y, z are positive reals such that $x + y + z = 1/2$ and $a = y + z$, $b = z + x$, $c = x + y$. Using Minkowski's inequality, we get

$$\begin{aligned}\left(a^n + b^n\right)^{1/n} &= \left((y+z)^n + (x+z)^n\right)^{1/n} \\ &\le \left(y^n + x^n\right)^{1/n} + \left(2z^n\right)^{1/n} \\ &< y + x + 2^{1/n}\, z \\ &= c + 2^{1/n}\, z.\end{aligned}$$

Similarly, we obtain

$$\left(b^n + c^n\right)^{1/n} < a + 2^{1/n}\, x, \quad \left(c^n + a^n\right)^{1/n} < b + 2^{1/n}\, y.$$

Adding, we get

$$\begin{aligned}&\left(a^n + b^n\right)^{1/n} + \left(b^n + c^n\right)^{1/n} + \left(c^n + a^n\right)^{1/n} \\ &\qquad < a + b + c + 2^{1/n}(x + y + z) = 1 + \frac{2^{1/n}}{2}.\end{aligned}$$

■

197. Let a, b, c, d be positive real numbers. Prove that

$$\sum_{\text{cyclic}} \frac{a}{b + 2c + d} \ge 1.$$

Solution: For any positive real numbers u, v, x, y, we have

$$\frac{u}{x} + \frac{v}{y} = \frac{uy + vx}{xy} \ge \frac{4(uy + vx)}{(x + y)^2}.$$

Thus, we get

$$\begin{aligned}\frac{a}{b + 2c + d} + \frac{c}{d + 2a + b} &\ge \frac{2a^2 + 2c^2 + ab + bc + cd + da}{(a + b + c + d)^2}, \\ \frac{b}{c + 2d + a} + \frac{d}{a + 2b + c} &\ge \frac{2b^2 + 2d^2 + ab + bc + cd + da}{(a + b + c + d)^2}.\end{aligned}$$

Adding, we get

$$\sum_{\text{cyclic}} \frac{a}{b+2c+d} \geq \frac{2\left(\sum_{\text{cyclic}} a^2\right) + 2(ab+bc+cd+da)}{(a+b+c+d)^2}$$

$$= \frac{\left(\sum_{\text{cyclic}} a\right)^2 + \left(\sum_{\text{cyclic}} a^2\right) - 2ac - 2bd}{(a+b+c+d)^2},$$

$$= 1 + \frac{(a-c)^2 + (b-d)^2}{(a+b+c+d)^2} \geq 1.$$

Hence the result follows. ∎

198. Let a, b, c be positive real numbers such that $(a+b)(b+c)(c+a) = 1$. Prove that

$$ab + bc + ca \leq \frac{3}{4}.$$

Solution: We have

$$\begin{aligned} 1 &= (a+b)(b+c)(c+a) \\ &= a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 + 2abc \\ &= (a+b+c)(ab+bc+ca) - abc. \end{aligned}$$

Thus, we obtain

$$ab + bc + ca = \frac{1+abc}{a+b+c}.$$

Using the AM-GM inequality, we have

$$\begin{aligned} a+b+c &= \frac{1}{2}\Big((a+b) + (b+c) + (c+a)\Big) \\ &\geq \frac{3}{2}\Big((a+b)(b+c)(c+a)\Big)^{1/3} = \frac{3}{2}. \end{aligned}$$

Similarly,

$$1 = (a+b)(b+c)(c+a) \geq 8\sqrt{ab}\,\sqrt{bc}\,\sqrt{ca} = 8abc.$$

This shows $abc \leq 1/8$. Finally,

$$ab + bc + ca = \frac{1+abc}{a+b+c} \leq \frac{1+1/8}{3/2} = \frac{3}{4}.$$

∎

199. Let ABC be a right-angled triangle with $A = 90°$. Let AD be the bisector of angle A, and I_a be the ex-centre opposite to A. Prove that

$$\frac{AD}{DI_a} \leq \sqrt{2} - 1.$$

Solution: Let h_a be the altitude from A on BC and r_a be the ex-radius corresponding to the vertex A. Then it is easy to see that

$$\frac{AD}{DI_a} = \frac{h_a}{r_a}.$$

But, we observe that

$$h_a = \frac{AB \cdot AC}{BC}.$$

Since ABC is right-angled at A, we know that $r_a = (a+b+c)/2$. Thus, we get

$$\frac{h_a}{r_a} = \frac{2bc}{a(a+b+c)}.$$

However

$$a(b+c) = (b+c)\sqrt{b^2+c^2} \geq \frac{(b+c)^2}{\sqrt{2}} \geq 2\sqrt{2}bc,$$

and

$$a^2 = b^2 + c^2 \geq 2bc.$$

Hence, we get

$$\frac{AD}{DI_a} \leq \frac{2bc}{a^2 + a(b+c)} \leq \frac{2bc}{2\sqrt{2}bc + 2bc} = \sqrt{2} - 1.$$

■

200. Let x, y, z be non-negative real numbers such that $x + y + z = 1$. Prove that

$$x^2 + y^2 + z^2 + 18xyz \leq 1.$$

Solution: Note that

$$x^2 + y^2 + z^2 = (x+y+z)^2 - 2(xy + yz + zx) = 1 - 2(xy + yz + zx).$$

Hence it is sufficient to prove that

$$9xyz \leq xy + yz + zx.$$

Introducing

$$a = y + z, \quad b = z + x, \quad c = x + y,$$

we see that a, b, c are the sides of a triangle, and

$$x = s - a, \quad y = s - b, \quad s = 1 - c,$$

where $s = (a+b+c)/2 = 1$. The inequality is

$$9(s-a)(s-b)(s-c) \leq (s-a)(s-b) + (s-b)(s-c) + (s-c)(s-a).$$

We use

$$\sum_{\text{cyclic}}(s-a)(s-b)=r(4R+r),\ (s-a)(s-b)(s-c)=r^2s,$$

to get

$$9r^2s\le r(4R+r).$$

But $s=1$ and this reduces to $2r\le R$, which is Euler's inequality. The result follows.

Alternate Solution: We use the homogenisation technique. We may write the inequality in the form

$$\big(x+y+z\big)\big(xy+yz+zx\big)+18xyz\le\big(x+y+z\big)^3.$$

Simplification gives

$$2\bigg(\sum_{\text{cyclic}}x^2y+\sum_{\text{cyclic}}xy^2-6xyz\bigg)\ge 0.$$

This follows from the AM-GM inequality.

$\blacksquare$

201. Let ABC be a triangle with circum-circle Γ, and G be its centroid. Extend AG, BG, CG to meet Γ in D, E, F respectively. Prove that

$$AG+BG+CG\le GD+GE+GF.$$

Solution: Let K,L,M be respectively the mid-points of BC, CA, AB. We have

$$AK\cdot KD=BK\cdot KC=\frac{a^2}{4}.$$

We thus obtain

$$KD=\frac{a^2}{4m_a}.$$

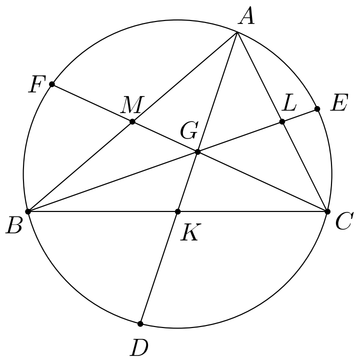

Fig. 6.25

This implies that

$$\begin{aligned} GD = GK + KD &= \frac{m_a}{3} + \frac{a^2}{4m_a} \\ &= \frac{4m_a^2 + 3a^2}{12m_a} \\ &= \frac{a^2+b^2+c^2}{6m_a}. \end{aligned}$$

Similarly, we obtain

$$GE = \frac{\sum_{\text{cyclic}} a^2}{6m_b}, \quad GF = \frac{\sum_{\text{cyclic}} a^2}{6m_c}.$$

Adding these, we get

$$GD + GE + GF = \left(\frac{a^2+b^2+c^2}{6}\right)\left(\frac{1}{m_a} + \frac{1}{m_c} + \frac{1}{m_c}\right).$$

But, we know that

$$4\big(m_a^2 + m_b^2 + m_c^2\big) = 3\big(a^2+b^2+c^2\big).$$

Thus, it follows that

$$GD + GE + GF = \frac{2}{9}\left(m_a^2 + m_b^2 + m_c^2\right)\left(\frac{1}{m_a} + \frac{1}{m_c} + \frac{1}{m_c}\right).$$

Hence, it is sufficient to prove that

$$\frac{1}{3}\left(\sum_{\text{cyclic}} m_a^2\right)\left(\sum_{\text{cyclic}} \frac{1}{m_a}\right) \geq \sum m_a,$$

which is a consequence of Chebyshev's inequality. ∎

202. Prove with usual notation that in a triangle ABC, the inequality

$$\big(a+b+c\big)\big(h_a+h_b+h_c\big) \geq 18\Delta.$$

Solution: If $a \leq b \leq c$, we see that $h_a \geq h_b \geq h_c$. Hence Chebyshev's inequality gives

$$\begin{aligned}\big(a+b+c\big)\big(h_a+h_b+h_c\big) &\geq 3\big(ah_a+bh_b+ch_c\big)\\ &= 3(6\Delta) = 18\Delta.\end{aligned}$$

■

203. Let a, b, c be three positive real numbers such that $ab+bc+ca = 1$. Prove that

$$\left(\frac{1}{a}+6b\right)^{1/3} + \left(\frac{1}{b}+6c\right)^{1/3} + \left(\frac{1}{c}+6a\right)^{1/3} \leq \frac{1}{abc}.$$

Solution: We use the concavity of $f(x) = x^{1/3}$ to get

$$\begin{aligned}&\left(\frac{1}{a}+6b\right)^{1/3} + \left(\frac{1}{b}+6c\right)^{1/3} + \left(\frac{1}{c}+6a\right)^{1/3}\\ &\qquad\qquad \leq \frac{3}{\sqrt[3]{3}}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+6(a+b+c)\right)^{1/3}.\end{aligned}$$

But, $ab+bc+ca = 1$ gives

$$3abc(a+b+c) \leq (ab+bc+ca)^2 = 1,$$

so that $(a+b+c) \leq 1/3abc$. Moreover

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{ab+bc+ca}{abc} = \frac{1}{abc}.$$

Hence, we obtain

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+6(a+b+c) \leq \frac{3}{abc}.$$

Thus we get

$$\left(\frac{1}{a}+6b\right)^{1/3} + \left(\frac{1}{b}+6c\right)^{1/3} + \left(\frac{1}{c}+6a\right)^{1/3} \leq \frac{3}{(abc)^{1/3}}.$$

Again, note that

$$1 = ab+bc+ca \geq 3(abc)^{2/3},$$

so that

$$\frac{3}{(abc)^{1/3}} \leq \frac{1}{(abc)^{2/3}} \cdot \frac{1}{(abc)^{1/3}} = \frac{1}{abc}.$$

This completes the proof. ■

204. Let ABC be a triangle. Show that there exists a point D on AB such that $CD^2 = AD \cdot BD$ if and only if $\sqrt{\sin A \sin B} \le \sin(C/2)$.

Solution: For a point D on AB, let $\angle ACD = \gamma_1$, $\angle BCD = \gamma_2$. We have

$$\frac{CD^2}{AD \cdot BD} = \frac{\sin A}{\sin \gamma_1} \cdot \frac{\sin B}{\sin \gamma_2}.$$

But, we have

$$\begin{aligned} \sin \gamma_1 \sin \gamma_2 &= \frac{1}{2}\Big\{ \cos(\gamma_1 - \gamma_2) - \cos C \Big\} \\ &\le \frac{1 - \cos C}{2} = \sin^2(C/2). \end{aligned}$$

Thus, it follows that

$$\frac{CD^2}{AD \cdot BD} \ge \frac{\sin A \sin B}{\sin^2(C/2)},$$

for any point D on AB. Hence, the range of values of $CD^2/AD \cdot BD$ is the interval $[\sin A \sin B / \sin^2(C/2), \infty)$. It follows that $CD^2 = AD \cdot BD$ if and only if 1 is in the range of values of $CD^2/AD \cdot BD$. This is equivalent to $\sqrt{\sin A \sin B} \le \sin(C/2)$. ■

205. Let a_1,a_2,a_3,...,a_n be $n > 1$ positive real numbers. For each k, $1 \le k \le n$, let $A_k = (a_1 + a_2 + \cdots + a_k)/k$. Let $g_n = (a_1 a_2 \cdots a_n)^{1/n}$ and $G_n = (A_1 A_2 \cdots A_n)^{1/n}$. Prove that

$$n\left(\frac{G_n}{A_n}\right)^{1/n} + \frac{g_n}{G_n} \le n + 1.$$

Find the cases of equality.

Solution: Put $A_0 = 0$ and define $c_k = A_{k-1}/A_k$ for $1 \le k \le n$. We observe that

$$\frac{a_k}{A_k} = \frac{kA_k - (k-1)A_{k-1}}{A_k} = k - (k-1)c_k.$$

We may write

$$\begin{aligned} \left(\frac{G_n}{A_n}\right)^{1/n} &= \left(c_2 c_3^2 \cdots c_n^{n-1}\right)^{1/n^2}, \\ \frac{g_n}{G_n} &= \left(\prod_{k=1}^{n} \left(k - (k-1)c_k\right)\right)^{1/n}. \end{aligned}$$

Using the AM-GM inequality, we have

$$\begin{aligned} n\left(1^{n(n+1)/2} c_2 c_3^2 \cdots c_n^{n-1}\right)^{1/n^2} &\leq \frac{1}{n}\left(\frac{n(n+1)}{2} + \sum_{k=2}^{n}(k-1)c_k\right) \\ &= \frac{n+1}{2} + \frac{1}{n}\sum_{k=1}^{n}(k-1)c_k. \end{aligned}$$

Similarly, the AM-GM inequality also gives

$$\left(\prod_{k=1}^{n}\left(k-(k-1)c_k\right)\right)^{1/n} \leq \frac{n+1}{2} - \frac{1}{n}\sum_{k=1}^{n}(k-1)c_k.$$

Adding these two, the required inequality is obtained. ■

206. Let x, y, z be real numbers in the interval $[0,1]$. Prove that

$$3\left(x^2y^2 + y^2z^2 + z^2x^2\right) - 2xyz\left(x+y+z\right) \leq 3.$$

Solution: We observe that

$$\begin{aligned} &3\left(x^2y^2 + y^2z^2 + z^2x^2\right) - 2xyz\left(x+y+z\right) \\ &\qquad = \left(xy+yz-zx\right)^2 + \left(-xy+yz+zx\right)^2 + \left(xy-yz+zx\right)^2. \end{aligned}$$

Note that

$$xy + yz - zx \geq -zx \geq -1,$$

and

$$\begin{aligned} xy + yz - zx &= y(z+x) - zx \\ &\leq x + z - zx \\ &= (x-1)(1-z) + 1 \leq 1. \end{aligned}$$

Thus, we obtain

$$\left(xy+yz-zx\right)^2 \leq 1.$$

Similarly, we may bound the remaining two terms also by 1 each. The inequality follows. ■

207. Let x, y, z be non-negative real numbers such that $x+y+z=1$. Prove that

$$7(xy+yz+zx) \leq 2 + 9xyz.$$

Solution: We use the technique of homogenisation to put it in the form

$$7(xy+yz+zx)(x+y+z) \leq 2(x+y+z)^3 + 9xyz.$$

This reduces to

$$7\Big(\sum_{\text{cyclic}} x^2y + \sum_{\text{cyclic}} xy^2 + 3xyz\Big)$$
$$\leq 2\Big(\sum_{\text{cyclic}} x^3 + 3\sum_{\text{cyclic}} x^2y + 3\sum_{\text{cyclic}} xy^2 + 6xyz\Big) + 9xyz.$$

Equivalently, we have to prove

$$2\Big(\sum_{\text{cyclic}} x^3\Big) \geq \sum_{\text{cyclic}} x^2y + \sum_{\text{cyclic}} xy^2.$$

But, this is a direct consequence Muirhead's theorem. ■

208. Let x, y, z be real numbers in the interval $[0, 1]$. Prove that

$$\frac{x}{yz+1} + \frac{y}{zx+1} + \frac{z}{xy+1} \leq 2.$$

Solution: First, observe that xyz cannot exceed each of xy, yz, zx. Thus, we see that

$$\begin{aligned}\frac{x}{yz+1} + \frac{y}{zx+1} + \frac{z}{xy+1} &\leq \frac{x}{xyz+1} + \frac{y}{yzx+1} + \frac{z}{zxy+1}\\ &= \frac{x+y+z}{xyz+1}.\end{aligned}$$

Hence, it is sufficient to prove that $(x+y+z) \leq 2(xyz+1)$. Since x, y are in $[0, 1]$, we have $(1-x)(1-y) \geq 0$. This reduces to $1 + xy \geq x + y$. Since $z \geq 0$, we get $z + xyz \geq zx + yz$. Similarly, we get $x + xyz \geq xy + zx$ and $y + xyz \geq xy + yz$. Adding these, we get

$$(x+y+z) + 3xyz \geq 2(xy+yz+zx).$$

Hence, it follows that

$$\begin{aligned}2xyz + 2 &\geq 2xyz + 2 + 2(xy+yz+zx) - 3xyz - (x+y+z)\\ &= 2(1-x)(1-y)(1-z) + xyz + (x+y+z)\\ &\geq x+y+z.\end{aligned}$$

This proves the inequality. ■

209. Let a, b, c, d be positive reals such that $a^3 + b^3 + 3ab = c + d = 1$. Prove that

$$\left(a + \frac{1}{a}\right)^3 + \left(b + \frac{1}{b}\right)^3 + \left(c + \frac{1}{c}\right)^3 + \left(d + \frac{1}{d}\right)^3 \geq 40.$$

Solution: We may write the relation in a, b in the form

$$a^3 + b^3 + (-1)^3 - 3(a)(b)(-1) = 0.$$

This is equivalent to

$$\big(a + b - 1\big)\big((a-b)^2 + (a+1)^2 + (b+1)^2\big) = 0.$$

Thus, either $a + b = 1$ or $a = b = -1$. Since a, b are positive, we conclude that $a + b = 1 = c + d$.

If x, y are positive real numbers, then

$$x^3 + y^3 \geq xy(x + y).$$

(In fact, this is equivalent to $(x-y)^2 \geq 0$.) We therefore get

$$\begin{aligned}\left(a+\frac{1}{a}\right)^3+\left(b+\frac{1}{b}\right)^3 &\geq \left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(a+\frac{1}{a}+b+\frac{1}{b}\right)\\ &= \geq 4\left(1+\frac{1}{ab}\right)\\ &\geq 4(1+4)=20;\end{aligned}$$

we have used $a + 1/a \geq 2$ and $ab \geq \big((a+b)/2\big)^2 = 1/4$. Similarly, using $c + d = 1$, we obtain

$$\left(c+\frac{1}{c}\right)^3+\left(d+\frac{1}{d}\right)^3 \geq 20.$$

Combining these two, we get the required inequality. ■

210. Let x, y, z be positive real numbers such that $x + y + z = xyz$. Prove that

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z^2}} \leq \frac{3}{2}.$$

Solution: As in the example 2.23 on page 75, we use trigonometric substitutions: $x = \tan A$, $y = \tan B$ and $z = \tan C$. Then A, B, C are the angles of a triangle and the inequality reduces to

$$\cos A + \cos B + \cos C \leq \frac{3}{2}.$$

This follows from **3.4.9** on page 119. ■

211. Let P be an interior point of a triangle ABC whose sides are a, b, c. Let $R_1 = PA$, $R_2 = PB$, and $R_3 = PC$. Prove that

$$\big(R_1R_2 + R_2R_3 + R_3R_1\big)\big(R_1 + R_2 + R_3\big) \geq a^2R_1 + b^2R_2 + c^2R_3.$$

When does equality hold?

Solution: The inequality may be written in the form

$$3 + \sum_{\text{cyclic}} \frac{R_1^2 + R_2^2 - c^2}{R_1 R_2} \geq 0.$$

Let $\angle BPC = \alpha$, $\angle CPA = \beta$ and $\angle APB = \gamma$. Then the cosine rule applied to triangles BPC, CPA and APB reduce the inequality to the form

$$\cos\alpha + \cos\beta + \cos\gamma \geq -\frac{3}{2}.$$

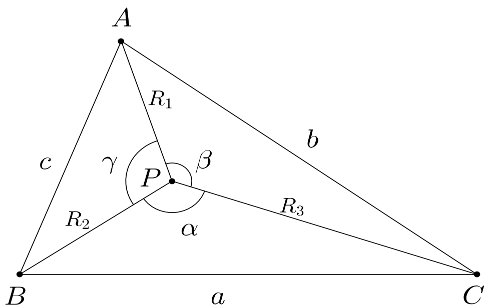

Fig. 6.26

We may assume $\alpha \leq \beta \leq \gamma$. Since $\gamma < \pi$, it follows that $\beta > \pi/2$. Now the function $f(x) = \cos x$ is convex in $[\pi/2, \pi]$. Hence Jensen's inequality gives

$$\cos\beta + \cos\gamma \geq 2\cos\left(\frac{\beta+\gamma}{2}\right) = -2\cos\left(\frac{\alpha}{2}\right).$$

Taking $\lambda = \cos(\alpha/2)$, we see that it is sufficient to prove that

$$2\lambda^2 - 1 - 2\lambda \geq -\frac{3}{2}.$$

This is equivalent to $(2\lambda - 1)^2 \geq 0$, which is obviously true. Equality holds if and only if $\beta = \gamma$ and $\lambda = 1/2$. This gives $\alpha = 120°$ and α being the least angle, we get $\alpha = \beta = \gamma = 120°$. Thus P is the Fermat's point of ABC. ■

212. Let t_1,t_2,t_3,...,t_n be positive real numbers such that

$$n^2 + 1 > \left(t_1 + t_2 + \cdots + t_n\right)\left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n}\right),$$

where $n \geq 3$ is an integer. Show that for each triple (j, k, l) with $1 \leq j < k < l \leq n$, there is a triangle with sides t_j, t_k, t_l.

Solution: We may assume that $t_1 \leq t_2 \leq \cdots \leq t_n$. Thus, it is enough to prove that $t_n < t_1 + t_2$. The given condition may be written as

$$\begin{aligned} n^2+1 &> n + \sum_{1\leq j<k\leq n} \left(\frac{t_j}{t_k} + \frac{t_k}{t_j}\right) \\ &= n + t_n\left(\frac{1}{t_1} + \frac{1}{t_2}\right) + \frac{1}{t_n}\big(t_1+t_2\big) \\ &\qquad + \sum_{\substack{1\leq j<k\leq n \\ (j,k)\neq(1,n),(2,n)}} \left(\frac{t_j}{t_k} + \frac{t_k}{t_j}\right). \end{aligned}$$

Using $\dfrac{t_j}{t_k} + \dfrac{t_k}{t_j} \geq 2$, it follows that

$$n^2+1 > n + t_n\left(\frac{1}{t_1} + \frac{1}{t_2}\right) + \frac{1}{t_n}\big(t_1+t_2\big) + 2\left\{\binom{n}{2} - 2\right\}.$$

This reduces to

$$t_n\left(\frac{1}{t_1} + \frac{1}{t_2}\right) + \frac{1}{t_n}\big(t_1+t_2\big) - 5 < 0.$$

We observe that

$$\frac{4t_n}{t_1+t_2} \leq t_n\left(\frac{1}{t_1} + \frac{1}{t_2}\right).$$

Thus, we get

$$\frac{4}{\lambda} + \lambda - 5 < 0,$$

where

$$\lambda = \frac{t_1+t_2}{t_n}.$$

This shows that $(\lambda-1)(\lambda-4) < 0$. We conclude that $1 < \lambda < 4$. Thus, we obtain

$$t_n < t_1 + t_2 < 4t_n,$$

and this completes the proof. ∎

213. Let x, y, z be non-negative real numbers. Prove that

$$x^3 + y^3 + z^3 \geq x^2\sqrt{yz} + y^2\sqrt{zx} + z^2\sqrt{xy}.$$

Solution: Put $x = a^2$, $y = b^2$, $z = c^2$. The inequality takes the form

$$a^6 + b^6 + c^6 \geq a^4bc + ab^4c + abc^4.$$

This follows from Muirhead's theorem, since $(4,1,1) \prec (6,0,0)$. ∎

214. Find all positive real numbers such that

$$4\big(ab+bc+ca\big) - 1 \geq a^2 + b^2 + c^2 \geq 3\big(a^3+b^3+c^3\big).$$

Solution: Chebyshev's inequality gives

$$\big(a+b+c\big)\big(a^2+b^2+c^2\big) \le 3\big(a^3+b^3+c^3\big).$$

Hence, $a+b+c \le 1$. On the other hand, we also have

$$4\big(ab+bc+ca\big)-1 \ge a^2+b^2+c^2 \ge ab+bc+ca.$$

This shows that $3\big(ab+bc+ca\big) \ge 1$. Thus,

$$1 \le 3\big(ab+bc+ca\big) \le \big(a+b+c\big)^2 \le 1.$$

We conclude that

$$a+b+c=1, \text{ and } 3\big(ab+bc+ca\big) = \big(a+b+c\big)^2.$$

It follows that $a=b=c=1/3$. ■

215. Let a,b,c be positive real numbers such that $abc=1$. Prove that

$$\frac{a}{(a+1)(b+1)}+\frac{b}{(b+1)(c+1)}+\frac{c}{(c+1)(a+1)} \ge \frac{3}{4}.$$

Solution: Clearing the denominators, this is equivalent to

$$4\big(ab+bc+ca+a+b+c\big) \ge 3\big(abc+ab+bc+ca+a+b+c+1\big).$$

This further reduces to

$$ab+bc+ca+a+b+c \ge 6.$$

Dividing by abc, this may be written in the form

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+a+b+c \ge 6.$$

Using $x+1/x \ge 2$, for positive x, the result follows. ■

216. Suppose a,b,c are positive real numbers such that $a^2+b^2+c^2=1$. Prove that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \ge 3+\frac{2\big(a^3+b^3+c^3\big)}{abc}.$$

Solution: Clearing the denominators, we get an equivalent inequality:

$$a^2b^2+b^2+c^2+c^2a^2 \ge 3a^2b^2c^2+2abc\big(a^3+b^3+c^3\big).$$

This may be homogenised:

$$\big(a^2+b^2+c^2\big)\big(a^2b^2+b^2+c^2+c^2a^2\big) \ge 3a^2b^2c^2+2abc\big(a^3+b^3+c^3\big).$$

Simplification gives

$$\sum_{\text{cyclic}} a^4b^2 + \sum_{\text{cyclic}} a^2b^4 \geq 2 \sum_{\text{cyclic}} a^4bc.$$

Since $(4,1,1) \prec (4,2,0)$, the result follows by Muirhead's theorem. ■

217. Let ABC be a triangle with circum-centre O and circum-radius R. Suppose the line AO, when extended, meets the circum-circle of OBC in D; similarly define E and F. Prove that

$$OD \cdot OE \cdot OF \geq 8R^3.$$

Solution: Let D', E' and F' be the points of intersection of AD, BE, and CF respectively with BC, CA and AB. Invert the whole configuration with the circum-circle of ABC. Then BC is mapped on to the circum-circle of BOC; CA on to that of OCA; and AB on to that of OAB. Moreover, D' moves to D; E' to E; and F' to F. The property of inversion shows that

$$OD \cdot OD' = OE \cdot OE' = OF \cdot OF' = R^2$$

Let $[OAB] = x$, $[OBC] = y$ and $[OCA] = z$. Observe that

$$\frac{AO}{OD'} = \frac{x+z}{y}, \frac{BO}{OE'} = \frac{x+y}{z}, \frac{CO}{OF'} = \frac{y+z}{x}.$$

Thus

$$\begin{aligned}\frac{AO \cdot BO \cdot CO}{OD' \cdot OE' \cdot OF'} &= \frac{(x+y)(y+z)(z+x)}{xyz} \\ &\geq \frac{2\sqrt{xy} \cdot 2\sqrt{yz} \cdot 2\sqrt{zx}}{x}yz \\ &\geq 8.\end{aligned}$$

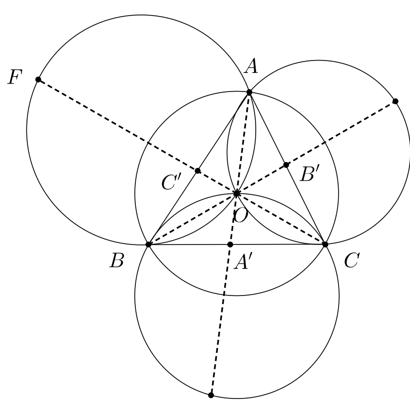

Fig. 6.27

This shows that

$$OD' \cdot OE' \cdot OF' \leq \frac{AO \cdot BO \cdot CO}{8} = \frac{R^3}{8}.$$

Hence, we get

$$OD \cdot OE \cdot OF = \frac{R^6}{OD' \cdot OE' \cdot OF'} \geq 8R^3.$$

■

218. Let x, y, z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

Solution: Suppose $x \leq y \leq z$. Then

$$(1+y)(1+z) \geq (1+z)(1+x) \geq (1+x)(1+y).$$

Hence, Chebyshev's inequality is applicable and

$$\begin{aligned} \sum_{\text{cyclic}} \frac{x^3}{(1+y)(1+z)} &\geq \frac{1}{3}\Big(\sum_{\text{cyclic}} x^3\Big)\Big(\sum_{\text{cyclic}} \frac{1}{(1+y)(1+z)}\Big) \\ &= \frac{1}{3} \frac{\big(x^3+y^3+z^3\big)\big(3+x+y+z\big)}{(1+x)(1+y)(1+z)} \end{aligned}$$

Thus, it is sufficient to prove that

$$4\big(x^3+y^3+z^3\big)\big(3+x+y+z\big) \geq 9(1+x)(1+y)(1+z).$$

Since

$$(1+x)(1+y)(1+z) \leq \left(\frac{3+x+y+z}{3}\right)^3,$$

all we need to prove is

$$12\big(x^3+y^3+z^3\big) \geq \big(3+x+y+z\big)^2.$$

By Hölder's inequality, we have

$$9\big(x^3+y^3+z^3\big) \geq \big(x+y+z\big)^3.$$

Thus, it is enough to prove that

$$4\big(x+y+z\big)^3 \geq 3\big(3+x+y+z\big)^2.$$

Putting $t = x + y + z$, this reduces to

$$4t^3 - 3t^2 - 18t - 27 \geq 0.$$

Equivalently $(t - 3)(4t^2 + 9t + 9) \geq 0$. We need to check that $t \geq 3$. This follows from

$$x + y + z \geq 3(xyz)^{1/3} = 3.$$

■

219. Let D, E, F be respectively the points of contact of the in-circle of a triangle ABC with its sides BC, CA, AB. Prove that

$$\frac{BC}{FD} + \frac{CA}{DE} + \frac{AB}{EF} \geq 6.$$

Solution: Note that $FD = 2(s - b)\sin(B/2)$. Thus

$$\frac{BC}{FD} = \frac{a}{2(s-b)\sin(B/2)} = \frac{a}{2r\cos(B/2)}.$$

But $\cos(B/2) = h_b/a$. We thus get

$$\frac{BC}{FD} = \frac{a^2 b}{4r\Delta}.$$

The inequality reduces to

$$a^2 b + b^2 c + c^2 a \geq 24r\Delta.$$

This may be written in the form

$$\frac{a}{c} + \frac{b}{a} + \frac{c}{b} \geq \frac{6r}{R}.$$

But

$$\frac{a}{c} + \frac{b}{a} + \frac{c}{b} \geq 3 \geq \frac{6r}{R},$$

because $2r \leq R$. ■

220. Let a, b, c, d be non-negative real numbers such that $ab + bc + cd + da = 1$. Show that

$$\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \geq \frac{1}{3}.$$

Solution: Observe that

$$a^2 + b^2 + c^2 + d^2 \geq ab + bc + cd + da = 1.$$

Note that the inequality is symmetric in a, b, c, d. Suppose $a \geq b \geq c \geq d$. Then

$$\frac{1}{b+c+d} \geq \frac{1}{c+d+a} \geq \frac{1}{d+a+b} \geq \frac{1}{a+b+c}.$$

Taking $x = b+c+d$, $y = c+d+a$, $z = d+a+b$ and $w = a+b+c$, Chebyshev's inequality gives

$$\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} + \frac{d^3}{w} \geq \frac{1}{4}\left(a^3 + b^3 + c^3 + d^3\right)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w}\right).$$

One more application of Chebyshev's inequality leads to

$$\begin{aligned} a^3 + b^3 + c^3 + d^3 &\geq \frac{1}{4}\left(a^2 + b^2 + c^2 + d^2\right)\left(a + b + c + d\right) \\ &\geq \frac{1}{4}(a + b + c + d). \end{aligned}$$

Thus, it follows that

$$\begin{aligned} &\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \\ &\quad\geq \frac{1}{16}\left(a + b + c + d\right)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w}\right) \\ &\quad= \frac{1}{48}\left(x + y + z + w\right)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w}\right) \\ &\quad\geq \frac{1}{3}, \end{aligned}$$

where we have used the AM-HM inequality in the last leg. ■

221. Find all real λ for which the inequality

$$x_1^2 + x_2^2 + x_3^2 \geq \lambda(x_1x_2 + x_2x_3),$$

holds for all real numbers x_1, x_2, x_3.

Solution: We show that the range of values of λ is $[-\sqrt{2}, \sqrt{2}]$. Taking $x_1 = x_3 = 1$ and $x_2 = p$, we have

$$2 + p^2 \geq 2p^2,$$

which shows that $p^2 \leq 2$. Thus $p \in [-\sqrt{2}, \sqrt{2}]$.

Conversely, suppose $p \in [-\sqrt{2}, \sqrt{2}]$, so that $p^2 \leq 2$. Consider x_1^2, $p^2 x_2^2$, x_3^2, where x_1, x_2, x_3 are arbitrary real numbers. Now the AM-GM inequality gives

$$\begin{aligned} x_1^2 + \frac{p^2}{4} x_2^2 &\geq |p x_1 x_2| \geq p x_1 x_2, \\ \frac{p^2}{4} x_2^2 + x_3^2 &\geq |p x_2 x_3| \geq p x_2 x_3. \end{aligned}$$

Adding these, we obtain

$$x_1^2 + \frac{p^2}{2} x_2^2 + x_3^2 \geq p(x_1 x_2 + x_2 x_3).$$

Since $p^2 \leq 2$, it follows that

$$x_1^2 + x_2^2 + x_3^2 \geq x_1^2 + \frac{p^2}{2} x_2^2 + x_3^2 \geq p(x_1 x_2 + x_2 x_3).$$

Thus for all values of $p \in [-\sqrt{2}, \sqrt{2}]$, the inequality

$$x_1^2 + x_2^2 + x_3^2 \geq p(x_1 x_2 + x_2 x_3),$$

holds for all choices of real x_1, x_2, x_3. ■

222. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Solution: Suppose $ab + bc + ca \geq a + b + c$. Using $1/a$, $1/b$, $1, c$ as weights, the weighted AM-HM inequality gives

$$\begin{aligned} \frac{a}{b} + \frac{b}{c} + \frac{c}{a} &\geq \frac{\left(\frac{1}{b} + \frac{1}{c} + \frac{1}{a}\right)^2}{\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}} \\ &= \frac{(ab + bc + ca)\left(\frac{1}{b} + \frac{1}{c} + \frac{1}{a}\right)}{a + b + c} \\ &\geq \frac{1}{b} + \frac{1}{c} + \frac{1}{a}. \end{aligned}$$

If $ab + bc + ca \leq a + b + c$, then we use a, b, c as weights to get

$$\begin{aligned} \frac{a}{b} + \frac{b}{c} + \frac{c}{a} &\geq \frac{(a + b + c)^2}{ab + bc + ca} \\ &\geq ab + bc + ca \\ &= \frac{1}{b} + \frac{1}{c} + \frac{1}{a}. \end{aligned}$$

■

223. Let a, b, c be non-negative reals such that

$$a + b \leq 1 + c, \quad b + c \leq 1 + a, \quad c + a \leq 1 + b.$$

Prove that

$$a^2 + b^2 + c^2 \leq 2abc + 1.$$

Solution: Adding $a + b \leq 1 + c$ and $b + c \leq 1 + a$, we get $2b \leq 2$ and hence $b \leq 1$. Similarly, we get $c \leq 1$ and $a \leq 1$. Take $\alpha = 1 - a$, $\beta = 1 - b$ and $\gamma = 1 - c$. Then $0 \leq \alpha, \beta, \gamma \leq 1$. Moreover $a = b \leq 1 + c$ implies that $\alpha \geq \gamma$. Similarly, $\beta + \gamma \geq \alpha$ and $\gamma + \alpha \geq \beta$. The inequality to be proved is:

$$(1-\alpha)^2 + (1-\beta)^2 + (1-\gamma)^2 \leq 2(1-\alpha)(1-\beta)(1-\gamma) + 1.$$

This reduces to

$$\alpha^2 + \beta^2 + \gamma^2 \leq 2(\alpha\beta + \beta\gamma + \gamma\alpha) - 2\alpha\beta\gamma.$$

We may assume that γ is the largest among the three; i.e., $\alpha \leq \gamma$ and $\beta \leq \gamma$. Then $\gamma \leq \alpha + \beta$ and hence $\gamma^2 \leq \gamma\alpha + \beta\gamma$. Moreover, $\alpha^2 \leq \alpha\gamma$ and $\beta^2 \leq \beta\gamma$. These two imply that

$$\alpha^2 + \beta^2 + \gamma^2 \leq 2(\beta\gamma + \gamma\alpha).$$

Thus it is sufficient to prove that $2\alpha\beta\gamma \leq 2\alpha\beta$. But this follows from $\gamma \leq 1$, and α, β are non-negative.

Alternate Solution: (Arpit Amar Goel and Utkarsh Tripathi) As in the earlier solution, we get $a \leq 1$, $b \leq 1$ and $c \leq 1$. By symmetry, we may assume c is the least and a the largest, so that $0 \leq c \leq b \leq a \leq 1$. We may write the inequality in the form

$$(a-b)^2 + c^2 \leq 2abc + 1 - 2bc,$$

or equivalently

$$(a-b)^2 \leq (1-c)(1+c-2ab).$$

Using $a + c \leq 1 + b$, we get $1 - c \geq a - b \geq 0$. Also

$$\begin{aligned} 1 + c - 2ab - a + b &= 1 + c - a(1+b) - b(a-1) \\ &\geq 1 + c - a(1+b), \ \text{(since } a \leq 1\text{)} \\ &= 1 + c - a - ab \\ &\geq 1 + c - a - b, \ \text{(since } a \leq 1\text{)} \\ &\geq 0, \end{aligned}$$

because $a + b \leq 1 + c$. Thus, we get $1 + c - 2ab \geq a - b \geq 0$. It follows that

$$(a-b)^2 \leq (1-c)(1+c-2ab).$$

Equality holds if and only if $a = 1$ and $b = c$, or $b = 1$ and $c = a$ or $c = 1$ and $a = b$. ■

224. If a, b, c are non-negative real numbers such that $a+b+c=1$ then show that

$$\frac{a}{1+bc}+\frac{b}{1+ca}+\frac{c}{1+ab} \geq \frac{9}{10}.$$

Solution: Using the weighted AM-GM inequality, we have

$$\frac{\dfrac{a}{1+bc}+\dfrac{b}{1+ca}+\dfrac{c}{1+ab}}{a+b+c} \geq \frac{a+b+c}{a(1+bc)+b(1+ca)+c(1+ca)}.$$

This reduces to

$$\frac{a}{1+bc}+\frac{b}{1+ca}+\frac{c}{1+ab} \geq \frac{1}{1+3abc}.$$

Using the AM-GM inequality, we also have

$$abc \leq \left(\frac{a+b+c}{3}\right) = \frac{1}{27}.$$

Thus, we get

$$1+3abc \leq \frac{10}{9}.$$

This implies that

$$\frac{a}{1+bc}+\frac{b}{1+ca}+\frac{c}{1+ab} \geq \frac{9}{10}.$$

More generally, if $x_1, x_2, \ldots, x_n$ are non-negative real numbers such that $x_1 + x_2 + \ldots + x_n = 1$, then

$$\sum_{j=1}^{n} \frac{x_j}{1+\prod\limits_{\substack{1\leq k\leq n\\ k\neq j}} x_k} \geq \frac{1}{1+n^{1-n}}.$$

■

225. Let ABC be a triangle with sides a, b, c, circum-radius R and in-radius r. Prove that

$$\frac{R}{2r} \geq \left(\frac{64a^2b^2c^2}{\left(4a^2-(b-c)^2\right)\left(4b^2-(c-a)^2\right)\left(4c^2-(a-b)^2\right)}\right)^2.$$

Solution: (Riddhipratim Basu) We define x, y, z by $s-a=x, s-b=y, s-c=z$, where $s=(a+b+c)/2$. Then x, y, z are positive, and $a=y+z$, $b=z+x$, $c=x+y$. We may express

$$\frac{R}{2r} = \frac{abcs}{4\Delta^2} = \frac{(x+y)(y+z)(z+x)}{8xyz}.$$

Similarly

$$4a^2 - (b-c)^2 = (3y+z)(3z+y), 4b^2 - (c-a)^2 = (3x+z)(3z+x),$$
$$4c^2 - (a-b)^2 = (3x+y)(3y+x).$$

Thus, we need to prove

$$\frac{(x+y)(y+z)(z+x)}{8xyz}$$
$$\geq \frac{64^2(x+y)^4(y+z)^4(z+x)^4}{(3x+y)^2(3y+x)^2(3y+z)^2(3z+y)^2(3x+z)^2(3z+x)^2}.$$

This may be written in the form

$$(3x+y)^2(3y+x)^2(3y+z)^2(3z+y)^2(3x+z)^2(3z+x)^2$$
$$\geq 32^3 xyz(x+y)^3(y+z)^(z+x)^3.$$

Using the AM-GM inequality, we have

$$\begin{aligned} (3x+y)(3y+x) &= 3x^2+3y^2+10xy \\ &= (x+y)^2+(x+y)^2+(x+y)^2+4xy \\ &\geq 4\Big\{(x+y)^6 \cdot 4xy\Big\}^{1/4}. \end{aligned}$$

It follows that $(3x+y)^2(3y+x)^2 \geq 32\sqrt{xy}(x+y)^3$ and similar expressions for the other products. Thus

$$(3x+y)^2(3y+x)^2(3y+z)^2(3z+y)^2(3x+z)^2(3z+x)^2$$
$$\geq 32^3 xyz(x+y)^3(y+z)^(z+x)^3,$$

which gives the desired inequality. ■

226. Let R denote the circum-radius of a triangle ABC; a, b, c its sides BC, CA, AB; and r_a, r_b, r_c its ex-radii opposite A, B, C. If $2R \leq r_a$, prove that

(i) $a > b$ and $a > c$;

(ii) $2R > r_b$ and $2R > r_c$.

Solution: We know that $2R = \dfrac{abc}{2\triangle}$ and $r_a = \dfrac{\triangle}{s-a}$, where a, b, c are the sides of the triangle ABC, $s = \dfrac{a+b+c}{2}$ and $\triangle$ is the area of ABC. Thus the given condition $2R \leq r_a$ translates to the condition

$$abc \leq \frac{2\triangle^2}{s-a}$$

Putting $s-a=p, s-b=q, s-c=r$, we get $a=q+r, b=r+p, c=p+q$ and the condition now is

$$p(p+q)(q+r)(r+p) \leq 2\triangle^2$$

But Heron's formula gives, $\triangle^2 = s(s-a)(s-b)(s-c) = pqr(p+q+r)$. We obtain $(p+q)(q+r)(r+p) \leq 2qr(p+q+r)$. Expanding and effecting some cancellations, we get

$$p^2(q+r)+p(q^2+r^2) \leq qr(q+r). \tag{226.1}$$

Suppose $a \leq b$. This implies that $q+r \leq r+p$ and hence $q \leq p$. This implies that $q^2r \leq p^2r$ and $qr^2 \leq pr^2$ giving $qr(q+r) \leq p^2r+pr^2 < p^2r+pr^2+p^2q+pq^2 = p^2(q+r)+p(q^2+r^2)$ which contradicts (226.1). Similarly, $a \leq c$ is also not possible. This proves (i).

Suppose $2R \leq r_b$. As above, this takes the form

$$q^2(r+p)+q(r^2+p^2) \leq pr(p+r). \tag{226.2}$$

Since $a > b$ and $a > c$, we have $q > p, r > p$. Thus $q^2r > p^2r$ and $qr^2 > pr^2$. Hence

$$q^2(r+p)+q(r^2+p^2) > q^2r+qr^2 > p^2r+pr^2 = pr(p+r)$$

which contradicts (226.2). Hence $2R > r_b$. Similarly, we can prove that $2R > r_c$. This proves (ii) ■

227. Given a square grid S containing 49 points in 7 rows and 7 columns, a subset T consisting of k points is selected. What is the maximum value of k such that no four points of T determine a rectangle R having sides parallel to the sides of S?

Solution: We show that $k=21$ is the maximum possible value. The following diagram shows that there is no rectangle for $k=21$.

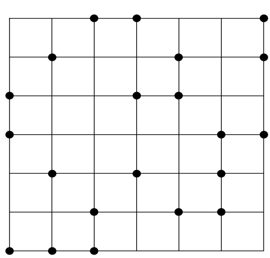

Fig. 6.28

We follow the argument in the example 4.3 on page 174. Let m_j, $1 \le j \le 7$, denote the number of elements of T in j-th row. In order to avoid a rectangle whose sides are parallel to the grids of T, we must have

$$\sum_{j=1}^{7} \binom{m_j}{2} \le \binom{7}{2}.$$

Taking $k = m_1 + m_2 + \cdots + m_7$, this may be written in the form

$$\sum_{j=1}^{7} m_j^2 \le 42 + k.$$

However, the Cauchy-Schwarz inequality gives

$$(m_1 + m_2 + \cdots + m_7)^2 \le 7 \sum_{j=1}^{7} m_j^2.$$

Thus, we obtain

$$\frac{k^2}{7} \le \sum_{j=1}^{7} m_j^2 \le 42 + k,$$

which reduces to $(k+14)(k-21) \le 0$. This shows that $k \le 21$. If $k = 21$, then we see that equality holds in the Cauchy-Schwarz inequality, forcing $m_j = 3$ for $1 \le j \le 7$. Since for $k = 21$, there is already an example, it follows that 21 is the maximum value of k which avoids a rectangle. ■

228. Let a, b, c, d be real numbers such that $0 \le a \le b \le c \le d$. Prove that

$$a^b b^c c^d d^a \ge b^a c^b d^c a^d.$$

Solution: We use the following property of convex functions:

If $f : [s,t] \to \mathbb{R}$ is a convex function and if x, y, z are three points in $[s,t]$ such that $x \le y \le z$, then

$$(y-z)f(x) + (z-x)f(y) + (x-y)f(z) \le 0.$$

Using the convexity of $f(x) = \ln x$ on $[0, \infty)$, we have for $0 \le x \le y \le z$,

$$(y-z)\ln x + (z-x)\ln y + (x-y)\ln z \ge 0.$$

This simplifies to

$$x^y y^z z^x \ge x^z y^x z^y.$$

Thus, we have

$$a^b b^c c^a \ge b^a c^b a^c, \quad a^c c^d d^a \ge c^a d^c a^d.$$

Multiplying these two and effecting some cancellations, we get the required inequality. ■

229. In a triangle ABC, let AA_1, CC_1 be the bisectors of the angles A, C respectively. Let M be point on the segment AC, and I be the in-centre of ABC. Draw a line through M, parallel to AA_1 and let it meet CC_1 in N and BC in Q. Similarly, let the line through M parallel to CC_1 meet AA_1 in H and AB in P. Let d_1, d_1, d_3 respectively denote the distances of H, I, N from the line PQ. Prove that

$$\frac{d_1}{d_2} + \frac{d_2}{d_3} + \frac{d_3}{d_1} \geq \frac{2ab}{a^2+bc} + \frac{2bc}{b^2+ca} + \frac{2ca}{c^2+ab}.$$

Solution: Join PQ. Let it intersect AA_1 in T and CC_1 in S respectively.

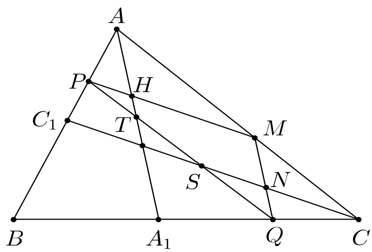

Fig. 6.29

Since MP is parallel to CC_1, we have

$$\frac{PH}{HM} = \frac{C_1I}{IC} = \frac{AC_1}{AC}.$$

But CC_1 is the bisector of angle C, so that $AC_1 = bc/(b+a)$. This gives

$$\frac{HM}{PH} = \frac{b+a}{c}, \quad \text{and} \quad \frac{PM}{PH} = \frac{a+b+c}{c}.$$

But PHT is similar to PMQ, so that

$$\frac{PT}{PQ} = \frac{PH}{PM} = \frac{c}{a+b+c}.$$

Similarly, we obtain

$$\frac{QS}{PQ} = \frac{a}{a+b+c}.$$

Thus, we have

$$\frac{TS}{PQ} = \frac{b}{a+b+c}.$$

Now, using the similarity of triangles PHT, SIT and SNQ, we get

$$\frac{d_1}{d_2} = \frac{c}{b}, \quad \frac{d_2}{d_3} = \frac{b}{a}, \quad \frac{d_3}{d_1} = \frac{a}{c}.$$

Thus, the inequality to be proved is::

$$\frac{c}{b} + \frac{b}{a} + \frac{a}{c} \geq \frac{2ab}{a^2+bc} + \frac{2bc}{b^2+ca} + \frac{2ca}{c^2+ab}.$$

However, we observe that

$$\frac{2ab}{a^2+bc} = \frac{2}{\dfrac{a}{b} + \dfrac{c}{a}} \leq \frac{1}{2}\left(\frac{b}{a} + \frac{a}{c}\right),$$

and similar expressions may be obtained. Therefore, we obtain

$$\frac{2ab}{a^2+bc} + \frac{2bc}{b^2+ca} + \frac{2ca}{c^2+ab} \leq \frac{c}{b} + \frac{b}{a} + \frac{a}{c}.$$

■

230. Let ABC be a triangle and h_a be the altitude through A. Prove that

$$(b+c)^2 \geq a^2 + 4h_a^2.$$

(As usual a, b, c denote the sides BC, CA, AB respectively.)

Solution: Draw a line l parallel to BC through A and reflect AC in this line to get AD. Let CD intersect l in P. Join BD. Observe that $CP = PD = AQ = h_a$, where AQ is the altitude through A. We have,

$$b + c = AC + AB = AD + AB \geq BD = \sqrt{CD^2 + CB^2} = \sqrt{4h_a^2 + a^2},$$

which give the desired result. Equality occurs if and only if B, A, D are collinear, i.e., if and only if $AD = AB$ (as AP is parallel to BC and bisects DC) and this is equivalent to $AC = BC$.

Alternate Solution:

The given inequality is equivalent to

$$(b+c)^2 - a^2 \;\geq\; 4h_a^2 \;=\; \frac{16\Delta^2}{a^2},$$

where Δ is the area of the triangle ABC. Using the identity

$$16\Delta^2 = [\,(b+c)^2 - a^2\,][\,a^2 - (b-c)^2\,]$$

we see that the inequality to be proved is $a^2 - (b-c)^2 \leq a^2$ (here we have used $a < b + c$) which is true. Observe that equality holds if and only if $b = c$. ■

231. Let a, b, c be three positive real numbers such that $a + b + c = 1$. Prove that among the three numbers $a - ab$, $b - bc$, $c - ca$ there is one which is at most $1/4$ and there is one which is at least $2/9$.

Solution: By the AM-GM inequality, we have

$$a(1-a) \le \left(\frac{a+1-a}{2}\right)^2 = \frac{1}{4}.$$

Similarly, we also have

$$b(1-b) \le \frac{1}{4} \quad \text{and} \quad c(1-c) \le \frac{1}{4}.$$

Multiplying these, we obtain

$$abc(1-a)(1-b)(1-c) \le \frac{1}{4^3}.$$

We may rewrite this in the form

$$a(1-b) \cdot b(1-c) \cdot c(1-a) \le \frac{1}{4^3}.$$

Hence, at least one of the factors from among $a(1-b), b(1-c), c(1-a)$ has to be less than or equal to $\frac{1}{4}$; otherwise **lhs** would exceed $\frac{1}{4^3}$.

Again, consider the sum $a(1-b) + b(1-c) + c(1-a)$. This is equal to $a+b+c-ab-bc-ca$. We observe that

$$3\big(ab+bc+ca\big) \le \big(a+b+c\big)^2,$$

which, in fact, is equivalent to $(a-b)^2 + (b-c)^2 + (c-a)^2 \ge 0$. This leads to the inequality

$$a+b+c-ab-bc-ca \ge (a+b+c) - \frac{1}{3}(a+b+c)^2 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Hence, at least one of the summands from among $a(1-b), b(1-c), c(1-a)$ has to be greater than or equal to $\frac{2}{9}$; (otherwise **lhs** would be less than $\frac{2}{3}$.) ■

232. Let x and y be positive real numbers such that $y^3 + y \le x - x^3$. Prove that

(a) $y < x < 1$; and

(b) $x^2 + y^2 < 1$.

Solution:

(a) Since x and y are positive, we have $y \le x - x^3 - y^3 < x$. We also have $x - x^3 \ge y + y^3 > 0$, so that $x(1-x^2) > 0$. This gives $x < 1$ and thus $y < x < 1$, proving part (a).

(b) Again, we have $x^3 + y^3 \le x - y$, so that

$$x^2 - xy + y^2 \le \frac{x-y}{x+y}.$$

This implies that

$$x^2 + y^2 \le \frac{x-y}{x+y} + xy = \frac{x - y + xy(x+y)}{x+y}.$$

But $xy(x+y) < 1 \cdot y \cdot (1+1) = 2y$ and hence

$$x^2 + y^2 < \frac{x - y + 2y}{x+y} = \frac{x+y}{x+y} = 1.$$

This proves (b).

■

233. Let a, b, c be three positive real numbers such that $a + b + c = 1$. Let

$$\lambda = \min\left\{a^3 + a^2bc,\ b^3 + ab^2c,\ c^3 + abc^2\right\}.$$

Prove that the roots of the equation $x^2 + x + 4\lambda = 0$ are real.

Solution: Suppose the equation $x^2 + x + 4\lambda = 0$ has no real roots. Then $1 - 16\lambda < 0$. This implies that

$$1 - 16\left(a^3 + a^2bc\right) < 0, \quad 1 - 16\left(b^3 + ab^2c\right) < 0, \quad 1 - 16\left(c^3 + abc^2\right) < 0.$$

Observe that

$$\begin{aligned}
1 - 16\left(a^3 + a^2bc\right) < 0 \quad &\Longrightarrow \quad 1 - 16a^2\left(a + bc\right) < 0 \\
&\Longrightarrow \quad 1 - 16a^2\left(1 - b - c + bc\right) < 0 \\
&\Longrightarrow \quad 1 - 16a^2(1-b)(1-c) < 0 \\
&\Longrightarrow \quad \frac{1}{16} < a^2(1-b)(1-c).
\end{aligned}$$

Similarly, we may obtain

$$\frac{1}{16} < b^2(1-c)(1-a), \quad \frac{1}{16} < c^2(1-a)(1-b).$$

Multiplying these three inequalities, we get

$$a^2b^2c^2(1-a)^2(1-b)^2(1-c)^2 > \frac{1}{16^3}.$$

However, $0 < a < 1$ implies that $a(1-a) \le 1/4$. Hence

$$a^2b^2c^2(1-a)^2(1-b)^2(1-c)^2 = \big(a(1-a)\big)^2\big(b(1-b)\big)^2\big(c(1-c)\big)^2 \le \frac{1}{16^3},$$

a contradiction. We conclude that the given equation has real roots. ■

234. If a, b, c are three positive real numbers, prove that

$$\frac{a^2+1}{b+c}+\frac{b^2+1}{c+a}+\frac{c^2+1}{a+b} \geq 3.$$

Solution: We use the trivial inequalities $a^2+1 \geq 2a$, $b^2+1 \geq 2b$ and $c^2+1 \geq 2c$. Hence we obtain

$$\frac{a^2+1}{b+c}+\frac{b^2+1}{c+a}+\frac{c^2+1}{a+b} \geq \frac{2a}{b+c}+\frac{2b}{c+a}+\frac{2c}{a+b}.$$

Thus, it is sufficient to prove that

$$\frac{2a}{b+c}+\frac{2b}{c+a}+\frac{2c}{a+b} \geq 3.$$

Adding 6 both sides, this is equivalent to

$$(2a+2b+2c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right) \geq 9.$$

Taking $x=b+c$, $y=c+a$, $z=a+b$, this is equivalent to

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9,$$

which is a consequence of the AM-GM inequality.

Alternate Solution:
The substitutions $b+c=x$, $c+a=y$, $a+b=z$ lead to

$$\sum_{\text{cyclic}} \frac{2a}{b+c}=\sum_{\text{cyclic}} \frac{y+z-x}{x}=\sum_{\text{cyclic}}\left(\frac{x}{y}+\frac{y}{x}\right)-3 \geq 6-3=3;$$

we have used the AM-GM inequality in the last leg. ■

235. If d is the largest among the positive numbers a, b, c, d, prove that

$$a(d-b)+b(d-c)+c(d-a) \leq d^2.$$

Solution: Consider the polynomial $P(x)=x^2-(a+b+c)x+(ab+bc+ca)$. Then we see that

$$\begin{aligned} P(d) &= d^2-(a+b+c)d+(ab+bc+ca) \\ &= d^2-a(d-b)-b(d-c)-c(d-a). \end{aligned}$$

Thus, we have to prove that $P(d) \geq 0$. On the other hand,

$$xP(x)-abc=(x-a)(x-b)(x-c).$$

Thus

$$P(d) = \frac{(d-a)(d-b)(d-c)+abc}{d} \geq \frac{abc}{d} \geq 0.$$

■

236. If x, y, z are positive real numbers, prove that

$$\left(x+y+z\right)^2\left(yz+zx+xy\right)^2 \leq 3\left(y^2+yz+z^2\right)\left(z^2+zx+x^2\right)\left(x^2+xy+y^2\right).$$

Solution: One solution using normalisation was given on page 91. We give here three more solutions.

Solution 1: We begin with the observation that

$$x^2+xy+y^2 = \frac{3}{4}(x+y)^2 + \frac{1}{4}(x-y)^2 \geq \frac{3}{4}(x+y)^2,$$

and similar bounds hold for y^2+yz+z^2, z^2+zx+x^2. Hence,

$$3\left(x^2+xy+y^2\right)\left(y^2+yz+z^2\right)\left(z^2+zx+x^2\right) \geq \frac{81}{64}\left(x+y\right)^2\left(y+z\right)^2\left(z+x\right)^2.$$

Thus, it is sufficient to prove that

$$\left(x+y+z\right)\left(xy+yz+zx\right) \leq \frac{9}{8}\left(x+y\right)\left(y+z\right)\left(z+x\right).$$

Equivalently, we need to prove that

$$8\left(x+y+z\right)\left(xy+yz+zx\right) \leq 9\left(x+y\right)\left(y+z\right)\left(z+x\right).$$

However, we note that

$$\left(x+y\right)\left(y+z\right)\left(z+x\right) = \left(x+y+z\right)\left(yz+zx+xy\right) - xyz.$$

Thus, the required inequality takes the form

$$\left(x+y\right)\left(y+z\right)\left(z+x\right) \geq 8xyz.$$

This follows from the AM-GM inequalities;

$$x+y \geq 2\sqrt{xy}, \quad y+z \geq 2\sqrt{yz}, \quad z+x \geq 2\sqrt{zx}.$$

Solution 2: Let us introduce $x+y=c$, $y+z=a$ and $z+x=b$. Then a, b, c are the sides of a triangle. If $s=(a+b+c)/2$, then it is easy to calculate $x=s-a$, $y=s-b$, $z=s-c$ and $x+y+z=s$. We also observe that

$$\begin{aligned}x^2+xy+y^2 &= (x+y)^2 - xy \\ &= c^2 - \frac{1}{4}(c+a-b)(c+b-a) = \frac{3}{4}c^2 + \frac{1}{4}(a-b)^2 \geq \frac{3}{4}c^2.\end{aligned}$$

Moreover, $xy + yz + zx = (s-a)(s-b) + (s-b)(s-c) + (s-c)(s-a)$. Thus it is sufficient to prove that

$$s\sum(s-a)(s-b) \le \frac{9}{8}abc.$$

But, $\sum(s-a)(s-b) = r(4R+r)$, where r, R are respectively the in-radius, the circum-radius of the triangle whose sides are a, b, c, and $abc = 4Rrs$. Thus the inequality reduces to

$$r(4R+r) \le \frac{9}{2}Rr.$$

This simply reduces to $2r \le R$, which is Euler's inequality.

Solution 3: Dividing throughout by $x^2y^2z^2$, the inequality may be written in the form

$$\left(3 + \frac{x}{y} + \frac{y}{x} + \frac{y}{z} + \frac{z}{y} + \frac{z}{x} + \frac{x}{z}\right)^2 \le 3\left(\frac{x}{y} + \frac{y}{x} + 1\right)\left(\frac{y}{z} + \frac{z}{y} + 1\right)\left(\frac{z}{x} + \frac{x}{z} + 1\right)$$

If we set

$$\frac{x}{y} + \frac{y}{x} = a + 2, \quad \frac{y}{z} + \frac{z}{y} = b + 2, \quad \frac{z}{x} + \frac{x}{z} = c + 2,$$

then a, b, c are non-negative and the inequality to be proved is

$$(9 + a + b + c)^2 \le 3(a+3)(b+3)(c+3).$$

This reduces to

$$a^2 + b^2 + c^2 \le 3abc + 7(ab + bc + ca) + 9(a + b + c).$$

However, it is easy to see that

$$\left(\frac{x}{y} + \frac{y}{x}\right)\left(\frac{y}{z} + \frac{z}{y}\right)\left(\frac{z}{x} + \frac{x}{z}\right) = 2 + \frac{x^2}{y^2} + \frac{y^2}{x^2} + \frac{y^2}{z^2} + \frac{z^2}{y^2} + \frac{z^2}{x^2} + \frac{x^2}{z^2}.$$

This takes the form

$$a^2 + b^2 + c^2 = abc + 2(ab + bc + ca).$$

Thus the inequality reduces to

$$2abc + 5(ab + bc + ca) + 9(a + b + c) \ge 0.$$

This follows from the non-negativity of a, b, c. ■

237. Suppose a, b, c are positive real numbers. Prove that

$$a^a b^b c^c \ge (abc)^{(a+b+c)/3}.$$

Solution: Suppose x and y are positive real numbers. Then rearrangement inequality implies that

$$x \ln x + y \ln y \geq x \ln y + y \ln x.$$

Exponentiation gives $x^x y^y \geq x^y y^x$. Thus, we have

$$a^a b^b \geq a^b b^a, \quad b^b c^c \geq b^c c^b, \quad c^c a^a \geq c^b a a^c.$$

Multiplying these out, we get

$$\left(a^a b^b c^c\right)^2 \geq a^{b+c} b^{c+a} c^{a+b}.$$

It follows that

$$\left(a^a b^b c^c\right)^3 \geq a^{b+c+a} b^{c+a+b} c^{a+b+c},$$

which reduces to

$$a^a b^b c^c \geq (abc)^{(a+b+c)/3}.$$

Alternately, we can use the weighted AM-HM inequality:

$$\begin{aligned}\left(a^a b^b c^c\right)^{1/(a+b+c)} \geq \frac{a+b+c}{\frac{a}{a}+\frac{b}{b}+\frac{c}{c}} &= \frac{a+b+c}{3} \\ &\geq (abc)^{1/3},\end{aligned}$$

where we have used the AM-GM inequality in the last stage. This gives the required inequality. ■

238. Find all real p and q for which the equation

$$x^4 - \frac{8p^2}{q} x^3 + 4qx^3 - 3px + p^2 = 0$$

has four positive roots.

Solution: Let the positive roots of

$$P(x) = x^4 - \frac{8p^2}{q} x^3 + 4qx^2 - 3px + p^2 = 0,$$

be a, b, c, d. Then we have

$$\begin{aligned} a+b+c+d &= \frac{8p^2}{q}, \\ ab+ac+ad+bc+bd+cd &= 4q, \\ abc+abd+acd+bcd &= 3p, \\ abcd &= p^2. \end{aligned}$$

Since a, b, c, d are positive, p, q are also positive. Now using the AM-GM inequality, we have

$$\frac{8p^2}{q} = a + b + c + d \geq 4(abcd)^{1/4} = 4\sqrt{p}.$$

Thus, $2p^2 \geq q\sqrt{p}$. Similarly,

$$\begin{aligned} 4q = ab + ac + ad + bc + bd + cd &\geq 6(abcd)^{1/2} = 6p, \\ 3p = abc + abd + acd + bcd &\geq 4(abcd)^{3/4} = 4p\sqrt{p}. \end{aligned}$$

Equality holds in each of these, if and only if $a = b = c = d$. Multiplying out all the three inequalities, we obtain

$$(2p^2)(4q)(3p) \geq (q\sqrt{p})(6p)(4p\sqrt{p}).$$

We see that this is an equality, so that $a = b = c = d$. Let us denote the common value by α. Thus we obtain

$$\begin{aligned} 4\alpha &= \frac{8p^2}{q}, \\ 6\alpha^2 &= 4q, \\ 4\alpha^3 &= 3p, \\ \alpha^4 &= p^2. \end{aligned}$$

Using the first two relations, we get

$$\alpha^3 = \frac{4p^2}{3}.$$

Using the third relation, we have

$$\frac{16p^2}{3} = 2p,$$

which gives $p = 9/16$. Thus, $\alpha = 3/4$ and hence $q = 27/32$. ■

239. Let a_1, a_2, a_3 be real numbers, each greater than 1. Let $S = a_1 + a_2 + a_3$ and suppose $S < a_j^2/(a_j - 1)$ for $j = 1, 2, 3$. Prove that

$$\frac{1}{a_1 + a_2} + \frac{1}{a_2 + a_3} + \frac{1}{a_3 + a_1} > 1.$$

Solution: Observe that $a_1^2/(a_1 - 1) > S$ is equivalent to $a_1^2 > (a_1 + a_2 + a_3)(a_1 - 1)$, which in turn is equivalent to

$$\frac{1}{a_2 + a_3} > \frac{a_1}{a_1 + a_2 + a_3}.$$

Using similar inequalities from the other two, we get

$$\sum_{\text{cyclic}} \frac{1}{a_1 + a_2} > \frac{a_1 + a_2 + a_3}{a_1 + a_2 + a_3} = 1.$$

■

240. Let a, b, c be positive real numbers such that $ab + bc + ca = 1/3$. Prove that

$$\frac{a}{a^2 - bc + 1} + \frac{b}{b^2 - ca + 1} + \frac{c}{c^2 - ab + 1} \geq \frac{1}{a + b + c}.$$

Solution: We write the **lhs** as

$$\begin{aligned}
&\frac{a}{a^2 - bc + 1} + \frac{b}{b^2 - ca + 1} + \frac{c}{c^2 - ab + 1} \\
&\qquad = \frac{a^2}{a^3 - abc + a} + \frac{b^2}{b^3 - abc + b} + \frac{c^2}{c^3 - abc + c}.
\end{aligned}$$

Using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
(a+b+c)^2 &= \left(\sum_{\text{cyclic}} \frac{a}{\sqrt{a^3 - abc + a}} \sqrt{a^3 - abc + a}\right)^2 \\
&\leq \left(\sum_{\text{cyclic}} \frac{a^2}{(a^3 - abc + a)}\right) \left(\sum_{\text{cyclic}} (a^3 - abc + a)\right)
\end{aligned}$$

Therefore

$$\left(\sum_{\text{cyclic}} \frac{a^2}{a^3 - abc + a}\right) \geq \frac{(a+b+c)^2}{a^3 + b^3 + c^3 + a + b + c - 3abc}.$$

But

$$\begin{aligned}
&a^3 + b^3 + c^3 - 3abc + a + b + c \\
&\qquad = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca + 1) \\
&\qquad = (a+b+c)^3,
\end{aligned}$$

because of $ab + bc + ca = 1/3$. Thus we get

$$\left(\sum_{\text{cyclic}} \frac{a^2}{a^3 - abc + a}\right) \geq \frac{(a+b+c)^2}{(a+b+c)^3} = \frac{1}{a+b+c}.$$

■

241. Suppose a, b, c are positive real numbers. Prove that

$$\frac{a^2b(b-c)}{a+b} + \frac{b^2c(c-a)}{b+c} + \frac{c^2a(a-b)}{c+a} \geq 0.$$

Solution: We can write the inequality in an equivalent form if we divide through out by abc:

$$\frac{a(b-c)}{c(a+b)} + \frac{b(c-a)}{a(b+c)} + \frac{c(a-b)}{b(c+a)} \geq 0.$$

Adding 3 both sides, this transforms to

$$\frac{a(b-c)}{c(a+b)} + 1 + \frac{b(c-a)}{a(b+c)} + 1 + \frac{c(a-b)}{b(c+a)} + 1 \geq 3.$$

Equivalently, this can be written as

$$\frac{b(c+a)}{c(a+b)} + \frac{c(a+b)}{a(b+c)} + \frac{a(b+c)}{b(c+a)} \geq 3.$$

This follows by AM-GM inequality.

■

242. Let $a_1, a_2, a_3, \ldots, a_n$ be $n > 2$ positive real numbers such that $a_1 + a_2 + a_3 + \cdots + a_n = 1$. Prove that

$$\sum_{j=1}^{n} \frac{a_1a_2\cdots a_{j-1}a_{j+1}\cdots a_n}{a_j + n - 2} \leq \frac{1}{(n-1)^2}.$$

Solution: Suppose $n \geq 4$. Then we have

$$\begin{aligned} \frac{a_1a_2\cdots a_{j-1}a_{j+1}\cdots a_n}{a_j+n-2} &\leq \frac{(a_1+a_2+\cdots+a_{j-1}+a_{j+1}+\cdots+a_n)^{(n-1)}}{(a_j+n-2)(n-1)^{(n-1)}} \\ &< \frac{1}{(n-1)^{(n-1)}(n-2)}. \end{aligned}$$

Therefore we obtain

$$\sum_{j=1}^{n} \frac{\prod_{k\neq j} a_k}{a_j+n-2} < \frac{n}{(n-1)^{(n-1)}(n-2)} \leq \frac{n}{(n-2)(n-1)^3},$$

since $n \geq 4$. Thus it is enough to prove that

$$\frac{n}{(n-2)(n-1)^3} \leq \frac{1}{(n-1)^2}.$$

Equivalently, we have to show that $b^2 - 4n + 2 \geq 0$. This follows from $n \geq 4$.

Suppose $n = 3$. We have to show that

$$\frac{bc}{1+a} + \frac{ca}{1+b} + \frac{ab}{1+c} \leq \frac{1}{4},$$

whenever a, b, c are positive and $a + b + c = 1$. Equivalently, we have to prove that

$$bc + ca + ab \leq \frac{1}{4} + abc\left(\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}\right),$$

whenever $a + b + c = 1$. Using AM-HM inequality, we have

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \geq \frac{9}{3 + (a+b+c)} = \frac{9}{4}.$$

Hence we need to prove that

$$ab + bc + ca \leq \frac{9abc}{4}.$$

By symmetry, we may assume $c \leq 1/3$. Eliminating a using $a + b + c = 1$, we have to prove that

$$bc + c(1-b-c) + b(1-b-c) \leq \frac{9}{4}bc(1-b-c),$$

for all positive b, c and $b + c < 1$. Treating this as a quadratic in b, we obtain

$$(4 - 9c)b^2 - (9c^2 - 13c + 4)b + (4c^2 - 4c + 1) \geq 0.$$

This must hold for all $b > 0$ and $b < 1$. It is sufficient to show that its discriminant is negative. Computing the discriminant, we get

$$D = 81c\left(c - \frac{1}{3}\right)^2\left(c - \frac{9}{4}\right).$$

Since $c \leq 1/3$, we see that $D < 0$.

■

243. Determine the largest value of k such that the inequality

$$\left(k + \frac{a}{b}\right)\left(k + \frac{b}{c}\right)\left(k + \frac{c}{ba}\right) \geq \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right)$$

holds for all positive real numbers a, b, c.

Solution: If we take $a = b = c$, we see that $k \leq \sqrt[3]{9} - 1$. We show that $\sqrt[3]{9} - 1$ is the largest value of k for which the given inequality holds for every positive reals a, b, c. Let us write

$$\alpha = \frac{a}{b} + \frac{b}{c} + \frac{c}{a}, \quad \beta = \frac{b}{a} + \frac{c}{b} + \frac{a}{c}.$$

Now we know by AM-GM inequality that $\alpha \geq 3$ and $\beta \geq 3$. Thus for any $k \geq 0$, we see that the following inequalities hold:

$$9(k^3 + 1) \leq (k^3 + 1)\alpha\beta, \quad 9k^2\alpha \leq 3k^2\alpha\beta, \quad 9k\beta \leq 3k\alpha\beta.$$

Adding these three inequalities, we get

$$9(k^3 + 1 + k^2\alpha + k\beta) \leq (k+1)^3\alpha\beta.$$

This is the same as

$$9\left(k + \frac{a}{b}\right)\left(k + \frac{b}{c}\right)\left(k + \frac{c}{a}\right) \leq (k+1)^3\alpha\beta.$$

Taking $k = \sqrt[3]{9} - 1$, this reduces to

$$9\left(k + \frac{a}{b}\right)\left(k + \frac{b}{c}\right)\left(k + \frac{c}{a}\right) \leq 9\alpha\beta.$$

This shows that $\sqrt[3]{9} - 1$ is the largest value of k satisfying the given inequality.

■

244. Let $x_1, x_2, x_3, \ldots, x_n$ be $n \geq 3$ positive real numbers. Prove that

$$\frac{x_1x_3}{x_1x_3 + x_2x_4} + \frac{x_2x_4}{x_2x_4 + x_3x_5} + \cdots + \frac{x_{n-1}x_1}{x_{n-1}x_1 + x_nx_2} + \frac{x_nx_2}{x_nx_2 + x_1x_3} \leq n - 1.$$

Solution: Each term is of the form

$$\frac{x_{j-1}x_{j+1}}{x_{j-1}x_{j+1} + x_jx_{j+2}},$$

where $1 \leq j \leq n$, $x_0 = x_n$ and $x_{n+1} = x_1$. Let us introduce

$$y_j = \frac{x_{j-1}x_{j+1}}{x_jx_{j+2}}, \quad 1 \leq j \leq n.$$

The inequality is transformed to

$$\frac{y_1}{y_1 + 1} + \frac{y_2}{y_2 + 1} + \frac{y_3}{y_3 + 1} + \cdots + \frac{y_n}{y_n + 1} \leq n - 1,$$

where $y_1, y_2, y_3, \ldots, y_n$ are positive real numbers whose product is 1. This can be re written in the form

$$\frac{1}{y_1 + 1} + \frac{1}{y_2 + 1} + \frac{1}{y_3 + 1} + \cdots + \frac{1}{y_n + 1} \geq 1.$$

We use induction on n to prove this. For $n = 3$, we have to show that

$$\frac{1}{y_1+1} + \frac{1}{y_2+1} + \frac{1}{y_3+1} \geq 1,$$

when $y_1 y_2 y_3 = 1$. Substituting $y_3 = 1/y_1 y_2$, the inequality is

$$\frac{1}{y_1+1} + \frac{1}{y_2+1} + \frac{y_1 y_2}{1+y_1 y_2} \geq 1$$

for all positive real numbers y_1, y_2. This follows from the inequality

$$\frac{1}{1+y_1} + \frac{1}{1+y_2} \geq \frac{1}{1+y_1 y_2},$$

which can be easily verified on cross multiplication. This proves the result for $n = 3$. Suppose the result holds for n numbers. If we take $n+1$ numbers $y_1, y_2, y_3, \ldots, y_n, y_{n+1}$, we obtain

$$\begin{aligned} \frac{1}{y_1+1} + \frac{1}{y_2+1} + \frac{1}{y_3+1} + \cdots + \frac{1}{y_{n-1}+1} + \frac{1}{y_n+1} + \frac{1}{y_{n+1}+1} \\ \geq \frac{1}{y_1+1} + \frac{1}{y_2+1} + \frac{1}{y_3+1} + \cdots + \frac{1}{y_{n-1}+1} + \frac{1}{y_n y_{n+1}+1}, \end{aligned}$$

because we know that

$$\frac{1}{y_n+1} + \frac{1}{y_{n+1}+1} \geq \frac{1}{y_n y_{n+1}+1}.$$

Now consider n numbers $y_1, y_2, y_3, \ldots, y_{n-1}, y_n y_{n+1}$. Their product is 1. Hence we can apply induction hypothesis to get

$$\frac{1}{y_1+1} + \frac{1}{y_2+1} + \frac{1}{y_3+1} + \cdots + \frac{1}{y_{n-1}+1} + \frac{1}{y_n y_{n+1}+1} \geq 1.$$

This completes induction and proves the result.

■

245. Let $a_1, a_2, a_3, \ldots, a_{2017}$ be positive real numbers. Prove that

$$\sum_{j=1}^{2017} \frac{a_j}{a_{j+1} + a_{j+2} + \cdots + a_{j+1008}} \geq \frac{2017}{1008},$$

where the indices are taken modulo 2017.

Solution: Let us consider the sum

$$S = \sum_{j=1}^{2017} \frac{a_j}{a_{j+1} + a_{j+2} + \cdots + a_{j+1008}}.$$

We begin with the following observation as a consequence of Cauchy-Schwarz inequality: for positive real numbers $x_1, x_2, x_3, \ldots, x_n$ and $y_1, y_2, y_3, \ldots, y_n$,

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \frac{x_3^2}{y_3} + \cdots + \frac{x_n^2}{y_n} \geq \frac{(x_1 + x_2 + \cdots + x_n)^2}{y_1 + y_2 + y_3 + \cdots + y_n}.$$

Here equality holds if and only if

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} = \frac{x_3}{y_3} = \cdots = \frac{x_n}{y_n}.$$

Thus

$$S \geq \frac{(a_1 + a_2 + a_3 + \cdots + a_n)^2}{\sum_{j=1}^{2017} a_j(a_{j+1} + a_{j+2} + \cdots + a_{j+1008})}.$$

Consider the sum in the denominator:

$$K = \sum_{j=1}^{2017} a_j(a_{j+1} + a_{j+2} + \cdots + a_{j+1008}).$$

We have

$$\begin{aligned} K &= \frac{1}{2} \sum_{\substack{j,k=1 \\ j \neq k}}^{2017} a_j a_k \\ &= \frac{1}{2}\Big((a_1 + a_2 + \cdots + a_{2017})^2 - (a_1^2 + a_2^2 + a_3^2 + \cdots + a_{2017}^2)\Big) \\ &\leq \frac{1}{2}\left(p^2 - \frac{1}{2017}p^2\right), \end{aligned}$$

where $p = a_1 + a_2 + a_3 + \cdots + a_{2017}$. We have used Cauchy-Schwarz inequality in the last leg. Thus we obtain

$$S \geq \frac{p^2}{\frac{1}{2}p^2\left(1 - \frac{1}{2017}\right)} = \frac{2 \times 2017}{2016} = \frac{2017}{1008}.$$

■

246. Let a, b, c be three positive real numbers such that $ab + bc + ca = 1$. Prove that

$$\sqrt{a + \frac{1}{a}} + \sqrt{b + \frac{1}{b}} + \sqrt{c + \frac{1}{c}} \geq 2(\sqrt{a} + \sqrt{b} + \sqrt{c}).$$

Solution: First we observe that

$$a + \frac{1}{a} = a + \frac{ab+bc+ca}{a} = a+b+c+\frac{bc}{a}$$
$$= (b+c) + \left(a + \frac{bc}{a}\right) \geq b+c+2\sqrt{bc} = \left(\sqrt{b}+\sqrt{c}\right)^2.$$

Thus we obtain

$$\sqrt{a+\frac{1}{a}} + \sqrt{b+\frac{1}{b}} + \sqrt{c+\frac{1}{c}} \geq 2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right).$$

■

247. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{a^3+2}{b+2} + \frac{b^3+2}{c+2} + \frac{c^3+2}{a+2} \geq 3.$$

Solution: Observe that

$$\frac{a^3+2}{b+2} = \frac{a^3+1+1}{b+2} \geq \frac{3a}{b+2},$$

with similar inequalities being true for the other two terms. Therefore, we obtain

$$\frac{a^3+2}{b+2} + \frac{b^3+2}{c+2} + \frac{c^3+2}{a+2} \geq 3\left(\frac{a}{b+2} + \frac{b}{c+2} + \frac{c}{a+2}\right).$$

Using Cauchy-Schwarz inequality, we also have

$$(a+b+c)^2 \leq \left(\frac{a^2}{a(b+2)} + \frac{b^2}{b(c+2)} + \frac{c^2}{c(a+2)}\right)\left(a(b+2)+b(c+2)+c(a+2)\right).$$

This gives

$$\frac{a}{b+2} + \frac{b}{c+2} + \frac{c}{a+2} \geq \frac{(a+b+c)^2}{ab+bc+ca+2(a+b+c)}.$$

But

$$ab+bc+ca \leq \frac{1}{3}(a+b+c)^2 = 3.$$

Therefore

$$\frac{a}{b+2} + \frac{b}{c+2} + \frac{c}{a+2} \geq \frac{9}{3+6} = 1.$$

This gives

$$\frac{a^3+2}{b+2} + \frac{b^3+2}{c+2} + \frac{c^3+2}{a+2} \geq 3.$$

Equality holds if and only if $a=b=c=1$. ■

248. Let a, b, c, d be real numbers such that $a^2 + b^2 + c^2 + d^2 = 4$. Prove that $(2 + a)(2 + b) \geq cd$.

Solution: We observe that

$$\begin{aligned} 2cd \leq c^2 + d^2 = 4 - a^2 - b^2 = 8 - (4 + a^2 + b^2) \\ = 2(2 + a)(2 + b) - (4a + 4b + 2ab + 4 + a^2 + b^2) \\ = 2(2 + a)(2 + b) - (a + b + 2)^2. \end{aligned}$$

This shows that $cd \leq (2 + a)(2 + b)$. ■

249. Find all real λ such that

$$\frac{a+b}{2} \geq \lambda\sqrt{ab} + (1 - \lambda)\sqrt{\frac{a^2 + b^2}{2}}$$

holds for all positive real numbers a, b.

Solution: Taking $a = b$, the inequality is

$$a \geq \lambda a + (1 - \lambda)a,$$

which is a true statement for any $\lambda \in \mathbb{R}$. Hence we may assume $a \neq b$. After some algebraic manipulation, we see that the inequality is equivalent to

$$2\lambda \geq 1 - \left(\frac{(a - b)^2}{\left(\sqrt{2a^2 + 2b^2} + (a + b)\right)\left(a + b + 2\sqrt{ab}\right)} \right).$$

This must hold for all $a \neq b$. Take $a = 1$ and $b = 1 + \epsilon$, where ϵ is an arbitrarily small positive real number. Then we must have

$$2\lambda \geq 1 - \frac{\epsilon^2}{16}.$$

Since this is true for every $\epsilon > 0$, it follows that $\lambda \geq \frac{1}{2}$.

Consider $\lambda = \frac{1}{2}$. The inequality is

$$a + b \geq \sqrt{ab} + \sqrt{\frac{a^2 + b^2}{2}}.$$

After squaring and simplifying, this can be rewritten as

$$(a + b)^2 \geq 4\sqrt{ab\left(\frac{a^2 + b^2}{2}\right)}.$$

This is equivalent to $(a+b)^4 \geq 8ab(a^2+b^2)$. But this follows from AM-GM inequality:

$$8ab(a^2+b^2) = 4 \times (a^2+b^2) \times (2ab) \leq 4\left(\frac{a^2+b^2+2ab}{2}\right)^2 = (a+b)^4.$$

Now we show that

$$f(\lambda) = \lambda\sqrt{ab} + (1-\lambda)\sqrt{\frac{a^2+b^2}{2}}$$

is a decreasing function of λ. If $\lambda_1 < \lambda_2$, we have

$$f(\lambda_2) - f(\lambda_1) = (\lambda_2 - \lambda_1)\left(\sqrt{ab} - \sqrt{\frac{a^2+b^2}{2}}\right)$$

But

$$\sqrt{ab} - \sqrt{\frac{a^2+b^2}{2}} \leq 0.$$

Therefore, for any $\lambda > \frac{1}{2}$, we see that

$$f(\lambda) \leq f(1/2) \leq \frac{a+b}{2},$$

as we have observed earlier. Hence it follows that for all $\lambda \geq \frac{1}{2}$, the given inequality holds.

■

250. Let a, b, c, d be real numbers having absolute value greater than 1 and such that $abc + abd + acd + bcd + a + b + c + d = 0$. Prove that

$$\frac{1}{a-1} + \frac{1}{b-1} + \frac{1}{c-1} + \frac{1}{d-1} > 0.$$

Solution: Let us introduce new symbols x, y, z, w:

$$x = \frac{a+1}{a-1}, \quad y = \frac{b+1}{b-1}, \quad z = \frac{c+1}{c-1}, \quad w = \frac{d+1}{d-1}.$$

Since $|a| > 1$, we see that $x > 0$. Similarly, $y > 0, z > 0$ and $w > 0$. The condition of the problem translates to $xyzw = 1$. Observe that

$$x - 1 = \frac{a+1}{a-1} - 1 = \frac{2}{a-1},$$

and similar expressions for $y-1, z-1$ and $w-1$. Thus we need to prove

$$(x-1) + (y-1) + (z-1) + (w-1) > 0.$$

This is equivalent to $x+y+z+w > 4$. Using AM-GM inequality, we have

$$x+y+z+w \geq 4(xyzw)^{1/4} = 4.$$

Here equality holds only if $x = y = z = w$. This implies $a = b = c = d$. The given condition now gives $a^3 + a = 0$ forcing $a = 0$. But this contradicts $|a| > 1$. We conclude that $x+y+z+w > 4$ and this leads to the required inequality.

■

251. Show that

$$\frac{1}{x+y+1} - \frac{1}{(x+1)(y+1)} < \frac{1}{11},$$

for all positive real numbers x, y.

Solution: We start with the inequality

$$4(x+1)(y+1) \leq (x+y+2)^2.$$

Thus it is sufficient to prove that

$$\frac{1}{x+y+1} - \frac{4}{(x+y+2)^2} < \frac{1}{11},$$

for $x > 0, y > 0$. Introducing $x+y+1 = t$, we have to prove that

$$\frac{1}{t} - \frac{4}{(t+1)^2} < \frac{1}{11}$$

for $t > 1$. Equivalently, we have to prove that $t^3 - 9t^2 + 23t - 11 > 0$ for $t > 1$. This takes the form $(t-1)(t-3)(t-5) + 4 > 0$ for $t > 1$. Note that $(t-1)(t-3)(t-5) \leq 0$ only when $t \in [3, 5]$. But in this case $0 < t-1 \leq 4$ and $(t-3)(t-5) = (t-4)^2 - 1 \geq -1$. Hence $(t-1)(t-3)(t-5) \geq -4$. Equality in the first inequality holds when $t = 5$. But then $(t-3)(t-5) = 0$. In the second inequality, equality occurs when $t = 4$. But then $t - 1 = 3$ and hence $(t-1)(t-3)(t-5) = -3$. Thus we see that $(t-1)(t-3)(t-5) + 4 > 0$ for $t > 1$. ■

252. Let a, b, c be three positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq \frac{3}{2}.$$

Solution: We use rearrangement inequality. Since the inequality is cyclically symmetric, we may assume $a \leq b \leq c$ or $a \geq b \geq c$. In both cases, we have

$$A = \frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq \frac{1}{c(a+b)} + \frac{1}{a(b+c)} + \frac{1}{b(c+a)} = B.$$

Therefore

$$\begin{aligned}
2A &\geq A+B \\
&= \frac{1}{b(a+b)}+\frac{1}{c(b+c)}+\frac{1}{a(c+a)}+\frac{1}{c(a+b)}+\frac{1}{a(b+c)}+\frac{1}{b(c+a)} \\
&= \frac{b+c}{bc(a+b)}+\frac{c+a}{ca(b+c)}+\frac{a+b}{ab(c+a)} \\
&\geq 3.
\end{aligned}$$

We have used AM-GM inequality and the given condition that $abc = 1$. It follows that $A \geq \frac{3}{2}$. Therefore

$$\frac{1}{b(a+b)}+\frac{1}{c(b+c)}+\frac{1}{a(c+a)} \geq \frac{3}{2}.$$

■

253. Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{a^2}{(b+c)^3}+\frac{b^2}{(c+a)^3}+\frac{c^2}{(a+b)^3} \geq \frac{9}{8}$$

Solution: We can write the inequality in the form

$$\frac{a^2}{(1-a)^3}+\frac{b^2}{(1-b)^3}+\frac{c^2}{(1-c)^3} \geq \frac{9}{8},$$

for positive reals a, b, c with $a+b+c=1$. Consider the function $f(x) = x^2/(1-x)^3$. We observe that

$$f'(x) = \frac{2x}{(1-x)^4}, \quad f''(x) = \frac{2(1+3x)}{(1-x)^5}.$$

Therefore $f''(x) > 0$ for $0 < x < 1$. It follows that $f(x)$ is a convex function on $(0,1)$. By Jensen's inequality, we get

$$f\left(\frac{a+b+c}{3}\right) \leq \frac{1}{3}\Big(f(a)+f(b)+f(c)\Big),$$

for all $a, b, c \in (0,1)$. This gives

$$\frac{a^2}{(1-a)^3}+\frac{b^2}{(1-b)^3}+\frac{c^2}{(1-c)^3} \geq 3f\left(\frac{a+b+c}{3}\right) = 3f\left(\frac{1}{3}\right) = \frac{9}{8}.$$

■

254. Suppose a, b, c are positive reals such that $ab+bc+ca \geq a+b+c$. Prove that

$$(a+b+c)(ab+bc+ca)+3abc \geq 4(ab+bc+ca).$$

Solution: Using Cauchy-Schwarz inequality, we have

$$\begin{aligned} 4(ab+bc+ca)^2 &= \big(a(b+c)+b(c+a)+c(a+b)\big)^2 \\ &\le (a+b+c)\big(a(b+c)^2+b(c+a)^2+c(a+b)^2\big). \end{aligned}$$

On the other hand $a+b+c \le ab+bc+ca$ gives

$$(ab+bc+ca)^2 \ge (ab+bc+ca)(a+b+c),$$

and

$$a(b+c)^2+b(c+a)^2+c(a+b)^2 = (a+b+c)(ab+bc+ca)+3abc.$$

It follows that

$$(a+b+c)(ab+bc+ca)+3abc \ge 4(ab+bc+ca).$$

■

255. Let a, b, c, d be four real numbers such that $a+b+c+d=0$. Prove that

$$(ab+ac+ad+bc+bd+cd)^2+12 \ge 6(abc+abd+acd+bcd).$$

Solution: Let us introduce $x=a+1$, $y=b+1$, $z=c+1$ and $w=d+1$. Then $x+y+z+w=4$. Then we see that

$$\begin{aligned} &(ab+ac+ad+bc+bd+cd) \\ &\quad= (xy+xz+xw+yz+yw+zw)-3(x+y+z+w)+6 \\ &\quad= (xy+xz+xw+yz+yw+zw)-6; \end{aligned}$$

and

$$\begin{aligned} &(abc+abd+acd+bcd) \\ &= (xyz+xyw+xzw+yzw)-2(xy+xz+xw+yz+yw+zw) \\ &\qquad +3(x+y+z+w)-4 \\ &= (xyz++xyw+xzw+yzw)-2(xy+xz+xw+yz+yw+zw)+8. \end{aligned}$$

The inequality to be proved is

$$(xy+xz+xw+yz+yw+zw)^2 \ge 6(xyz+xyw+xzw+yzw).$$

If $xyz+xyw+xzw+yzw<0$, the result is immediate. Suppose $xyz+xyw+$

$xzw + yzw \geq 0$. We write

$$\begin{aligned}
&(xy + xz + xw + yz + yw + zw)^2 \\
&\quad = \quad (xy)^2 + (xz)^2 + (xw)^2 + (yz)^2 + (yw)^2 + (zw)^2 + 2xyz(x + y + z) \\
&\qquad\quad +2xyw(x + y + w) + 2xzw(x + z + w) + 2yzw(y + z + w) + 6xyzw \\
&\quad = \quad (xy - zw)^2 + (2xyz + 2xyw + 2xzw + 2yzw)(x + y + z + w) \\
&\qquad\qquad + (xz)^2 + (xw)^2 + (yz)^2 + (yw)^2 \\
&\quad \geq \quad 8(xyz + xyw + xzw + yzw) \\
&\quad > \quad 6(xyz + xyw + xzw + yzw).
\end{aligned}$$

■

256. Consider the expression

$$P = \frac{x^3y^4z^3}{(x^4 + y^4)(xy + z^2)^3} + \frac{y^3z^4x^3}{(y^4 + z^4)(yz + x^2)^3} + \frac{z^3x^4y^3}{(z^4 + x^4)(zx + y^2)^3}.$$

Find the maximum value of P when x, y, z vary over the set of all positive real numbers.

Solution: We show that the maximum value of P is $\frac{3}{16}$. First we observe that

$$x^4 + y^4 \geq xy(x^2 + y^2), \quad (xy + z^2)^2 \geq 4xyz^2.$$

Therefore, it follows that

$$\begin{aligned}
(x^4 + y^4)(xy + z^2)^3 \quad &\geq \quad 4x^2y^2z^2(x^2 + y^2)(x^2 + yz) \\
&\geq \quad 4x^2y^2z^2(x^2z^2 + y^2z^2 + 2x^2y^2).
\end{aligned}$$

We have used the fact that $x^2 + y^2 \geq 2xy$. This gives

$$\begin{aligned}
\frac{x^3y^4z^3}{(x^4 + y^4)(xy + z^2)^3} \quad &\leq \quad \frac{x^3y^2z^3}{4x^2y^2z^2(x^2z^2 + y^2z^2 + 2x^2y^2)} \\
&= \quad \frac{xy^2z}{4(x^2z^2 + y^2z^2 + 2x^2y^2)}.
\end{aligned}$$

Thus it is sufficient to prove that

$$\sum_{\text{cyclic}} \frac{xy^2z}{4(x^2z^2 + y^2z^2 + 2x^2y^2)} \leq \frac{3}{4},$$

for all positive reals x, y, z. Let us introduce $a = xy$, $b = yz$, $c = zx$. The required inequality is now

$$\sum_{\text{cyclic}} \frac{ab}{2a^2 + b^2 + c^2} \leq \frac{3}{4},$$

for positive real numbers a, b, c. If $a \geq b \geq c$, we see that $ab \geq ac \geq bc$ and

$$\frac{1}{2c^2+a^2+b^2} \geq \frac{1}{2b^2+c^2+a^2} \geq \frac{1}{2a^2+b^2+c^2}.$$

Hence rearrangement inequality gives

$$\sum_{\text{cyclic}} \frac{ab}{2a^2+b^2+c^2} \leq \sum_{\text{cyclic}} \frac{ab}{a^2+b^2+2c^2}.$$

On the other hand AM-GM inequality leads to

$$4\sum_{\text{cyclic}} \frac{ab}{a^2+b^2+2c^2} \leq \sum_{\text{cyclic}} \frac{(a+b)^2}{a^2+b^2+2c^2} \leq \sum_{\text{cyclic}} \left(\frac{a^2}{c^2+a^2} + \frac{b^2}{c^2+b^2}\right) = 3.$$

We have used the following:

$$\frac{(a+b)^2}{\lambda+\mu} \leq \frac{a^2}{\lambda} + \frac{b^2}{\mu},$$

which can be easily verified. Thus it follows that $P \leq \frac{3}{16}$. Equality holds when $x = y = z = 1$. ■

257. Let a, b, c be the sides of an acute-angled triangle. Prove that

$$\sqrt{a^2+b^2-c^2} + \sqrt{b^2+c^2-a^2} + \sqrt{c^2+a^2-b^2} \leq \sqrt{3(ab+bc+ca)}.$$

Solution: We introduce new positive numbers ;

$$x^2 = a^2+b^2-c^2, \quad y^2 = b^2+c^2-a^2, \quad z^2 = c^2+a^2-b^2.$$

This is possible since the triangle is acute. We can solve for a, b, c;

$$a = \sqrt{\frac{z^2+x^2}{2}}, \quad b = \sqrt{\frac{x^2+y^2}{2}}, \quad c = \sqrt{\frac{y^2+z^2}{2}}.$$

The inequality to be proved is:

$$(x+y+z)^2 \leq \\ \frac{3}{2}\left(\sqrt{(x^2+y^2)(z^2+x^2)} + \sqrt{(y^2+z^2)(x^2+y^2)} + \sqrt{(z^2+x^2)(y^2+z^2)}\right).$$

Cauchy-Schwarz inequality gives

$$a^2 + bc \leq \sqrt{a^2+b^2}\sqrt{a^2+c^2}.$$

Thus we obtain

$$\sqrt{\frac{3}{2}\left(\sqrt{(x^2+y^2)(z^2+x^2)}+\sqrt{(y^2+z^2)(x^2+y^2)}+\sqrt{(z^2+x^2)(y^2+z^2)}\right)}$$
$$\geq \sqrt{\frac{3}{2}(x^2+y^2+z^2+xy+yz+zx)}.$$

Therefore it is enough to prove that

$$2(x+y+z)^2 \leq 3(x^2+y^2+z^2+xy+yz+zx).$$

This reduces to $x^2+y^2+z^2 \geq xy+yz+zx$, which follows from

$$(x-y)^2+(y-z)^2+(z-x)^2 \geq 0.$$

Alternately, we can use trigonometry and reduce the problem. Let α, β, γ be the angles of the triangle opposite to the sides a, b, c, respectively. Then the **lhs** of the inequality is

$$\sqrt{2ab\cos\gamma}+\sqrt{2bc\cos\alpha}+\sqrt{2ca\cos\beta}.$$

and by Cauchy-Schwarz inequality we see that this is not larger than

$$\sqrt{(ab+bc+ca)}\sqrt{2(\cos\alpha+\cos\beta+\cos\gamma)}.$$

Thus it is good enough to prove that

$$\sqrt{(ab+bc+ca)}\sqrt{2(\cos\alpha+\cos\beta+\cos\gamma)} \leq \sqrt{3(ab+bc+ca)}.$$

Equivalently, this reduces to the geometrical inequality:

$$\cos\alpha+\cos\beta+\cos\gamma \leq \frac{3}{2}.$$

This is a standard inequality; see **3.4.9**. ■

258. Let $x_1, x_2, x_3, \ldots, x_n$ be Positive real numbers such that $x_1x_2x_3\cdots x_n = 1$. Let $S = x_1^3+x_2^3+x_3^3+\cdots+x_n^3$. Prove that

$$\frac{x_1}{S-x_1^3+x_1^2}+\frac{x_2}{S-x_2^3+x_2^2}+\frac{x_3}{S-x_3^3+x_3^2}+\cdots+\frac{x_n}{S-x_n^3+x_n^2} \leq 1.$$

Solution: By the Cauchy-Schwarz inequality, we get

$$(x_1^2+x_2^2+x_3^2+\cdots+x_n^2)^2$$
$$\leq (x_1^2+x_2^3+x_3^3+\cdots x_n^3)(x_1^2+x_2+x_3+\cdots+x_n).$$

Thus it follows that

$$S - x_1^3 + x_1^2 \geq \frac{(x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2)^2}{(x_1^2 + x_2 + x_3 + \cdots + x_n)}.$$

This gives

$$\frac{1}{S - x_1^3 + x_1^2} \leq \frac{(x_1^2 + x_2 + x_3 + \cdots + x_n)}{(x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2)^2}.$$

Thus we see that

$$\sum_{j=1}^{n} \frac{x_j}{S - x_j^3 + x_j^2} \leq \frac{\sum_{j=1}^{n} x_j(x_1 + x_2 + \cdots + x_{j-1} + x_j^2 + x_{j+1} + \cdots x_n)}{(x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2)^2}.$$

But we can write

$$\sum_{j=1}^{n} x_j(x_1 + x_2 + \cdots + x_{j-1} + x_j^2 + x_{j+1} + \cdots + x_n) = \sum_{j=1}^{n} x_j^3 + 2 \sum_{\substack{j,k=1 \\ j<k}}^{n} x_j x_k.$$

Here we use the following result.

Lemma: If $a_1, a_2, \ldots, a_n$ are n positive real numbers such that $a_1 a_2 \cdots a_n = 1$, then

$$\sum_{j=1}^{n} a_j^m \leq \sum_{j=1}^{n} a_j^{m+1},$$

for any integer $m \geq 1$.

Proof of Lemma: We have for any j, $1 \leq j \leq n$,

$$\begin{aligned} & a_1^{m+1} + a_2^{m+1} + a_3^{m+1} + \cdots + a_{j-1}^{m+1} + (nm+1)a_j^{m+1} + a_{j+1}^{m+1} + \cdots + a_n^{m+1} \\ & \quad \geq n(m+1)\Big(a_1^{m+1} a_2^{m+1} a_3^{m+1} \cdots a_{j-1}^{m+1} a_j^{(nm+1)} a_{j+1}^{m+1} \cdots a_n^{m+1}\Big)^{1/n(m+1)} \end{aligned}$$

Since the product of $a_1, a_2, \ldots, a_n$ is 1, this reduces

$$\begin{aligned} & a_1^{m+1} + a_2^{m+1} + a_3^{m+1} + \cdots + a_{j-1}^{m+1} + (nm+1)a_j^{m+1} + a_{j+1}^{m+1} + \cdots + a_n^{m+1} \\ & \qquad \geq n(m+1)\Big(a_j^{nm(m+1))}\Big)^{1/(n(m+1))} = n(m+1)a_j^m. \end{aligned}$$

Summing over j, we get

$$n(m+1) \sum_{j=1}^{n} a_j^{m+1} \geq n(m+1) \sum_{j=1}^{n} a_j^m.$$

This completes the proof of the lemma.

By lemma, it follows that

$$\sum_{j=1}^{n} x_j^3 \le \sum_{j=1}^{n} x_j^4, \quad 2\sum_{\substack{j,k=1\\ j<k}}^{n} x_j x_k \le 2\sum_{\substack{j,k=1\\ j<k}}^{n} x_j^2 x_k^2.$$

Therefore

$$\begin{aligned}\sum_{j=1}^{n} x_j(x_1+x_2+\cdots+x_{j-1}+x_j^2+x_{j+1}+\cdots+x_n) &\le \sum_{j=1}^{n} x_j^4 + 2\sum_{\substack{j,k=1\\ j<k}}^{n} x_j^2 x_k^2 \\ &= \left(\sum_{j=1}^{n} x_j^2\right)^2.\end{aligned}$$

This estimation shows that

$$\sum_{j=1}^{n} \frac{x_j}{S - x_j^3 + x_j^2} \le 1.$$

Here equality can occur when $x_1 = x_2 = x_3 = \ldots = x_n = 1$.

■

259. Let $a_1, a_2, a_3, \ldots, a_n$ be $n(>1)$ positive real numbers whose sum is 1. Define $b_k = \frac{a_k^2}{\sum_{j=1}^{n} a_j^2}$, $1 \le k \le n$. Prove that

$$\sum_{k=1}^{n} \frac{a_k}{1-a_k} \le \sum_{k=1}^{n} \frac{b_k}{1-b_k}.$$

Solution: For $p \ge 1$, consider the function $f(x) = x^p$ defined on $[0, \infty)$. This is a strictly increasing function. In fact

$$f'(x) = px^{p-1} > 0, \text{ for } x > 0.$$

Hence, it is convex and we can apply Jensen's inequality. If $x_1, x_2, \ldots, x_n$ are non-negative and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are non-negative real numbers such that $\sum_{j=1}^{n} \alpha_j = 1$, then

$$\left(\sum_{j=1}^{n} \alpha_j x_j\right)^p \le \sum_{j=1}^{n} \alpha_j x_j^p.$$

Taking $x_j = \alpha_j = a_j$ and $p = 2m-1$ with $n \ge 1$, we obtain

$$\left(\sum_{j=1}^{n} a_j^2\right)^{2m-1} \le \sum_{j=1}^{n} a_j^{2m}.$$

Similarly, taking $x_j = a_j^{n-1}$, $\alpha_j = a_j$, $1 \le j \le n$ and $p = (2m-1)/(m-1)$ we get

$$\left(\sum_{j=1}^{n} a_j^m\right)^{(2m-1)/(m-1)} \le \sum_{j=1}^{n} a_j^{2m}.$$

Combining both the inequalities, we obtain

$$\left(\sum_{j=1}^{n} a_j^2\right)^m \left(\sum_{j=1}^{n} a_j^m\right) \le \sum_{j=1}^{n} a_j^{2m}.$$

This is valid for any $m \ge 1$. Using

$$a_j^2 = b_j \left(\sum_{k=1}^{n} a_k^2\right),$$

for $1 \le j \le n$, we obtain

$$\sum_{k=1}^{n} a_k^m \le \sum_{k=1}^{n} b_k^m,$$

for all $m \ge 1$. Now we use the known result

$$\sum_{m=1}^{\infty} x^m = \frac{x}{1-x}$$

for $|x| < 1$. This leads to

$$\sum_{k=1}^{n} \frac{a_k}{1-a_k} \le \sum_{k=1}^{n} \frac{b_k}{1-b_k}.$$

Alternate solution: We may assume that $a_1 \ge a_2 \ge a_3 \ge \cdots \ge a_n$. Let

$$S = \sum_{j=1}^{n} a_j^2.$$

Then $b_k = a_k^2/S$ so that $b_1 \ge b_2 \ge b_3 \ge \cdots \ge b_n$. We observe that $b_1 + b_2 + \cdots + b_n = 1$. For $1 \le k \le n$, let

$$D_k = \sum_{j=1}^{k} (b_j - a_j).$$

Observe $D_n = 0$, since

$$\sum_{j=1}^{n} a_j = \sum_{j=1}^{n} b_j = 1.$$

For $1 \leq k < n$, we have

$$S \cdot D_k = \sum_{j=1}^{k} \sum_{l=1}^{n} a_j^2 a_l - \sum_{j=1}^{k} \sum_{l=1}^{n} a_j a_l^2 = \sum_{\substack{1 \leq j < l \\ l < k \leq n}} a_j a_l (a_j - a_l) \geq 0.$$

Here we have used the ordering of a's. Since $S > 0$, it follows that $D_k\ geq 0$ for $1 \leq k \leq n$. Since $a_1 a_n (a_1 - a_n) > 0$ unless $a_1 = a_2 = \ldots = a_n$, we see that $D_k > 0$ for $1 \leq k < n$ unless $a_1 = a_2 = \ldots = a_n$. Now for any $a \neq 1$ and $b \neq 1$, we have

$$\frac{b}{1-b} - \frac{a}{1-a} = \frac{(b-a)}{(1-a)(1-b)}.$$

Taking

$$c_j = \frac{1}{(1-a_j)(1-b_j)}, \quad 1 \leq j \leq n,$$

we obtain

$$\begin{aligned} D = \sum_{j=1}^{n} \frac{b_j}{1-b_j} - \sum_{j=1}^{n} \frac{a_j}{1-a_j} &= \sum_{j=1}^{n} c_j (b_j - a_j) \\ &= \sum_{j=1}^{n-1} (c_j - c_{j+1}) D_j, \end{aligned}$$

where we have used $D_n = 0$. Since $a_j \geq a_{j+1}$ and $b_j \geq b_{j+1}$, we can check that $c_j \geq c_{j+1}$, for $1 \leq j \leq n$. It follows the sum above is non-negative. Thus we obtain

$$\sum_{j=1}^{n} \frac{b_j}{1-b_j} - \sum_{j=1}^{n} \frac{a_j}{1-a_j} \geq 0.$$

■

260. Suppose a, b, c, d are positive real numbers. Prove that

$$\sum_{\text{cyclic}} \frac{a^4}{a^3 + a^2 b + ab^2 + b^3} \geq \frac{a+b+c+d}{4}.$$

Solution: We first observe that

$$\sum_{\text{cyclic}} \frac{a^4 - b^4}{a^3 + a^2 b + ab^2 + b^3} = \sum_{\text{cyclic}} (a-b) = 0.$$

Hence

$$\frac{a^4}{a^3 + a^2 b + ab^2 + b^3} = \frac{1}{2} \left(\frac{a^4 + b^4}{a^3 + a^2 b + ab^2 + b^3} \right).$$

It is sufficient to prove that

$$\sum_{\text{cyclic}} \frac{a^4+b^4}{a^3+a^2b+ab^2+b^3} \geq \frac{a+b+c+d}{2}.$$

But we see that

$$\frac{a^4+b^4}{a^3+a^2b+ab^2+b^3} \geq \frac{a+b}{4}.$$

This follows from

$$\begin{aligned} 4(a^4+b^4) &- (a+b)(a^3+a^2b+ab^2+b^3) \\ &= (a^2-b^2)^2 + 2(a-b)(a^3-b^3) \\ &\geq (a^2-b^2)^2 + 2(a-b)^2(a^2+ab+b^2) \geq 0. \end{aligned}$$

■

261. Let a, b, c be non-negative real numbers satisfying $a^2+b^2+c^2=1$. Prove that

$$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a} \geq 5abc+2.$$

Solution: We prove this inequality in two stages: we first prove that

$$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a} \geq \sqrt{7(a+b+c)-3}, \tag{261.1}$$

and then we prove that

$$\sqrt{7(a+b+c)-3} \geq 5abc+2. \tag{261.2}$$

Let us write $a+b+c=x$. Then $ab+bc+ca=(x^2-1)/2$. By Cauchy-Schwarz inequality, we obtain

$$1 = a^2+b^2+c^2 \leq (a+b+c)^2 \leq 3(a^2+b^2+c^2) = 3.$$

This shows that $1 \leq x \leq \sqrt{3}$. We can write

$$\begin{aligned} \left(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\right)^2 &= 2(a+b+c) + \sum_{\text{cyclic}} \sqrt{(a+b)(b+c)} \\ &= 2x + \sum_{\text{cyclic}} \sqrt{a^2+\frac{x^2-1}{2}}. \end{aligned}$$

We prove that

$$\sqrt{a^2+\frac{x^2-1}{2}} \geq a + \frac{x-1}{2}. \tag{261.3}$$

Equivalently,

$$\frac{x^2-1}{2} \geq a(x-1) + \frac{(x-1)^2}{4}.$$

This further reduces to $(x-1)(x+3-4a) \geq 0$. Since $a \leq 1 \leq x$, this result is true there by proving (261.3). Therefore

$$\begin{aligned}\left(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\right)^2 &= 2x+2\sum_{\text{cyclic}}\sqrt{a^2+\frac{x^2-1}{2}} \\ &\geq 2x+2(a+b+c)+3(x-1) \\ &= 7x-3=7(a+b+c)-3.\end{aligned}$$

This proves (261.1).

We know by AM-GM inequality that

$$ab+bc+ca \geq 3(a^2b^2c^2)^{1/3}.$$

This gives

$$abc \leq \left(\frac{ab+bc+ca}{3}\right)^{3/2} = \left(\frac{x^2-1}{6}\right)^{3/2},$$

and therefore

$$2+5abc \leq 2+5\left(\frac{x^2-1}{6}\right)^{3/2}.$$

Thus it is sufficient to show that

$$\left(2+5\left(\frac{x^2-1}{6}\right)^{3/2}\right)^2 \leq 7x-3. \tag{261.4}$$

This further reduces to

$$\left(\frac{25(x^2-1)^2(x+1)}{216}+\frac{5\sqrt{6}(x^2-1)^{1/2}(x+1)}{9}-7\right)(x-1) \leq 0.$$

Since $1 \leq x \leq \sqrt{3}$, it is sufficient to prove that

$$\frac{25(x^2-1)^2(x+1)}{216}+\frac{5\sqrt{6}(x^2-1)^{1/2}(x+1)}{9} \leq 7. \tag{261.5}$$

But the left side $f(x)$ is an increasing function of x. Hence it is enough to verify (261.5) at $x=\sqrt{3}$. However, we see that

$$f(\sqrt{3}) = \frac{205+85\sqrt{3}}{54} < 7.$$

This proves (261.4) and completes the proof of the main result.

■

262. Let x, y, z be positive real numbers such that $x^2+y^2+z^2 \leq x+y+z$. Prove that

$$\frac{x^2+3}{x^3+1}+\frac{y^2+3}{y^3+1}+\frac{z^2+3}{z^3+1} \geq 6.$$

Solution: Let us introduce $f(t) = (t^2+3)/(t^3+1)$, for $t > 0$. Since $t^2+1 \geq 2t$, we get

$$f(t) \geq \frac{2t+2}{t^3+1} = \frac{2}{t^2-t+1}.$$

Introduce new variables: $x^2 - x + 1 = a$, $y^2 - y + 1 = b$ and $z^2 - z + 1 = c$. Then

$$a + b + c = (x^2 + y^2 + z^2) - (x + y + z) + 3 \leq 3,$$

by the given condition. Therefore

$$f(x) + f(y) + f(z) \geq \frac{2}{a} + \frac{2}{b} + \frac{2}{c} \geq \frac{18}{a+b+c} \geq 6;$$

we have used AM-HM inequality here. ■

263. For any three positive real numbers a, b, c, prove that

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} \geq \frac{3a+2b-c}{4}.$$

Solution: AM-GM inequality gives

$$\frac{a^2}{a+b} + \frac{a+b}{4} \geq a, \qquad \frac{b^2}{b+c} + \frac{b+c}{4} \geq b.$$

Adding both, we obtain

$$\frac{a^2}{a+b} + \frac{a+b}{4} + \frac{b^2}{b+c} + \frac{b+c}{4} \geq a + b.$$

Therefore

$$\begin{aligned} \frac{a^2}{a+b} + \frac{b^2}{b+c} \quad &\geq \quad a + b - \left(\frac{a+b}{4} + \frac{b+c}{4}\right) \\ &= \quad \frac{3a+2b-c}{4}. \end{aligned}$$

■

264. Let a, b, c be the sides of a triangle with perimeter equal to 1. Prove that

$$\sqrt{a^2+b^2} + \sqrt{b^2+c^2} + \sqrt{c^2+a^2} < 1 + \frac{\sqrt{2}}{2}.$$

Solution: Without loss of generality, we may assume $a \geq b \geq c$. We are given that $a + b + c = 1$. Therefore $a < b + c = 1 - a$ so that $a < \frac{1}{2}$. Since $b \geq a$, we have

$$\sqrt{a^2+b^2} \leq \sqrt{2a^2} = \sqrt{2}\, a < \frac{\sqrt{2}}{2}.$$

Since $c \le b$, we also have

$$b^2 + c^2 \le b^2 + bc < b^2 + bc + \frac{c^2}{4} = \left(b + \frac{c}{2}\right)^2,$$

which gives

$$\sqrt{b^2 + c^2} < b + \frac{c}{2}.$$

Similarly, we can also get

$$\sqrt{c^2 + a^2} < a + \frac{c}{2}.$$

Adding all three inequalities, we obtain

$$\sqrt{a^2 + b^2} + \sqrt{b^2 + c^2} + \sqrt{c^2 + a^2} < a + b + c + \frac{\sqrt{2}}{2} = 1 + \frac{\sqrt{2}}{2}.$$

■

265. Let a, b, c be the sides of a triangle. Prove that

$$\sqrt{a^2 + ab + b^2} + \sqrt{b^2 + bc + c^2} + \sqrt{c^2 + ca + a^2} \\ \le \sqrt{5(a^2 + b^2 + c^2) + 4(ab + bc + ca)}.$$

Solution: Let us introduce

$$x = a^2 + ab + b^2, \quad y = b^2 + bc + c^2, \quad z = c^2 + ca + a^2.$$

We have to show that

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \le \sqrt{3S + x + y + z},$$

where $S = a^2 + b^2 + c^2 + ab + bc + ca$. Using Cauchy-Schwarz inequality, we get

$$(\sqrt{x} + \sqrt{y} + \sqrt{z})^2 \le \left(\frac{x}{x+S} + \frac{y}{y+S} + \frac{z}{z+S}\right)(x + y + z + 3S).$$

This gives

$$\frac{(\sqrt{x} + \sqrt{y} + \sqrt{z})^2}{3S + x + y + z} \le \frac{x}{x+S} + \frac{y}{y+S} + \frac{z}{z+S}.$$

Thus it is enough to prove

$$\frac{x}{x+S} + \frac{y}{y+S} + \frac{z}{z+S} \le 1.$$

Equivalently, we have to show that

$$2xyz + S(xy + yz + zx) \le S^3. \tag{265.1}$$

But

$$S^2 - (xy + yz + zx) = \sum_{\text{sym}} a^3b + \sum_{\text{cyclic}} a^2bc.$$

Here **sym** means one has to take symmetric sum. Thus we have to prove that

$$\begin{aligned}
&2(a^2 + ab + b^2)(b^2 + bc + c^2)(x^2 + ca + a^2) \\
&\qquad \leq (a^2 + b^2 + c^2 + ab + bc + ca)\left(\sum_{\text{sym}} a^3b + \sum_{\text{cyclic}} a^2bc\right).
\end{aligned}$$

After considerable simplification, this reduces to

$$\begin{aligned}
&2\left(3a^2b^2c^2 + 2\sum_{\text{sym}} a^3b^2c + \sum_{\text{sym}} a^4b^2 + \sum_{\text{cyclic}} a^4bc + \sum_{\text{sym}} a^3b^3\right) \\
&\quad \leq \sum_{\text{sym}} a^5b + 2\sum_{\text{sym}} a^3b^3 + 4\sum_{\text{sym}} a^3b^2c + \sum_{\text{cyclic}} a^4bc + \sum_{\text{sym}} a^4b^2 + 3a^2b^2c^2.
\end{aligned}$$

This is equivalent to

$$\sum_{\text{sym}} a^4b^2 + 3a^2b^2c^2 \leq \sum_{\text{sym}} a^5b + \sum_{\text{cyclic}} a^4bc. \tag{265.2}$$

By AM-GM inequality, we know that

$$3a^2b^2c^2 \leq \sum_{\text{cyclic}} a^4bc. \tag{265.3}$$

We also observe that

$$a^5b + ab^5 - (a^4b^2 + a^2b^4) = ab(a-b)^2(a^2 + ab + b^2) \geq 0.$$

Hence

$$\sum_{\text{sym}} a^4b^2 \leq \sum_{\text{sym}} a^5b. \tag{265.4}$$

Adding both (265.3) and (265.4), we get (265.2) and hence (265.1). This completes the proof. ■

266. Suppose a, b, c are non-negative real numbers such that $a^3+b^3+c^3+abc = 4$. Prove that

$$a^3b + b^3c + c^3a \leq 3.$$

Solution: We may assume that $a = \max\{a, b, c\}$. Suppose $b = \min\{a, b, c\}$. We can write

$$c(a^3 + b^3 + c^3) + abc^2 = 4c.$$

We can write the inequality in an equivalent form:

$$a^3 b + b^3 c + c^3 a \leq 3 + (a^3 c + b^3 c + c^4 + abc^2 - 4c).$$

This is again equivalent to

$$ac^3 + a^3 b \leq a^3 c + abc^2 + (c^4 - 4c + 3).$$

By AM-GM inequality, we have

$$c^4 + 3 = c^4 + 1 + 1 + 1 \geq 4c.$$

Moreover

$$ac^3 + a^3 b - a^3 c - abc^2 = ac^2(c - b) - a^3(c - b) = a(c^2 - a^2)(c - b) \leq 0,$$

since b is the minimum among a, b, c. A similar, proof works when $c = \min\{a, b, c\}$.

■

267. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(a + \frac{1}{b}\right)^2 + \left(b + \frac{1}{c}\right)^2 + \left(c + \frac{1}{a}\right)^2 \geq 3(a + b + c + 1).$$

Solution: By Cauchy-Schwarz inequality, we have

$$\sum_{\text{cyclic}} \left(a + \frac{1}{b}\right)^2 \geq \sum_{\text{cyclic}} \left(a + \frac{1}{b}\right)\left(b + \frac{1}{c}\right).$$

But

$$\sum_{\text{cyclic}} \left(a + \frac{1}{b}\right)\left(b + \frac{1}{c}\right) = ab + bc + ca + \frac{a}{c} + \frac{b}{a} + \frac{c}{b} + 3 + a + b + c.$$

Here we have used $abc = 1$. However, AM-GM inequality gives

$$ab + \frac{b}{a} \geq 2a, \quad bc + \frac{c}{b} \geq 2c, \quad ca + \frac{c}{a} \geq 2c.$$

Thus we get

$$\sum_{\text{cyclic}} \left(a + \frac{1}{b}\right)^2 \geq 2a + 2b + 2c + 3 + a + b + c = 3(a + b + c + 1).$$

■

268. Let a, b, c be positive real numbers with $abc = 1$. Prove that

$$\frac{a}{c(a+1)} + \frac{b}{a(b+1)} + \frac{c}{b(c+1)} \geq \frac{3}{2}.$$

Solution: Since $abc = 1$, we can find positive reals x, y, z such that

$$a = \frac{x}{y}, \quad b = \frac{y}{z}, \quad c = \frac{z}{x}.$$

The inequality transforms to

$$\frac{x^2}{z(x+y)} + \frac{y^2}{x(y+z)} + \frac{z^2}{y(z+x)} \geq \frac{3}{2}.$$

By Cauchy-Schwarz inequality, we have

$$(x+y+z)^2 \leq \left(\frac{x^2}{z(x+y)} + \frac{y^2}{x(y+z)} + \frac{z^2}{y(z+x)}\right)\Big(2(xy+yz+zx)\Big).$$

Therefore,

$$\frac{x^2}{z(x+y)} + \frac{y^2}{x(y+z)} + \frac{z^2}{y(z+x)} \geq \frac{(x+y+z)^2}{2(xy+yz+zx)} \geq \frac{3(xy+yz+zx)}{2(xy+yz+zx)} = \frac{3}{2}.$$

We have used $(x+y+z)^2 \geq 3(xy+yz+zx)$ which simply reduces to

$$(x-y)^2 + (y-z)^2 + (z-x)^2 \geq 0.$$

Equality holds if and only if $x = y = z$ and hence $a = b = c = 1$. ■

269. Let a, b, c be positive real number such that $abc = 1$. Prove that

$$\frac{1}{1+a^{2014}} + \frac{1}{1+b^{2014}} + \frac{1}{1+c^{2014}} > 1.$$

Solution: Taking $x = a^{2014}$, $y = b^{2014}$, $z = c^{2014}$, we see that $xyz = 1$. The inequality to be proved is

$$\sum_{\text{cyclic}} (1+x)(1+y) > (1+x)(1+y)(1+z).$$

This reduces to

$$3 + (xy+yz+zx) + 2(x+y+z) > 1 + (x+y+z) + (xy+yz+zx) + xyz.$$

Equivalently, this takes the form

$$1 + x + y + z > 0,$$

which is true by positivity of x, y, z. ■

270. For positive real numbers a, b, c, prove the inequality

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\right) \geq \frac{9}{1+abc}.$$

Solution: Using AM-GM inequality,

$$\begin{aligned}
\sum_{\text{cyclic}} \frac{1}{1+a} &= \sum_{\text{cyclic}} \frac{1/a}{1+1/a} \\
&\geq 3\sqrt[3]{\frac{1/abc}{(1+1/a)(1+1/b)(1+1/c)}} \\
&= \frac{3}{\sqrt[3]{abc}\sqrt[3]{(1+1/a)(1+1/b)(1+1/c)}} \\
&\geq \frac{3}{\sqrt[3]{abc}}\,\frac{3}{3+(1/a)+(1/b)+(1/c)}.
\end{aligned}$$

Hence

$$\left(\sum_{\text{cyclic}} \frac{1}{a}\right)\left(\sum_{\text{cyclic}} \frac{1}{1+a}\right) \geq \frac{9}{\sqrt[3]{abc}\left(1+\frac{3}{\sum_{\text{cyclic}}(1/a)}\right)}.$$

Using

$$\frac{3}{\sum_{\text{cyclic}}(1/a)} \leq \sqrt[3]{abc},$$

we obtain

$$\left(\sum_{\text{cyclic}} \frac{1}{a}\right)\left(\sum_{\text{cyclic}} \frac{1}{1+a}\right) \geq \frac{9}{\sqrt[3]{abc}\left(\sqrt[3]{abc}+1\right)}.$$

However, we observe that $(1+x^3) - x(1+x) = (x+1)(x-1)^2 \geq 0$ for any non-negative real x. It follows that $x(x+1) \leq 1+x^3$ and hence

$$\sqrt[3]{abc}\left(\sqrt[3]{abc}+1\right) \leq 1+abc.$$

This proves the given inequality. ■

271. Let x, y, z be positive real numbers such that $x+y+z=3$. Prove that

$$\sqrt{x}+\sqrt{y}+\sqrt{z} \geq xy+yz+zx.$$

Solution: We have

$$3(x+y+z) = (x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx).$$

Hence it follows that

$$xy+yz+zx = \frac{1}{2}(3x-x^2+3y-y^2+3z-z^2).$$

Then

$$\begin{aligned}
&x+y+z-(xy+yz+zx)\\
&\quad= (\sqrt{x}+\sqrt{y}+\sqrt{z})+\frac{1}{2}(x^2-3x+y^2-3y+z^2-3z)\\
&\quad= \frac{1}{2}\Big((x^2-3x+2\sqrt{x})+(y^2-3y+2\sqrt{y})+(z^2-3z+2\sqrt{z})\Big)\\
&\quad= \sqrt{x}\left(\sqrt{x}-1\right)^2\left(\sqrt{x}+2\right)+\sqrt{y}\left(\sqrt{y}-1\right)^2\left(\sqrt{y}+2\right)\\
&\qquad\quad +\sqrt{z}\left(\sqrt{z}-1\right)^2\left(\sqrt{z}+2\right)\\
&\quad\geq 0.
\end{aligned}$$

Therefore

$$\sqrt{x}+\sqrt{y}+\sqrt{z}\geq xy+yz+zx.$$

■

272. Let a, b, c be positive real numbers. Prove that

$$\frac{9abc}{2(a+b+c)}\leq \frac{ab^2}{a+b}+\frac{bc^2}{b+c}+\frac{ca^2}{c+a}\leq \frac{a^2+b^2+c^2}{2}.$$

Solution: We begin with the observation $ab+bc+ca\leq a^2+b^2+c^2$. Therefore

$$\begin{aligned}
\frac{ab^2}{a+b}+\frac{bc^2}{b+c}+\frac{ca^2}{c+a} &= \frac{1}{\frac{1}{b^2}+\frac{1}{ab}}+\frac{1}{\frac{1}{c^2}+\frac{1}{bc}}+\frac{1}{\frac{1}{a^2}+\frac{1}{ca}}\\
&\leq \frac{b^2+ab}{4}+\frac{c^2+bc}{4}+\frac{c^2+ca}{4}\\
&= \frac{a^2+b^2+c^2+ab+bc+ca}{4}\\
&\leq \frac{a^2+b^2+c^2+a^2+b^2+c^2}{4}\\
&= \frac{a^2+b^2+c^2}{2}.
\end{aligned}$$

We have used AM-HM inequality. This proves the right side inequality. On the other hand, AM-GM inequality gives

$$\frac{ab^2}{a+b}+\frac{bc^2}{b+c}+\frac{ca^2}{c+a}\geq \frac{3abc}{\sqrt[3]{(a+b)(b+c)(c+a)}}.$$

Hence it is good enough to prove that

$$\frac{3abc}{\sqrt[3]{(a+b)(b+c)(c+a)}}\geq \frac{9abc}{2(a+b+c)}.$$

Equivalently, we need to show

$$2(a+b+c) \geq 3\sqrt[3]{(a+b)(b+c)(c+a)}.$$

This is a direct consequence of AM- GM inequality.

■

273. For positive real numbers a, b, c, prove that

$$\frac{abc}{(1+a)(a+b)(b+c)(c+16)} \leq \frac{1}{81}.$$

Solution: We write the product in the denominator as

$$\begin{aligned}
&(1+a)(a+b)(b+c)(c+16) \\
&= \left(1+\frac{a}{2}+\frac{a}{2}\right)\left(a+\frac{b}{2}+\frac{b}{2}\right)\left(b+\frac{c}{2}+\frac{c}{2}\right)(c+8+8) \\
&\geq 3\sqrt[3]{\frac{a^2}{4}} \cdot 3\sqrt[3]{\frac{ab^2}{4}} \cdot 3\sqrt[3]{\frac{bc^2}{4}} \cdot 3\sqrt[3]{\frac{64c}{4}} \\
&= 81abc.
\end{aligned}$$

This gives the required inequality. ■

274. Let a, b, c, d be positive real numbers such that a + b + c + d = 4. Prove that

$$\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}+\frac{1}{d^2+1} \geq 2.$$

Solution: We have

$$\frac{1}{a^2+1} = 1-\frac{a^2}{a^2+1} \geq 1-\frac{a^2}{2a} = 1-\frac{a}{2}.$$

Similarly, we get

$$\frac{1}{b^2+1} \geq 1-\frac{b}{2}, \quad \frac{1}{c^2+1} \geq 1-\frac{c}{2}, \quad \frac{1}{d^2+1} \geq 1-\frac{d}{2}.$$

Adding these inequalities we obtain

$$\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}+\frac{1}{d^2+1} \geq 4-\frac{a+b+c+d}{2} = 4-2 = 2.$$

Equality occurs if and only if $a=b=c=d=1$. ■

275. Let a, b, c be the sides of a triangle. Prove that inequality

$$64(s-a)(s-b)(s-c) \leq (a+b)(b+c)(c+a),$$

where $s=(a+b+c)/2$ is the semi-perimeter of the triangle.

Solution: Using AM-GM inequality, we have

$$(a+b)(b+c)(c+a) \geq 2\sqrt{ab} \cdot 2\sqrt{bc} \cdot 2\sqrt{ca} = 8abc.$$

So, it suffices to show that

$$8(s-a)(s-b)(s-c) \leq abc.$$

This follows from (3.4.1). Equality occurs if and only if a = b = c.

■

276. Let a, b, c be positive real numbers. Prove that

$$\frac{1+ab}{c} + \frac{1+bc}{a} + \frac{1+ca}{b} \geq \sqrt{a^2+2} + \sqrt{b^2+2} + \sqrt{c^2+2}.$$

Solution: We can write the **lhs** as

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + abc\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right).$$

But we know that

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \leq \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

Thus we get

$$abc\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \geq abc\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right) = a+b+c.$$

Therefore

$$\frac{1+ab}{c} + \frac{1+bc}{a} + \frac{1+ca}{b} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + a+b+c.$$

But we also have

$$a + \frac{1}{a} > \sqrt{a^2+2},$$

as can be seen by squaring both sides. Similar relations hold for the other sums also. Thus we get

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + a+b+c > \sqrt{a^2+2} + \sqrt{b^2+2} + \sqrt{c^2+2}.$$

■

277. Let I be the in-centre of a triangle ABC. Let R_A, R_B, R_C be respectively the circum-radii of triangles BCI, CAI, ABI. If R is the circum-radius of $\triangle ABC$, prove that

$$R_A + R_B + R_C \leq 3R.$$

Solution: Using Sine rule, we have

$$\frac{a}{\sin A} = 2R, \qquad \frac{a}{\sin\left(\frac{B}{2}+\frac{C}{2}\right)} = 2R_A.$$

But

$$\sin\left(\frac{B}{2}+\frac{C}{2}\right) = \sin\left(90^\circ - \frac{A}{2}\right) = \cos\frac{A}{2}.$$

Thus we obtain

$$\frac{R_A}{R} = \frac{\sin A}{\cos(A/2)} = 2\sin(A/2).$$

Similarly, we get

$$\frac{R_B}{R} = 2\sin(B/2), \qquad \frac{R_C}{R} = 2\sin(C/2).$$

Thus we have to show that

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \le \frac{3}{2}.$$

This follows from (3.4.10).

■

278. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{a^2}{b^3+c^4+1} + \frac{b^2}{c^3+a^4+1} + \frac{c^2}{a^3+b^4+1} > \frac{1}{5}.$$

Solution: First we observe that $a, b, c \in (0,1)$. Hence $b^3 < b$, $c^4 < c$ and therefore, $b^3 + c^4 + 1 < b + c + 1 = 2 - a$. This gives

$$\frac{a^2}{b^3+c^4+1} > \frac{a^2}{2-a^2} = -2 - a + \frac{4}{2-a}.$$

Similarly,

$$\frac{b^2}{c^3+a^4+1} > -2 - b + \frac{4}{2-b}, \qquad \frac{c^2}{a^3+b^4+1} > -2 - c + \frac{4}{2-c}.$$

Together they give

$$\begin{aligned} &\frac{a^2}{b^3+c^4+1} + \frac{b^2}{c^3+a^4+1} + \frac{c^2}{a^3+b^4+1} \\ &\qquad > \quad -6 - (a+b+c) + \frac{4}{2-a} + \frac{4}{2-b} + \frac{4}{2-c} \\ &\qquad = \quad -7 + 4\left(\frac{1}{2-a} + \frac{1}{2-b} + \frac{1}{2-c}\right). \end{aligned}$$

But $2-a, 2-b, 2-c$ are positive numbers. Using AM-HM inequality, we get

$$\frac{1}{2-a}+\frac{1}{2-b}+\frac{1}{2-c} \geq \frac{9}{6-(a+b+c)}=\frac{9}{5}.$$

Therefore, we finally get

$$\frac{a^2}{b^3+c^4+1}+\frac{b^2}{c^3+a^4+1}+\frac{c^2}{a^3+b^4+1}>-7+\frac{36}{5}=\frac{1}{5}.$$

Alternate Solution: We can use Cauchy-Schwarz inequality:

$$\begin{aligned}&(a+b+c)^2\\ &\leq\left(\frac{a^2}{b^3+c^4+1}+\frac{b^2}{c^3+a^4+1}+\frac{c^2}{a^3+b^4+1}\right)\Big(\sum_{\text{cyclic}} a^3+\sum_{\text{cyclic}} a^4+3\Big).\end{aligned}$$

However, we have

$$\sum_{\text{cyclic}} a^3+\sum_{\text{cyclic}} a^4+3<3+(a+b+c)+(a+b+c)=5.$$

It follows that

$$\frac{a^2}{b^3+c^4+1}+\frac{b^2}{c^3+a^4+1}+\frac{c^2}{a^3+b^4+1} \geq \frac{(a+b+c)^2}{5}=\frac{1}{5}.$$

■

279. Let x, y, z be three positive real numbers such that $xy+yz+zx=3xyz$. Prove that

$$x^2y+y^2z+z^2x \geq 2(x+y+z)-3.$$

Solution: We can write the given condition as

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=3.$$

Using AM-GM inequality, we observe that

$$x^2y+\frac{1}{y} \geq 2x, \quad y^2z+\frac{1}{z} \geq 2y, \quad z^2x+\frac{1}{x} \geq 2z.$$

Therefore

$$\begin{aligned} x^2y+y^2z+z^x &\geq 2(x+y+z)-\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\\ &= 2(x+y+z)-3.\end{aligned}$$

Equality holds if and only if $x=y=z=1$.

■

280. Let a, b, c and x, y, z be two sets of positive real numbers. Prove that

$$\frac{x}{y+z}(b+c)+\frac{y}{z+x}(c+a)+\frac{z}{x+y}(a+b) \geq \sqrt{3(ab+bc+ca)}.$$

Solution: Since the inequality is homogeneous in a, b, c, we may assume that $a+b+c=1$. Thus the inequality can be written in the form

$$\frac{x}{y+z}(1-a)+\frac{y}{z+x}(1-b)+\frac{z}{x+y}(1-c) \geq \sqrt{3(ab+bc+ca)}.$$

We write this in the form

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y} \geq \sqrt{3(ab+bc+ca)}+\frac{ax}{y+z}+\frac{by}{z+x}+\frac{cz}{x+y} \tag{280.1}$$

Using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\frac{ax}{y+z}+\frac{by}{z+x}+\frac{cz}{x+y}+\sqrt{3(ab+bc+ca)} \\
&\qquad \leq \sqrt{\left(\frac{x}{y+z}\right)^2+\left(\frac{y}{z+x}\right)^2+\left(\frac{z}{x+y}\right)^2}\sqrt{(a^2+b^2+c^2)} \\
&\qquad\qquad +\sqrt{\frac{3}{4}}\sqrt{ab+bc+ca}+\sqrt{\frac{3}{4}}\sqrt{ab+bc+ca}.
\end{aligned}$$

One more use of Cauchy-Schwarz inequality gives

$$\begin{aligned}
&\frac{ax}{y+z}+\frac{by}{z+x}+\frac{cz}{x+y}+\sqrt{3(ab+bc+ca)} \\
&\qquad \leq \sqrt{\left(\frac{x}{y+z}\right)^2+\left(\frac{y}{z+x}\right)^2+\left(\frac{z}{x+y}\right)^2+\frac{3}{2}} \\
&\qquad\qquad \times \sqrt{(a^2+b^2+c^2)+2(ab+bc+ca)} \\
&\qquad = \sqrt{\left(\frac{x}{y+z}\right)^2+\left(\frac{y}{z+x}\right)^2+\left(\frac{z}{x+y}\right)^2+\frac{3}{2}}
\end{aligned}$$

Hence (280.1) is proved once we prove

$$\sqrt{\left(\frac{x}{y+z}\right)^2+\left(\frac{y}{z+x}\right)^2+\left(\frac{z}{x+y}\right)^2+\frac{3}{2}} \leq \frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}.$$

On squaring, this further reduces to

$$\sum_{\text{cyclic}} \frac{yz}{(x+y)(x+z)} \geq \frac{3}{4}. \tag{280.2}$$

However, (280.2) is the same as

$$\sum_{\text{sym}} x^2 y \geq 6xyz. \tag{280.3}$$

But (280.3) is a direct consequence of AM-GM inequality. ■

281. Let x, y, z be positive real numbers such that $xy + yz + zx = 1$. Prove that

$$\frac{x}{x^2+1} + \frac{y}{y^2+1} + \frac{z}{z^2+1} \leq \frac{3\sqrt{3}}{4}.$$

Solution: We have $1 + x^2 = xy + yz + zx + x^2 = (x+y)(x+z)$. Similarly, we obtain $1 + y^2 = (y+x)(y+z)$, $1 + z^2 = (z+x)(z+y)$. Thus we get

$$\begin{aligned}
&\frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} \\
&\qquad = \frac{x}{(x+y)(x+z)} + \frac{y}{(y+z)(y+x)} + \frac{z}{(z+x)(z+y)} \\
&\qquad = \frac{x(y+z) + y(z+x) + z(x+y)}{(x+y)(y+z)(z+x)} \\
&\qquad = \frac{2}{(x+y)(y+z)(z+x)}.
\end{aligned}$$

But

$$\begin{aligned}
(x+y)(y+z)(z+x) &= (x+y)(xy+yz+zx+z^2) \\
&= (x+y)(1+z^2) = x+y+z(zx+zy) = x+y+z-xyz.
\end{aligned}$$

Using $(x+y+z)^2 \geq 3(xy+yz+zx) = 3$, we get $x+y+z \geq \sqrt{3}$. Besides

$$1 = xy + yz + zx \geq 3\big(xyz\big)^{2/3},$$

so that

$$xyz \leq \frac{1}{3\sqrt{3}}.$$

Therefore

$$(x+y)(y+z)(z+x) = x+y+z-xyz \geq \sqrt{3} - \frac{1}{3\sqrt{3}} = \frac{8}{3\sqrt{3}}.$$

Finally, we get

$$\frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} = \frac{2}{(x+y)(y+z)(z+x)} \leq \frac{4}{3\sqrt{3}}.$$

■

282. Suppose x, y, z are positive real numbers such that $x + y + z = 1$. Prove that

$$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right) \geq 64.$$

Solution: Equivalently, we have to prove that

$$2 + (xy + yz + zx) \geq 63xyz. \tag{282.1}$$

But we know that $xy + yz + zx \geq 3(xyz)^{2/3}$. Therefore it is sufficient to prove that

$$2 + 3(xyz)^{2/3} \geq 63xyz.$$

Taking $(xyz)^{1/3} = \lambda$, this reduces to

$$63\lambda^3 - 3\lambda^2 - 2 \leq 0.$$

Or

$$(3\lambda - 1)(21\lambda^2 + 6\lambda + 2) \leq 0. \tag{282.2}$$

Since $x + y + z = 1$, AM-GM inequality gives

$$\lambda = (xyz)^{1/3} \leq \frac{x+y+z}{3} = \frac{1}{3}.$$

Hence $3\lambda \leq 1$ and this implies (282.2). This in turn gives (282.1).

■

283. Let x, y, z be positive real numbers such that $x + y + z = 1$. Prove that

$$\frac{1}{1-xy} + \frac{1}{1-yz} + \frac{1}{1-zx} \leq \frac{27}{8}.$$

Solution: We have to prove

$$\sum_{\text{cyclic}} 8(1-xy)(1-yz) \leq 27(1-xy)(1-yz)(1-zx). \tag{283.1}$$

This reduces to

$$3 - 11(xy + yz + zx) + 19xyz - (xyz)^2 \leq 0.$$

By AM-GM inequality, we have

$$xy + yz + zx \leq 3(xyz)^{2/3}.$$

So it is sufficient to prove that

$$3 - 33(xyz)^{2/3} + 19xyz - (xyz)^2 \leq 0.$$

Taking $\lambda = (xyz)^{1/3}$, and using

$$(xyz)^{1/3} \leq \frac{x+y+z}{3} = \frac{1}{3},$$

we have to prove that

$$27\lambda^6 - 19\lambda^3 + 3\lambda^2 - 3 \leq 0$$

for $\lambda \in (0, 1/3]$. Since

$$27\lambda^6 - 19\lambda^3 + 3\lambda^2 - 3 = (3\lambda - 1)(9\lambda^5 + 3\lambda^4 + \lambda^3 - 6\lambda^2 - \lambda + 3),$$

we need to show that

$$9\lambda^5 + 3\lambda^4 + \lambda^3 - 6\lambda^2 - \lambda + 3 \geq 0$$

for $\lambda \in (0, 1/3]$. However, we see that

$$3 - \lambda - 6\lambda^2 \geq 3 - 3\lambda - 6\lambda^2 = 3(1+\lambda)(1-2\lambda) \geq 0,$$

since $1 - 2\lambda > 0$ in the interval $(0, 1/3]$. This completes the proof of (283.1). ■

284. Let x, y, z be positive real numbers such that $x + y + z = 1$. Show that

$$\frac{z - xy}{x^2 + xy + y^2} + \frac{x - yz}{y^2 + yz + z^2} + \frac{y - zx}{z^2 + zx + x^2} \geq 2.$$

Solution: Let us introduce the symmetric variables: $p = x + y + z$, $q = xy + yz + zx$ and $r = xyz$. We have

$$x^2 + 2xy + y^2 = (x+y)^2 = (1-z)^2 = 1 - z - z(1-z) = 1 - z - z(x+y).$$

Hence $x^2 + xy + y^2 = 1 - z - q$. Similarly, we obtain $y^2 + yz + z^2 = 1 - x - q$ and $z^2 + zx + x^2 = 1 - y - q$. Besides, we also have

$$z - xy = 1 - (x+y) - q + z(x+y) = (1-q) - (1-z)^2,$$

and similar relations for the other two expressions. Thus the inequality is

$$\sum_{\text{cyclic}} \frac{(1-q) - (1-z)^2}{(1-q-z)} \geq 2.$$

By a tedious algebra, we can reduce this to

$$q^3 + q^2 - 4q + 4r + 3qr + 1 \geq 0.$$

But we have observed that (refer 2.31)

$$p^3 - 4pq + 9r \geq 0.$$

Since $p = 1$, this is same as $1 - 4q + 9r \geq 0$. Hence

$$r \geq \frac{4q - 1}{9}.$$

Therefore, it is sufficient to prove that

$$9q^3 + 9q^2 - 36q + (3q + 4)(4q - 1) + 9 \geq 0.$$

However,

$$q = xy + yz + zx \geq 3(xyz)^{2/3} = 3r^{2/3}.$$

And

$$\begin{aligned} &9q^3 + 9q^2 - 36q + (3q + 4)(4q - 1) + 9 \\ &\qquad = 9q^3 + 21q^2 - 23q + 5 = (3q - 1)(3q^2 + 8q - 5). \end{aligned}$$

This shows that it is enough to prove

$$3q^2 + 8q - 5 \leq 0.$$

Since $q \leq 1/3$, we see that

$$3q^2 + 8q - 5 \leq \frac{3}{9} + \frac{8}{9} - 5 = -2 < 0.$$

■

285. Let a, b, c be positive real numbers define

$$u = a + b + c, \quad \frac{u^2 - v^2}{3} = ab + bc + ca, \quad w = abc,$$

where $v \geq 0$. Then

$$\frac{(u + v)^2(u - 2v)}{27} \leq r \leq \frac{(u - v)^2(u + 2v)}{27}.$$

Solution: We have to find the maximum and the minimum of $w = abc$ in terms of $ab + bc + ca = (u^2 - v^2)/3$. If $v = 0$, then

$$ab + bc + ca = \frac{(a + b + c)^2}{3}.$$

Therefore $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0$ giving $a = b = c$. In this case

$$w = abc = \frac{u^3}{27}.$$

Suppose $v \neq 0$. Then

$$ab + bc + ca = \frac{u^2 - v^2}{3} < \frac{u^2}{3} = \frac{(a + b + c)^2}{3}.$$

In this case $(a-b)^2+(b-c)^2+(c-a)^2>0$. Hence a,b,c are not all equal. Consider the function

$$f(x)=(x-a)(x-b)(x-c)=x^3-ux^2+\frac{u^2-v^2}{3}x-w.$$

Observe that

$$f'(x)=3x^2-2ux+\frac{u^2-v^2}{3}.$$

Hence $f'(x)=0$ has two roots:

$$x_1=\frac{u+v}{3}, \quad x_2=\frac{u-v}{3}.$$

Hence $f'(x)<0$ for $x_1<x<x_2$ and $f'(x)>0$ for $x<x_1$ and $x>x_2$. Besides

$$f''(x)=6x-2u.$$

Hence

$$f''(x_1)=6x_1-2u=2u+2v-2u=2v>0$$

Hence $f'(x)$ has a local minimum at x_1. Similarly,

$$f''(x_2)=-2v<0,$$

and $f'(x)$ has a local maximum at x_2. Since $f(x)$ has zeros at a,b,c, if we assume $a\le b\le c$, then $f(x)\ge 0$ between a and b; and $f(x)\le 0$ between b and c. Therefore $a\le (u-v)/3\le b$ and $b\le (u+v)/3\le c$. It follows that

$$f(x_2)\ge 0, \quad f(x_1)\le 0.$$

Hence

$$f\left(\frac{u-v}{3}\right)=\frac{(u-v)^2(u+2v)}{27}-w\ge 0,$$

and

$$f\left(\frac{u+v}{3}\right)=\frac{(u+v)^2(u-2v)}{27}-w\le 0.$$

Combining both, we get

$$\frac{(u+v)^2(u-2v)}{27}\le w\le \frac{(u-v)^2(u+2v)}{27}.$$

■

286. Let a,b,c be positive real numbers. Prove that

$$a^4+b^4+c^4\ge abc(a+b+c).$$

Solution: As in problem 3.6, let us introduce $p = a + b + c$, $ab + bc + ca = (p^2 - q^2)/3$ and $r = abc$. We may also assume that $p = 1$ so that $ab + bc + ca = (1 - q^2)/3$.

Then we have

$$\begin{aligned} a^4 + b^4 + c^4 &= (a^2 + b^2 + c^2)^2 - 2(a^2b^2 + b^2c^2 + c^2a^2) \\ &= \left(\frac{1 + 2q^2}{3}\right)^2 - \frac{1}{9}(1 - 2q^2 + 4q^4 - 18r) \\ &= \frac{2q^4 + 8q^2 - 1}{9} + 4r. \end{aligned}$$

The required inequality is therefore

$$\frac{2q^4 + 8q^2 - 1}{9} + 3r \geq 0.$$

Equivalently, we have

$$2q^4 + 8q^2 - 1 + 27r \geq 0.$$

In view of the conclusion of the problem 3.6, it is sufficient to prove that

$$2q^4 + 8q^2 - 1 + (1 + q)^2(1 - 2q) \geq 0.$$

This reduces to

$$q^2(2q^2 - 2q + 5) \geq 0.$$

But

$$2q^2 - 2q + 5 = 2\left(q - \frac{1}{2}\right)^2 + \frac{9}{2} > 0.$$

This proves the required inequality.

■

287. Let a, b, c be real numbers such that $a^2 + b^2 + c^2 = 9$. Prove that

$$2(a + b + c) - abc \leq 10.$$

Solution: We employ technique similar to the one used in problem 3.6. We introduce $p = (a + b + c)$, $\frac{(p^2 - q^2)}{3} = ab + bc + ca$ and $r = abc$. Using $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$, we observe that

$$a^2 + b^2 + c^2 = p^2 - \frac{2}{3}(p^2 - q^2) = \frac{p^2 + 2q^2}{3}.$$

Thus the given condition is

$$p^2 + 2q^2 = 27.$$

Using problem 3.6, we have

$$\begin{aligned} 2(a+b+c) - abc = 2p - r &\leq 2p - \frac{(p+q)^2(p-2q)}{27} \\ &= \frac{p(27+5q^2)+2q^3}{27}. \end{aligned}$$

It is sufficient to prove that

$$\frac{p(27+5q^2)+2q^3}{27} \leq 10.$$

This simplifies to

$$p(27+5q^2) \leq 270 - 2q^3. \tag{287.1}$$

But

$$\begin{aligned} &(270-2q^3)^2 - p^2(27+5q^2)^2 \\ &= (270-2q^3)^2 - (27-2q^2)(27+5q^2)^2 \\ &= 27(q-3)^2\left(2q^4+12q^3+49q^2+146q+219\right), \end{aligned}$$

since $p^2 = 27 - 2q^2$ by the given condition. Since the right hand side is non-negative, this proves the inequality (287.1) and completes the proof of the required inequality.

■

288. Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$a^2+b^2+c^2+3abc \geq \frac{9}{4}.$$

Solution: We will homogenise the inequality, using $a+b+c=1$, as follows:

$$9(a+b+c)(a^2+b^2+c^2)+27abc \geq 4(a+b+c)^3 \tag{288.1}$$

This simplifies to

$$5(a^3+b^3+c^3)+3abc \geq 3(ab(a+b)+bc(b+c)+ca(c+a)).$$

By Schur's inequality, we have

$$a^3+b^3+c^3+3abc \geq ab(a+b)+bc(b+c)+ca(c+a).$$

Since $(2,1,0) \prec (3,0,0)$, Muirhead's theorem gives

$$4(a^3+b^3+c^3) \geq 2(ab(a+b)+bc(b+c)+ca(c+a)).$$

Adding these two inequalities we get (288.1). ■

289. Determine the maximum value of λ such that

$$a+b+c \geq \lambda$$

for all positive reals a, b, c with $a\sqrt{bc}+b\sqrt{c}+c\sqrt{a} \geq 1$.

Solution: We give here two solutions.

Solution 1. We show that the maximum of λ is $\sqrt{3}$. Taking $a = b = c = 1/\sqrt{3}$, we see that

$$a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} = 1, \quad \text{and} \quad a + b + c = \sqrt{3}.$$

Therefore the largest value of λ is $\leq \sqrt{3}$. We show that $a+b+c \geq \sqrt{3}$ whenever $a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} \geq 1$.

Using AM-GM inequality, we have

$$a\left(\frac{b+c}{2}\right) + b\left(\frac{c+a}{2}\right) + c\left(\frac{a+b}{2}\right) \geq a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} \geq 1.$$

Hence $ab + bc + ca \geq 1$. Therefore

$$(a+b+c)^2 \geq 3(ab+bc+ca) \geq 3.$$

Hence $a + b + c \geq \sqrt{3}$.

Solution 2. Again we have

$$\begin{aligned}
\frac{(a+b+c)^4}{9} &= \left(\frac{a+b+c}{3}\right)^3 \cdot 3(a+b+c) \\
&\geq 3abc(a+b+c) \\
&= (abc+abc+abc)(a+b+c) \\
&\geq \left(a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}\right)^2 \geq 1.
\end{aligned}$$

We have used Cauchy- Schwarz inequality at the end. Again we get $a+b+c \geq \sqrt{3}$. ■

290. If a, b, c are real numbers such that $a + b + c = 1$, prove that

$$10(a^3 + b^3 + c^3) - 9(a^5 + b^5 + c^5) \geq 1.$$

Solution: We know that

$$\begin{aligned}
a^3 + b^3 + c^3 &= (a+b+c)^3 - 3(a+b)(b+c)(c+a) \\
&= 1 - 3(a+b)(b+c)(c+a).
\end{aligned}$$

Similarly, we can show that

$$a^5 + b^5 + c^5 = 1 - 5(a+b)(b+c)(c+a)(a^2 + b^2 + c^2 + ab + bc + ca).$$

Therefore

$$\begin{aligned}
&10\left(\sum_{\text{cyclic}} a^3\right) - 9\left(\sum_{\text{cyclic}} a^5\right) \geq 1 \\
&\quad\Leftrightarrow\quad 10\left(1 - 3\prod_{\text{cyclic}}(a+b)\right) - 9\left(1 - 5\prod_{\text{cyclic}}(a+b)\Big(\sum_{\text{cyclic}}(a^2+ab)\Big)\right) \geq 1 \\
&\quad\Leftrightarrow\quad 45(a+b)(b+c)(c+a)\Big(\sum_{\text{cyclic}} a^2 + \sum_{\text{cyclic}} ab\Big) \geq 30(a+b)(b+c)(c+a) \\
&\quad\Leftrightarrow\quad \sum_{\text{cyclic}} a^2 \geq \sum_{\text{cyclic}} ab.
\end{aligned}$$

But this is immediate.

■

291. Suppose $a_1, a_2, a_3, \ldots, a_n$ are n positive real numbers such that

$$\Big(a_1 + a_2 + a_3 + \cdots + a_n\Big)\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n}\right) < n^2 + 1.$$

Show that for any three distinct numbers j, k, l the numbers a_j, a_k, a_l form the sides of a triangle.

Solution: Suppose the contrary that for some choice of distinct j, k, l the numbers a_j, a_k, a_l do not form a triangle. We may assume $j = 1, k = 2, l = 3$. We may also assume that $a_1 + a_2 \leq a_3$. Thus we have

$$\begin{aligned}
&\left(\sum_{k=1}^{n} a_j\right)\left(\sum_{l=1}^{n} \frac{1}{a_k}\right) \\
&\quad = n + \sum_{k<l}\left(\frac{a_k}{a_l} + \frac{a_l}{a_k}\right) \\
&\quad = n + \left(\frac{a_1}{a_3} + \frac{a_3}{a_1}\right) + \left(\frac{a_2}{a_3} + \frac{a_3}{a_2}\right) + \sum_{\substack{1\leq k<l\leq n \\ (k,l)\neq(1,3),(2,3)}}\left(\frac{a_k}{a_l} + \frac{a_l}{a_k}\right) \\
&\quad = n + \frac{a_1 + a_2}{a_3} + a_3\left(\frac{1}{a_1} + \frac{1}{a_2}\right) + \sum_{\substack{1\leq k<l\leq n \\ (k,l)\neq(1,3),(2,3)}}\left(\frac{a_k}{a_l} + \frac{a_l}{a_k}\right) \\
&\quad \geq n + \frac{a_1 + a_2}{a_3} + \frac{4a_3}{a_1 + a_2} + 2\left(\binom{n}{2} - 2\right).
\end{aligned}$$

Taking $t = a_3/(a_1 + a_2)$, we see that

$$4t + \frac{1}{t} - 5 = \frac{(t-1)(4t-1)}{t} \geq 0,$$

since $t \geq 1$ by our assumption. Thus we obtain

$$\begin{aligned}\left(\sum_{k=1}^{n} a_j\right)\left(\sum_{l=1}^{n}\frac{1}{a_k}\right) &\geq 2\left(\binom{n}{2}-2\right)+5+n \\ &= n(n-1)+1+n = n^2+1.\end{aligned}$$

This contradicts the given hypothesis. We conclude that for every choice of distinct indices j, k, l, the numbers a_j, a_k, a_l form the sides of a triangle. ■

292. Let a, b, c be positive real numbers. Prove that

$$24abc \leq \left|a^3+b^3+c^3-(a+b+c)^3\right| \leq \frac{8}{9}(a+b+c)^3.$$

Show further that equality holds in both the inequalities if and only if $a = b = c$.

Solution: We observe that

$$a^3+b^3+c^3-(a+b+c)^3 = -3(a+b)(b+c)(c+a).$$

Therefore $\left|a^3+b^3+c^3-(a+b+c)^3\right| = 3(a+b)(b+c)(c+a)$. By AM-GM inequality, we have

$$(a+b)(b+c)(c+a) \leq \left(\frac{a+b+b+c+c+a}{3}\right)^3 = \frac{8}{27}(a+b+c)^3.$$

Here equality holds if and only if $a = b = c$.This gives the right side inequality. We also have

$$(a+b)(b+c)(c+a) \geq \left(2\sqrt{ab}\right)\left(2\sqrt{bc}\right)\left(2\sqrt{ca}\right) = 8abc.$$

Here again equality holds if and only if $a = b = c$. We get the left side inequality.

■

293. Two circles Γ_1, Γ_2 with respective centre S_1, S_2 and radii r_1, r_2 are externally tangent to each other and lie in a square $ABCD$ of side a units so that Γ_1 touches DC, DA while Γ_2 touches CD, CB. Prove that the area of at least one of the triangles AS_1S_2 and BS_1S_2 is no more than $\frac{3}{16}a^2$ units.

Solution: First we observe that S_1 lies on the diagonal BD and S_2 lies on AC. Hence $AS_2 \perp BS_1$. Let P be the point of intersection of the diagonals AC and BD. We also observe that $DS_1 = r_1\sqrt{2}$ and $CS_2 = r_2\sqrt{2}$. Hence

$$BS_1 = (a-r_1)\sqrt{2}, \quad PS_1 = \left(\frac{a}{2}-r_1\right)\sqrt{2},$$

$$AS_2 = (a-r_2)\sqrt{2}, \quad PS_2 = \left(\frac{a}{2}-r_2\right)\sqrt{2}.$$

Thus

$$\begin{aligned}
[AS_1S_2] &= \frac{1}{2}AS_1 \cdot PS_1 = (a-r_2)\left(\frac{a}{2}-r_1\right), \\
[BS_1S_2] &= \frac{1}{2}BS_1 \cdot PS_2 = (a-r_1)\left(\frac{a}{2}-r_2\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
[AS_1S_2] + [BS_1S_2] &= (a-r_2)\left(\frac{a}{2}-r_1\right) + (a-r_1)\left(\frac{a}{2}-r_2\right) \\
&= a^2 - \frac{3}{2}a(r_1+r_2) + 2r_1r_2.
\end{aligned}$$

Let K be the point at which Γ_1 touches AD; H, L be the points at which Γ_2 touches CD, CB, respectively. Let M be the point of intersection of KS_1 and HS_2. (See Fig 6.30)

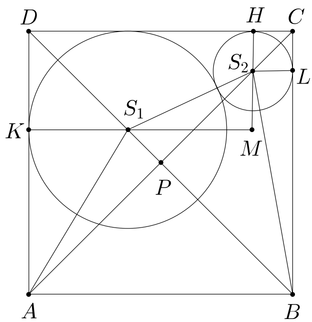

Fig 6.30

By Pythagoras' theorem for triangle S_1MS_2, we have

$$(a-r_1-r_2)^2 + (r_1-r_2)^2 = (r_1+r_2)^2.$$

Therefore

$$a - r_1 - r_2 = 2\sqrt{r_1r_2}.$$

This gives

$$a = r_1 + r_2 + 2\sqrt{r_1r-2} = (\sqrt{r_1}+\sqrt{r_2})^2 \geq 4\sqrt{r_1r_2}.$$

We obtain

$$r_1r_2 \leq \frac{a^2}{16}.$$

We also observe that the length of the side DC is not greater than the length of the polygonal segment KS_1S_2L. Therefore $a \leq 2r_1 + 2r_2$. Hence

$$\begin{aligned}
[AS_1S_2] + [BS_1S_2] &= a^2 - \frac{3}{2}a(r_1+r_2) + 2r_1r_2 \\
&\leq a^2 - \frac{3}{2}a^2 + \frac{1}{8}a^2 = \frac{3}{8}a^2.
\end{aligned}$$

This implies that at least one of the areas cannot exceed $\frac{3}{16}a^2$.

$\blacksquare$

294. Find all $\lambda > 0$ such that the inequality

$$\sqrt{a^2 + \lambda b^2} + \sqrt{b^2 + \lambda a^2} \geq a + b + (\lambda - 1)\sqrt{ab}$$

holds for all positive real numbers a and b.

Solution: If $a = b = 1$, then we must have $2\sqrt{\lambda + 1} \geq \lambda + 1$. Hence $\lambda \leq 3$. We show that for any $\lambda \in (0, 3]$, the inequality holds for any choice of real numbers a, b. If $\lambda \in (0, 1]$, the result is immediate. Suppose $\lambda \in (1, 3]$. Using Minkowski inequality, we obtain

$$\sqrt{a^2 + pb^2} + \sqrt{b^2 + pa^2} \geq \sqrt{1 + p}(a + b).$$

Using AM-GM inequality, we also have

$$a + b + (p - 1)\sqrt{ab} \leq a + b + (p - 1)\left(\frac{a + b}{2}\right) = \frac{(p + 1)(a + b)}{2}.$$

Now we must check that

$$(a + b)\sqrt{1 + p} \geq \frac{(p + 1)(a + b)}{2}.$$

This is equivalent to $\sqrt{1 + p} \leq 2$. Since $p \in (1, 3]$, the conclusion follows.

$\blacksquare$

295. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$a + b + c \geq \sqrt{\frac{1}{3}(a + 2)(b + 2)(c + 2)}.$$

Solution: By AM-GM inequality, we have

$$a^2 + 1 \geq 2a, \quad b^2 + 1 \geq 2b, \quad c^2 + 1 \geq 2c.$$

Therefore

$$a^2 + b^2 + c^2 + 3 \geq 2(a + b + c).$$

The AM-GM inequality also gives

$$ab + bc + ca \geq 3(abc)^{2/3} = 3.$$

Similarly,

$$a^2 + b^2 + c^2 \geq 3.$$

Therefore

$$3(a^2+b^2+c^2)+6+4(ab+bc+ca) \geq 4(a+b+c)+12+3.$$

Adding $2(ab+bc+ca)-6$ both sides, we obtain

$$3(a+b+c)^2 \geq 2(ab+bc+ca)+4(a+b+c)+9.$$

Using $abc=1$, we write

$$\begin{aligned}
&2(ab+bc+ca)+4(a+b+c)+9 \\
&\qquad = 8+4(a+b+c)+2(ab+bc+ca)+abc \\
&\qquad\qquad = (a+2)(b+2)(c+2).
\end{aligned}$$

Thus we get

$$3(a+b+c)^2 \geq (a+2)(b+2)(c+2),$$

which leads to the required inequality

$$(a+b+c) \geq \sqrt{\frac{1}{3}(a+2)(b+2)(c+2)}.$$

■

296. Let $x_1, x_2, x_3, \ldots, x_n$ be $n \geq 3$ positive real numbers with $x_1x_2x_3\cdots x_n = 1$. Prove that

$$\sum_{j=1}^{n} \frac{x_j^8}{x_{j+1}(x_j^4+x_{j+1}^4)} \geq \frac{n}{2},$$

where $x_{n+1}=x_1$.

Solution: We first observe that for any two positive reals a, b the inequality

$$(a^3+b^3)^2 \geq 2ab(a^4+b^4) \tag{296.1}$$

holds. In fact, we see that

$$\begin{aligned}
(a^3+b^3)^2 - 2ab(a^4+b^4) &= a^6+2a^3b^3+b^6-2a^5b-2ab^5 \\
&= (a-b)^2(a^4-a^2b^2+b^4) \geq 0,
\end{aligned}$$

since

$$a^4-a^2b^2+b^4 \geq a^4-2a^2b^2+b^4 = (a^2-b^2)^2 \geq 0.$$

Using (296.1), we have

$$\begin{aligned}
\sum_{j=1}^{n} \frac{x_j^8}{(x_j^4+x_{j+1}^4)x_{j+1}} &= \sum_{j=1}^{n} \frac{x_j^9}{(x_j^4+x_{j+1}^4)x_jx_{j+1}} \\
&\geq \sum_{j=1}^{n} \frac{2x_j^9}{\left(x_j^3+x_{j+1}^3\right)^2} \\
&= \sum_{j=1}^{n} \frac{2\left(x_j^3\right)^3}{\left(x_j^3+x_{j+1}^3\right)^2}.
\end{aligned}$$

Using Hölder's inequality with $p = 3$ and $q = 3/2$, we have

$$\begin{aligned}\left(\sum_{j=1}^{n} x_j^3\right) &= \sum_{j=1}^{n} \frac{x_j^3}{\left(x_j^3 + x_{j+1}^3\right)^{2/3}} \left(x_j^3 + x_{j+1}^3\right)^{2/3} \\ &\le \left(\sum_{j=1}^{n} \frac{\left(x_j^3\right)^3}{\left(x_j^3 + x_{j+1}^3\right)^2}\right)^{1/3} \left(\sum_{j=1}^{n} \left(x_j^3 + x_{j+1}^3\right)\right)^{2/3}.\end{aligned}$$

This gives

$$\sum_{j=1}^{n} \frac{\left(x_j^3\right)^3}{\left(x_j^3 + x_{j+1}^3\right)^2} \ge \frac{\left(\sum_{j=1}^{n} x_j^3\right)^3}{4\left(\sum_{j=1}^{n} x_j^3\right)^2}$$

Combining both, we obtain

$$\begin{aligned}\sum_{j=1}^{n} \frac{x_j^8}{(x_j^4 + x_{j+1}^4)x_{j+1}} &\ge \frac{1}{2}\left(\sum_{j=1}^{n} x_j^3\right) \\ &\ge \frac{1}{2}\, n\, (x_1^3 x_2^3 x_3^3 \cdots x_n^3)^{1/n} \\ &= \frac{n}{2}.\end{aligned}$$

■

297. Let a, b, c be positive real number such that $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = 1$. Prove that

$$\frac{a^2 + b^2 + c^2 + ab + bc + ca - 3}{5} \ge \frac{a}{b} + \frac{b}{c} + \frac{c}{a}.$$

Solution: The condition that

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = 1$$

gives

$$a + b + c = abc.$$

Hence the given inequality is equivalent to

$$(a^2 + b^2 + c^2 + ab + bc + ca)(a + b + c) \ge 5(a^2c + b^2a + c^2b). \qquad (297.1)$$

This reduces to

$$a^3 + b^3 + c^3 \ge 3(a^2c + b^a + c^2b) - 2(a^2b + b^2c + c^2a).$$

However, we observe that

$$3(a^2c + b^a + c^2b) - 2(a^2b + b^2c + c^2a) = a^2c + b^2a + c^2b + 2(a - b)(b - c)(c - a).$$

Hence we have to prove that

$$a^3 + b^3 + c^3 \geq a^2c + b^2a + c^2b + 2(a-b)(b-c)(c-a). \tag{297.2}$$

We prove (297.2), considering the sign of $(a-b)(b-c)(c-a)$.

Suppose $(a-b)(b-c)(c-a) \geq 0$. By rearrangement inequality, we already know that

$$a^3 + b^3 + c^3 \geq a^2c + b^2a + c^2b.$$

Hence (297.2) follows if $(a-b)(b-c)(c-a) \leq 0$.

Suppose $(a-b)(b-c)(c-a) > 0$. We may take a to be the least among a, b, c. Then $a < b$ and $a < c$. Thus $c - a > 0$ and $a - b < 0$. We must therefore have $b - c < 0$ and therefore $a < b < c$. Take $b = a + x$, $c = a + x + y$, where x, y are positive. In this case (297.2) transforms to

$$\begin{aligned} &a^3 + (a+x)^3 + (a+x+y)^3 \\ &\qquad \geq a^2(a+x+y) + (a+x)^2a + (a+x+y)^2(a+x) + 2xy(x+y). \end{aligned}$$

A further simplification gives

$$a\Big(2x^2 + 2y(x+y)\Big) + \Big(x^3 + y^3 - x^2y\Big) \geq 0.$$

But observe that

$$x^3 + y^3 - x^2y = (x+y)(x-y)^2 + y^2x \geq 0.$$

This completes the proof of (297.2) and hence that of (297.1). ■

298. Show that for all positive real numbers x, y, z, the inequality

$$\frac{x(2x-y)}{y(2z+x)} + \frac{y(2y-z)}{z(2x+y)} + \frac{z(2z-x)}{x(2y+z)} \geq 1.$$

Solution: We may write

$$\frac{x(2x-y)}{y(2z+x)} = 1 + \frac{2(x^2+yz)}{y(2z+x)},$$

and similar identities to the other two expressions. Thus the inequality is equivalent to

$$\sum_{\text{cyclic}} \frac{2(x^2+yz)}{y(2z+x)} \geq 2. \tag{298.1}$$

Using Cauchy-Schwarz inequality, we have

$$\sum_{\text{cyclic}} \frac{x^2}{y(2z+x)} \geq \frac{(x+y+z)^2}{3(xy+yz+zx)}.$$

Similarly, we also have

$$\sum_{\text{cyclic}} \frac{z}{(2z+x)} \geq \frac{(x+y+z)^2}{2(x^2+y^2+z^2)+(xy+yz+zx)}.$$

Adding these two we get

$$\begin{aligned}
&\sum_{\text{cyclic}} \frac{2(x^2+yz)}{y(2z+x)} \\
&\geq \frac{(x+y+z)^2}{3(xy+yz+zx)} + \frac{(x+y+z)^2}{2(x^2+y^2+z^2)+(xy+yz+zx)} \\
&= \frac{2(x+y+z)^4}{3\Big(xy+yz+zx\Big)\Big(2(x^2+y^2+z^2)+(xy+yz+zx)\Big)} \\
&= \frac{2(x+y+z)^4}{3\Big(xy+yz+zx\Big)\Big(2(x+y+z)^2-3(xy+yz+zx)\Big)} \\
&\geq \frac{2(x+y+z)^4}{\left(\dfrac{3(xy+yz+zx)+\big(2(x^2+y^2+z^2)+(xy+yz+zx)\big)}{2}\right)^2} \\
&= 2.
\end{aligned}$$

This proves (298.1) and hence proves the required inequality. ■

299. Suppose

$$\frac{z(xz+yz+y)}{xy_y^2+z^2+1} \leq K,$$

for all real numbers $x, y, z \in (-2, 2)$ with $x^2+y^2+z^2+xyz = 4$. Find the smallest value of K.

Solution: We claim $K = 1$ is the smallest value for which the inequality holds. Taking

$$x = \frac{1+\sqrt{5}}{2}, \quad y = \frac{1-\sqrt{5}}{2}, \quad z = \frac{1-\sqrt{5}}{2},$$

we see that x, y, z satisfy the conditions of the problem and

$$\frac{z(xz+yz+y)}{xy+y^2+z^2+1} = 1.$$

We show that

$$\frac{z(xz+yz+y)}{xy+y^2+z^2+1} \leq 1 \tag{299.1}$$

for all $x, y, z \in (-2, 2)$ which satisfy $x^2 + y^2 + z^2 + xyz = 4$.

We first observe that

$$xy + y^2 + z^2 + 1 = \left(y + \frac{x}{2}\right)^2 + z^2 + \left(1 - \frac{x^2}{4}\right) > 0.$$

If $z(xz + yz + y) \leq 0$, the inequality (299.1) is obviously true. Hence we may assume that $z(xz + yz + y) > 0$ and we have to show that

$$xy + y^2 + z^2 + 1 - z(xz + yz + y) \geq 0. \tag{299.2}$$

But we can write the **lhs** as a sum of two non-negative quantities as follows:

$$xy + y^2 + z^2 + 1 - z(xz + yz + y) = S_1 + S_2, \tag{299.3}$$

where

$$\begin{aligned} S_1 &= \left(x + y - \frac{x + z^2}{2}\right)^2, \\ S_2 &= \frac{4 - x^2}{4}\left(1 - \frac{z(xz + 2y)}{4 - x^2}\right)^2. \end{aligned}$$

Indeed,

$$\left(x + y - \frac{x + z^2}{2}\right)^2 = y^2 + xy - yz^2 - \frac{xz^2}{4} + \frac{x^2}{4} + \frac{z^4}{4},$$

and

$$\begin{aligned} \frac{4 - x^2}{4}\left(1 - \frac{z(xz + 2y)}{4 - x^2}\right)^2 &= \frac{4 - x^2}{4} - \frac{z(xz + 2y)}{2} + \frac{z^2(xz + 2y)^2}{4(4 - x^2)} \\ &= \frac{4 - x^2}{4} - \frac{z(xz + 2y)}{2} + \frac{z^2(4 - z^2)}{4}. \end{aligned}$$

We have used $x^2 + y^2 + z^2 + xyz = 4$ in the last equality. Adding these, we get (299.3). This implies the inequality (299.2), and in turn proves (299.1). ■

300. Suppose a, b, c are positive real numbers such that $a^3 + b^3 + c^3 = a^4 + b^4 + c^4$. Prove that

$$\frac{a}{a^2 + b^3 + c^3} + \frac{b}{b^2 + c^3 + a^3} + \frac{c}{c^2 + c^3 + a^3} \geq 1.$$

Solution: By Cauchy-Schwarz inequality

$$(a + b + c)^2 \leq \left(\sum_{\text{cyclic}} \frac{a}{a^2 + b^3 + c^3}\right)\left(\sum_{\text{cyclic}} a(a^2 + b^3 + c^3)\right).$$

But

$$\sum_{\text{cyclic}} a(a^2+b^3+c^3) = (a+b+c)(a^3+b^3+c^3)$$

since $a^3+b^3+c^3 = a^4+b^4+c^4$. Hence it is enough to prove that $a+b+c \geq a^3+b^3+c^3$. Again Cauchy-Schwarz inequality gives

$$\begin{aligned}(a^2+b^2+c^2)^2 &\leq (a+b+c)(a^3+b^3+c^3),\\ (a^3+b^3+c^3)^2 &\leq (a^2+b^2+c^2)(a^4+b^4+c^4).\end{aligned}$$

Using $a^3+b^3+c^3 = a^4+b^4+c^4$ once again, we obtain

$$a^3+b^3+c^3 \leq a+b+c.$$

■

301. Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{a^4+5b^4}{a(a+2b)} + \frac{b^4+5c^4}{b(b+2c)} + \frac{c^4+5a^4}{c(c+2a)} \geq 1-(ab+bc+ca).$$

Solution: Note that it is enough to prove

$$\frac{a^4+5b^4}{a(a+2b)} + \frac{b^4+5c^4}{b(b+2c)} + \frac{c^4+5a^4}{c(c+2a)} \geq a^2+b^2+c^2+ab+bc+ca,$$

since $a+b+c=1$. Using Cauchy-Schwarz inequality, we have

$$(a^2+b^2+c^2)^2 \leq \left(\sum_{\text{cyclic}} \frac{a^4}{a(a+2b)}\right)(a+b+c)^2.$$

Therefore

$$\sum_{\text{cyclic}} \frac{a^4}{a(a+2b)} \geq (a^2+b^2+c^2)^2.$$

Similarly, we can also obtain

$$\sum_{\text{cyclic}} \frac{b^4}{a(a+2b)} \geq (a^2+b^2+c^2)^2.$$

Together, these give

$$\sum_{\text{cyclic}} \frac{a^4+5b^4}{a(a+2b)} \geq 6(a^2+b^2+c^2)^2.$$

Now we show that

$$6(a^2+b^2+c^2)^2 \geq a^2+b^2+c^2+ab+bc+ca.$$

But we know that $a^2+b^2+c^2 \geq ab+bc+ca$. Hence

$$a^2+b^2+c^2+ab+bc+ca \leq 2(a^2+b^2+c^2) \leq 6(a^2+b^2+c^2)^2,$$

because

$$a^2+b^2+c^2 \geq \frac{1}{3}(a+b+c)^2 = \frac{1}{3}.$$

■

302. Let $a_0, a_1, a_2, \ldots, a_n$ be real numbers in the interval $\left(0, \frac{\pi}{2}\right)$ such that

$$\tan\left(a_0 - \frac{\pi}{4}\right)\tan\left(a_1 - \frac{\pi}{4}\right)\tan\left(a_2 - \frac{\pi}{4}\right)\cdots\tan\left(a_n - \frac{\pi}{4}\right) \geq n-1.$$

Prove that

$$\tan(a_0)\tan(a_1)\tan(a_2)\cdots\tan(a_n) \geq n^{n+1}.$$

Solution: We have $\tan(x) = \tan\left(\left(x-\frac{\pi}{4}\right)+\frac{\pi}{4}\right) = \frac{y+1}{1-y}$ where $y = \tan\left(x - \frac{\pi}{4}\right)$. Taking

$$b_j = \tan\left(a_j - \frac{\pi}{4}\right), \quad 0 \leq j \leq n,$$

The given condition is $b_0+b_1+b_2+\cdot+b_n \geq n-1$. We have to show that

$$\prod_{j=0}^{n} \frac{b_j+1}{1-b_j} \geq n^{n+1}.$$

The given condition can be written as

$$1+b_j = \sum_{k\neq j}(1-b_k),$$

for any $j \in \{0,1,2,\ldots,n\}$. By AM-GM inequality, we have

$$\frac{1+b_j}{n} \geq \prod_{k\neq j}(1-b_k)^{1/n}.$$

Taking product over j, we obtain

$$\prod_{j=0}^{n}\frac{1+b_j}{n} \geq \prod_{j=0}^{n}\prod_{k\neq j}(1-b_k)^{1/n} = \prod_{j=0}^{n}(1-b_j).$$

Hence

$$\prod_{j=0}^{n}\frac{1+b_j}{1-b_j} \geq n^{n+1}.$$

■

303. Let x, y, z be positive real numbers. Prove that

$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}.$$

Solution: Here we give three solutions.

1. The given inequality is equal to

$$4(xy + yz + zx)\sum_{\text{cyclc}}(x+y)^2(y+z)^2 \geq 9(x+y)^2(y+z)^2(z+x)^2. \qquad (303.1)$$

Let us introduce $p = x + y + z$, $q = xy + yz + zx$ and $r = xyz$. We see that

$$(x+y)^2(y+z)^2(z+x)^2 = (pq - r)^2,$$

and

$$(x+y)^2(y+z)^2 + (y+z)^2(z+x)^2 + (z+x)^2(x+y)^2(y+z)$$
$$= (p^2 + q)^2 - 4p(pq - r).$$

Then the above inequality reduces to

$$4q\Big((p^2 + q)^2 - 4p(pq - r)\Big) \geq 9(pq - r)^2.$$

This further reduces to

$$3pq(p^3 - 4pq + 9r) + q(p^4 - 5p^2q + 4q^2 + 6pr) + r(pq - 9r) \geq 0. \qquad (303.2)$$

If we introduce

$$v_1 = \frac{p}{3}, \quad v_2^2 = \frac{q}{3}, \quad v_3^3 = r,$$

the inequalities for symmetric functions give

$$v_3 \leq v_2 \leq v_1.$$

(Refer theorem 11.) This gives $pq - 9r \geq 0$. Now using results in example 2.31 we get (303.2).

2. We introduce new variables a, b, c by $x + y = a$, $y + z = b$ and $z + x = c$. We can get x, y, z in terms of a, b, c by

$$x = \frac{c + a - b}{2}, \quad \frac{y}{=}\frac{a + b - c}{2}, \quad z = \frac{b + c - a}{2}.$$

The positivity of x, y, z show that a, b, c are the sides of a triangle: $a + b > c$, $b + c > a$ and $c + a > b$. It is easy to obtain

$$xy + yz + zx = \frac{2ab + 2bc + 2ca - a^2 - b^2 - c^2}{4}.$$

The inequality is now transformed to

$$\left(2ab + 2bc + 2ca - a^2 - b^2 - c^2\right)\left(a^2b^2 + b^2c^2 + c^2a^2\right) - 9a^2b^2c^2 \geq 0. \quad (303.3)$$

This can be rewritten in the form

$$\sum_{\text{cyclic}} \left(\frac{2}{ab} - \frac{1}{c^2}\right)(a-b)^2 \geq 0. \quad (303.4)$$

Suppose $a = b$ so that $x = z$. Then (303.3) reduces to

$$(2a^2 + 4ac - 2a^2 - c^2)(a^4 + 2a^2c^2) - 9a^4c^2 \geq 0.$$

This reduces to

$$4a(a-c)^2 + 2c^2(2a-c) \geq 0.$$

Since $2a = a + b > c$, this result is true. Hence the given inequality holds if $x = z$. Similarly, we can show that the result is true if any two of x, y, z are equal. This observation reduces the problem to distinct a, b, c. Since (303.3) is symmetric in a, b, c, we may assume $a > b > c$. We consider two different cases.

Case 1. Suppose $2c^2 \geq ab$. We claim that $2a^2 \geq bc$ and $2b^2 \geq ac$. If $2a^2 < bc$, then

$$2a^2 - bc < 0 \leq 2c^2 - ab.$$

This implies that $(a-c)(2c + 2a + b) < 0$, forcing $a < c$. This contradict $a > b > c$. Similarly, we can show that $2b^2 < ac$ is also not possible. So in this case

$$2c^2 \geq ab, \quad 2a^2 \geq bc, \quad 2b^2 \geq ac.$$

Thus all the terms on the left side of the inequality (303.4) are non-negative, which in turn implies the inequality.

Case 2. Suppose $2c^2 < ab$. Here again we prove $2a^2 \geq bc$ and $2b^2 \geq ac$. If $2a^2 < bc$, then we get $2(c^2 + a^2) < (ab + bc)$. But then we have

$$(c+a)^2 \leq 2(c^2 + a^2) < b(c+a),$$

which gives $c + a < b$ and this contradicts that a, b, c are the sides of a triangle. Hence $2a^2 \geq bc$. Similarly, we prove $2b^2 \geq ca$. Now we can put the inequality (303.4) in the form

$$\left(\frac{2}{ac} - \frac{1}{b^2}\right)(a-c)^2 + \left(\frac{2}{bc} - \frac{1}{a^2}\right)(b-c)^2 \geq \left(\frac{1}{c^2} - \frac{2}{ab}\right)(a-b)^2. \quad (303.5)$$

Here we observe that

$$(a-c)^2 = (a - b + b - c)^2 > (a-b)^2 + (b-c)^2,$$

since $(a-b)$ and $(b-c)$ are both positive. Hence it is enough to prove

$$\left(\frac{2}{ac}-\frac{1}{b^2}\right)\left((a-b)^2+(b-c)^2\right)+\left(\frac{2}{bc}-\frac{1}{a^2}\right)(b-c)^2 \geq \left(\frac{1}{c^2}-\frac{2}{ab}\right)(a-b)^2.$$

Equivalently, we need to prove

$$\left(\frac{2}{ac}-\frac{1}{b^2}-\frac{1}{a^2}+\frac{2}{bc}\right)(b-c)^2 \geq \left(\frac{1}{c^2}-\frac{2}{ab}-\frac{2}{ac}+\frac{1}{b^2}\right)(a-b)^2 \qquad (303.6)$$

However Using $b+c>a$, we observe that

$$\frac{1}{c^2}-\frac{2}{ab}-\frac{2}{ac}+\frac{1}{b^2}<\left(\frac{1}{b}-\frac{1}{c}\right)^2.$$

Hence it is enough to prove that

$$\left(\frac{2}{ac}-\frac{1}{b^2}-\frac{1}{a^2}+\frac{2}{bc}\right) \geq \frac{(a-b)^2}{b^2c^2}.$$

However, observe that

$$\frac{1}{ac}>\frac{1}{a^2}, \quad \frac{1}{bc}>\frac{1}{b^2}.$$

This further reduces our task to prove

$$\frac{a+b}{abc} \geq \frac{(a-b)^2}{b^2c^2}.$$

Simplification gives $(a+b)bc>a(a-b)^2$. But $a-b<c$. Hence

$$a(a-b)^2<ac^2<(a+b)bc,$$

since $ac<ab<b(a+b)$. This completes the proof.

3. Since the inequality (303.1) is homogeneous, we may assume $xy+yz+zx=3$. Thus we have to prove

$$\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2} \geq \frac{3}{4}.$$

Setting $x+y+z=3a$, we see that

$$9a^2=(x+y+z)^2 \geq 3(xy+yz+zx)=9.$$

Hence $a \geq 1$. The inequality (303.1) takes the form

$$\frac{1}{(3a-z)^2}+\frac{1}{(3a-x)^2}+\frac{1}{(3a-y)^2} \geq \frac{3}{4}.$$

This can be written in the form

$$3(12a^2-1)(3a^2-4)+xyz(34a-xyz)\geq 0. \tag{303.7}$$

We can also write this in another form:

$$12(3a^2-1)^2+208a^2\geq (17a-xyz)^2. \tag{303.8}$$

If $3a^2-4\geq 0$, then $12a^2-1\geq 0$ so that (303.7) holds. Suppose $3a^2-4<0$. We consider (303.8). Schur's inequality gives

$$(x+y+z)^3-4(x+y+z)(xy+yz+zx)+9xyz\geq 0.$$

Using $xy+yz+zx=3$ and $x+y+z=3a$, this reduces to $3a^3-4a+xyz\geq 0$. Therefore $17a-xyz\leq 13a+3a^3$. Since $3a^2-4<0$, we can check that $17a-xyz>0$ Hence

$$\begin{aligned}
&12(3a^2-1)^2+208a^2-(17a-xyz)^2\\
&\qquad\geq\ 12(3a^2-1)^2+208a^2-(13a+3a^3)^2\\
&\qquad=\ 3(4-11a^2+10a^4-3a^6)\\
&\qquad=\ 3(1-a^2)^2(4-3a^2)^2\geq 0.
\end{aligned}$$

This completes the solution. ■

304. Suppose a,b,c are positive real numbers such that $abc=1$. Prove that

$$\sum_{\text{cyclic}}\frac{a^2+bc}{a^2(b+c)}\geq ab+bc+ca.$$

Solution: We first observe that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{ab+bc+ca}{abc}=ab+bc+ca.$$

So we need to prove

$$\sum_{\text{cyclic}}\frac{a^2+bc}{a^2(b+c)}\geq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

However

$$\frac{a^2+bc}{a^2(b+c)}-\frac{1}{a}=\frac{(a-b)(a-c)}{a^2(b+c)}.$$

We use the following generalisation of Schur's inequality.

Lemma: Let a,b,c be non-negative real numbers. Suppose a,b,c are three positive real numbers such that $a\geq b\geq c$ or $a\leq b\leq c$. Then

$$a(x-y)(x-z)+b(y-z)(y-x)+c(z-x)(z-y)\geq 0.$$

Proof of Lemma: We may assume that $x \geq y \geq z$. Suppose $a \geq b \geq c$. Then $c(z-x)(z-y) \geq 0$. Consider

$$a(x-z) - b(y-z) = (ax - by) + z(b-a).$$

Since $x \geq y$, we see that $ax - by \geq ay - by = y(a-b)$, so that

$$a(x-z) - b(y-z) \geq y(a-b) + z(b-a) = (y-z)(a-b) \geq 0.$$

Multiplying by $x - y \geq 0$, we get

$$a(x-y)(x-z) + b(y-z)(y-x) \geq 0.$$

Similarly, we can prove for $a \leq b \leq c$. Therefore the lemma follows.

Suppose $a \geq b \geq c$. Consider

$$a^2(b+c) - b^2(c+a) = ab(a-b) + c(a+b)(a-b) = (a-b)(a+b+c) \geq 0.$$

Hence $a^2(b+c) \geq b^2(c+a)$. Similarly, we prove $b^2(c+a) \geq c^2(a+b)$. Hence

$$\frac{1}{a^2(b+c)} \leq \frac{1}{b^2(c+a)} \leq \frac{1}{c^2(a+b)}.$$

Now the lemma gives

$$\sum_{\text{cyclic}} \frac{(a-b)(a-c)}{a^2(b+c)} \geq 0.$$

This implies the required inequality.

■

305. Let a, b, c be non-negative real numbers. Prove that

$$4\left(a^3 + b^3 + c^3\right) + 15abc \geq (a+b+c)^3.$$

Solution: Since it is homogeneous in a, b, c, we may take $a + b + c = 2$. The inequality reduces to

$$a^3 + b^3 + c^3 + 3abc \geq \sum_{\text{sym}} a^2 b.$$

But this is just Schur's inequality:

$$a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b) \geq 0.$$

■

306. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{1}{a^4+b+c} + \frac{1}{b^4+c+a} + \frac{1}{c^4+a+b} \leq \frac{3}{a+b+c}.$$

Solution: The Cauchy-Schwarz inequality gives

$$(a^2+b^2+c^2)^2 \le (a^4+b+c)(1+b^3+c^3).$$

Therefore

$$\frac{1}{a^4+b+c} \le \frac{1+b^3+c^3}{(a^2+b^2+c^2)}.$$

Thus we have

$$\sum_{\text{cyclic}} \frac{1}{a^4+b+c} \le \frac{1}{(a^2+b^2+c^2)} \sum_{\text{cyclic}} (1+b^3+c^3) = \frac{3+2(a^3+b^3+c^3)}{(a^2+b^2+c^2)^2}.$$

Equivalently, we need to prove

$$\left(3+2(a^3+b^3+c^3)\right)\left(a+b+c\right) \le 3\left(a^2+b62+c^2\right)^2.$$

Introducing $p=a+b+c$, $q=ab+bc+ca$ and $r=abc$, this can be written in the form

$$p\left(3+2p(p^2-2q)+6r\right) \le \left(p^2-2q\right)^2.$$

Simplification brings this to the form

$$(p^2-3q)^2+q^2-3p+2(q^2-3pr) \ge 0.$$

The given condition is $4abc = a+b+c+1 \ge 4(abc)^{1/4}$. Therefore $r \ge 1$. This implies that $q^2-2p \ge q^2-3pr$. Thus it is sufficient to prove that

$$(p^2-3q)^2+3(q^2-3pr) \ge 0.$$

But

$$q^2-3pr = (ab+bc+ca)^2 - 3abc(a+b+c) \ge 0.$$

Hence the result follows.

■

307. Let a, b, c be positive reals. Prove that

$$a^4(b+c)+b^4(c+a)+c^4(a+b) \le \frac{1}{12}(a+b+c)^5.$$

Solution: We use the transformations $p=a+b+c$, $q=ab+bc+ca$ and $r=abc$. Observe that

$$\begin{aligned}
a^4(b+c)+b^4(c+a)+c^4(a+b) &= a^3(ab+ac)+b^3(bc+ba)+c^3(ca+cb)\\
&= a^3(q-bc)+b^3(q-ca)+c^3(q-ab)\\
&= q(a^3+b^3+c^3)-abc(a^2+b^2+c^2)\\
&= q\left(p(p^2-3q)+3r\right)-r(p^2-2q)\\
&= q(1-3q)+r(5q-1).
\end{aligned}$$

Thus we need to prove that

$$q(1-3q)+r(5q-1) \le \frac{1}{12}.$$

Using $1 = p^2 \le 3q$, we get $q \le 1/3$. If $q \le 1/5$, then $r(5q-1) \le 0$ and

$$\begin{aligned} q(1-3q)+r(5q-1) \le q(1-3q) &= \frac{1}{3}3q(1-3q) \\ &= \le \frac{1}{3}\left(\frac{1-3q+3q}{2}\right)^2 = \frac{1}{12}. \end{aligned}$$

Suppose $q > 1/5$ so that $\frac{1}{5} < q \le \frac{1}{3}$. Consider the function

$$f(q) = q(1-3q)+r(5q-1).$$

Its derivative is $f'(q) = 1-6q+5r$. Since $pq \ge 96$, we have $q \ge 9r$. Hence

$$f'(q) \le 1-6q+\frac{5}{9}q = 1-\frac{49}{9}q < 0.$$

Hence f is strictly decreasing on $\left(\frac{1}{5}, \frac{1}{3}\right]$. Therefore we get

$$f(q) f\left(\frac{1}{5}\right).$$

This gives

$$q(1-3q)+r(5q-1) < \frac{1}{5}\left(1-\frac{3}{5}\right) = \frac{2}{25} < \frac{1}{12}.$$

■

308. Suppose a, b, c are positive reals such that $ab+bc+ca = 1$. Prove that

$$\frac{1}{a+b} + \frac{b+c}{+}\frac{1}{c+a} - \frac{1}{a+b+c} \ge 2.$$

Solution: Clearing the denominators, the inequality is

$$\frac{(a+b)(b+c)+(b+c)(c+a)+(c+a)(a+b)}{(a+b)(b+c)(c+a)} \ge \frac{1}{a+b+c} + 2.$$

We introduce $p = a+b+c$, $q = ab+bc+ca$ and $r = abc$. However, we know that

$$(a+b)(b+c)+(b+c)(c+a)+(c+a)(a+b) = p^2q, (a+b)(b+c)(c+a) = pq-r.$$

Thus we have to prove that

$$\frac{p^2+q}{pq-r} - \frac{1}{p} \ge \frac{1}{2}.$$

Using $q = ab + bc + ca = 1$, we get the equivalent inequality:

$$p^2(p-2) + r(2p+1) \geq 0.$$

If $p \geq 2$, this is obviously true. Suppose $p < 2$. We use $p^3 - 4pq + 9r \geq 0$. Since $q = 1$, this is just $p^3 - 4p + 9r \geq 0$, or $r \geq (4p - p^3)/9$. Thus it is enough to prove

$$p^3 - 2p^2 + \frac{(4p - p^3)}{9}(2p+1) \geq 0.$$

This reduces to

$$-p(p-2)(p-1)^2 \geq 0.$$

Since $p < 2$, this is true. ■

309. Let a, b, c be positive real numbers such that $ab + bc + ca = 1$. Prove that

$$\frac{1 + a^2b^2}{(a+b)^2} + \frac{1 + b^2c^2}{(b+c)^2} + \frac{1 + c^2a^2}{(c+a)^2} \geq \frac{5}{2}.$$

Solution: We homogenise the inequality using $ab + bc + ca = 1$ and write it in th form

$$\sum_{\text{cyclic}} \frac{(ab + bc + ca)^2 + a^2b^2}{(a+b)^2} \geq \frac{5}{2}(ab + bc + ca).$$

This can be written in the form (after some rearrangement of terms)

$$f_a(b-c)^2 + f_b(c-a)^2 + f_c(a-b)^2 \geq 0,$$

where

$$\begin{aligned} f_a &= 1 - \frac{ab + bc + ca}{(b+c)^2}, \\ f_b &= 1 - \frac{ab + bc + ca}{(c+a)^2}, \\ f_c &= 1 - \frac{ab + bc + ca}{(a+b)^2}. \end{aligned}$$

We may assume $a \geq b \geq c$ so that $f_a \leq f_b \leq f_c$. Moreover

$$f_b = \frac{(c+a)^2 - (ab + bc + ca)}{(c+a)^2} = \frac{c^2 + (a-b)(a+c)}{(c+a)^2} > 0.$$

Therefore $f_c \geq f_b > 0$. It is easy to check that $\frac{a-c}{b-c} \geq \frac{a}{b}$. We also observe that

$$\begin{aligned} a^2 f_b + b^2 f_a &= a^2\left(1 - \frac{ab + bc + ca}{(c+a)^2}\right) + b^2\left(1 - \frac{ab + bc + ca}{(b+c)^2}\right) \\ &= a^2\left(\frac{c^2 + (c+a)(a-b)}{(c+a)^2}\right) + b^2\left(\frac{c^2 + (c+b)(b-a)}{(b+c)^2}\right) \\ &= c^2\left(\frac{a^2}{(c+a)^2} + \frac{b^2}{(c+b)^2}\right) + (a-b)^2\frac{ab + bc + ca}{(c+a)(c+b)} > 0. \end{aligned}$$

Therefore

$$\begin{aligned} f_a(b-c)^2+f_b(c-a)^2+f_c(a-b)^2 &\geq f_a(b-c)^2+f_b(c-a)^2 \\ &= (b-c)^2\left(f_a+f_b\left(\frac{(a-c)^2}{(b-c)^2}\right)\right) \\ &\geq (b-c)^2\left(\frac{b^2f_a+a^2f_b}{b^2}\right)\geq 0. \end{aligned}$$

equality holds if and only if $a=b=c=1/\sqrt{3}$. ■

310. Let a,b,c,d be non-negative real numbers. Prove that

$$a^4+b^4+c^4+d^4+2abcd \geq a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2.$$

Solution: This is known as Turkevicius inequality. We may assume that $a\geq b\geq c\geq d$. Let us write

$$f(a,b,c,d)=a^4+b^4+c^4+d^4+2abcd \\ -a^2b^2-a^2c^2-a^2d^2-b^2c^2-b^2d^2-c^2d^2.$$

It is easy to check that

$$f(a,b,c,d)-f(\sqrt{ac},b,\sqrt{ac},d)=(a-c)^2\big((a+c)^2-(b^2+d^2)\big)\geq 0.$$

Thus $f(a,b,c,d)\geq f(\sqrt{ac},b,\sqrt{ac},d)$. Hence it is enough to prove that

$$f(a,a,a,x)\geq 0 \text{ when } a\geq x.$$

But

$$f(a,a,a,x)=3a^4+x^4+2a^3x-(3a^4+3x^2a^2)=x^4+2a^3x-3a^2x^2.$$

By AM-GM inequality, we have

$$x^4+2a^3x=x^4+a^3x+a^3x\geq 3a^2x^2.$$

Hence we get $f(a,a,a,x)\geq 0$.

■

311. Let a,b,c be positive real numbers. Prove that

$$3+a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3\left(\frac{(a+1)(b+1)(c+1)}{1+abc}\right).$$

Solution: The inequality can be reduced to proving

$$a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3\left(\frac{a+b+c+ab+bc+ca}{1+abc}\right).$$

This may be written in the form

$$abc\left(\sum_{\text{cyclic}} a\right)+\sum_{\text{cyclic}}\frac{1}{a}+\sum_{\text{cyclic}} a^2c+\sum_{\text{cyclic}}\frac{a}{b}\geq 2\left(\sum_{\text{cyclic}} a+\sum_{\text{cyclic}} ab\right).$$

However we have,

$$a^2bc+\frac{b}{c}\geq 2ab,\quad b^2ca+\frac{c}{a}\geq 2bc,\quad c^2ab+\frac{a}{b}\geq 2ca$$

and

$$a^2c+\frac{1}{c}\geq 2a,\quad b^2a+\frac{1}{a}\geq 2b,\quad c^2b+\frac{1}{b}\geq 2c.$$

Adding all these, we get the required inequality. ■

312. Let a,b,c be distinct positive real numbers such that $abc=1$. Prove that

$$\sum_{\text{cyclic}}\frac{a^6}{(a-b)(a-c)}>15.$$

Solution: Consider a cubic polynomial whose roots are a,b,c. We get

$$P(x)=x^3-px^2+qx-r,$$

where $p=a+b+c$, $q=ab+bc+ca$ and $r=abc$. We observe that

$$\frac{a^n}{(a-b)(a-c)}+\frac{b^n}{(b-c)(b-a)}+\frac{c^n}{(c-a)(c-b)}=\sum_{\text{cyclic}}\frac{-a^n(b-c)}{(a-b)(b-c)(c-a)}.$$

Let us write

$$S_n=\sum_{\text{cyclic}}\frac{a^n(b-c)}{-(a-b)(b-c)(c-a)}.$$

We have

$$S_1=\frac{a(b-c)+b(c-a)+c(a-b)}{-(a-b)(b-c)(c-a)}=0.$$

Moreover,

$$S_2=\frac{a^2(b-c)+b^2(c-a)+c^2(a-b)}{-(a-b)(b-c)(c-a)}=1.$$

(In fact, we have

$$\begin{aligned} a^2(b-c)+b^2(c-a)+c^2(a-b) &= (a^2b-ab^2)+(b^2c-a^2c)+c^2(a-b) \\ &= (a-b)(ab-c(a+b)+c^2) = -(a-b)(b-c)(c-a). \end{aligned}$$

Since a, b, c are the roots of $P(x)=0$, we have

$$\begin{aligned} a^3-pa^2+qa-r &= 0, \\ b^3-pb^2+qb-r &= 0, \\ c^3-pc^2+qc-r &= 0. \end{aligned}$$

Multiply the first by $b-c$, the second by $(c-a)$ and the third by $(a-b)$, we obtain

$$\begin{aligned} a^3(b-c)-pa^2(b-c)+qa(b-c)-r(b-c) &= 0, \\ b^3(c-a)-pb^2(c-a)+qb(c-a)-r(c-a) &= 0, \\ c^3(a-b)-pc^2(a-b)+qc(a-b)-r(a-b) &= 0. \end{aligned}$$

Adding all these and dividing the sum by $-(a-b)(b-c)(c-a)$, we obtain

$$S_3-pS_2+qS_1=0.$$

Hence

$$S_3=p.$$

Now multiply the first by a , the second by b and the third by c and divide through out by $-(a-b)(b-c)(c-a)$ to get

$$S_4-pS_3+qS_2-rS_1=0.$$

Hence

$$S_4=p^2-q.$$

Similarly, we get

$$S_5-pS_4+qS_3-rS_2=0,$$

or $S_5=p(p^2-q)-qp+r=p^3-2pq+r$. Now we also get

$$S_6-pS_5+qS_4-rS_3=0.$$

This gives

$$S_6=p(p^3-2pq+r)-q(p^2-q)+rp=p^4-3p^2q+2pr+q^2.$$

We can write it as

$$S_6=p^2(p^2-3q)+2pr+q^2.$$

But

$$p^2 - 3q = (a+b+c)^2 - 3(ab+bc+ca) = a^2+b^2+c^2-ab-bc-ca > 0.$$

Hence

$$S_6 > 2pr + q^2 = 2abc(a+b+c) + (ab+bc+ca)^2 \geq 6+9 = 15.$$

■

313. Let a, b, c be real numbers such that $a^2+b^2+c^2=1$. Prove that

$$a+b+c \leq 2abc + \sqrt{2}.$$

Solution: We have $2ab \leq a^2+b^2 \leq a^2+b^2+c^2 = 1$. Similarly, $2bc \leq 1$ and $2ca \leq 1$. We have

$$\begin{aligned}
&2-(a+b+c-2abc)^2 \\
&\quad = \quad 1+a^2+b^2+c^2-(a+b+c)^2+4abc(a+b+c)-4a^2b^2c^2 \\
&\quad = \quad 1-2(ab+bc+ca)+4abc(a+b+c)-4a^2b^2c^2 \\
&\quad = \quad (1-2ab)(1-2bc)(1-2ca)+4a^2b^2c^2 \geq 0.
\end{aligned}$$

It follows that

$$(a+b+c-2abc)^2 \leq 2.$$

This gives $a+b+c \leq 2abc+\sqrt{2}$. Equality holds if and only if one of a, b, c is zero and the other two are equal to $1/\sqrt{2}$ each. ■

314. Let a, b, c be positive real numbers. Prove that

$$\frac{(b+c-a)^2}{a^2+(b+c)^2} + \frac{(c+a-b)^2}{b^2+(c+a)^2} + \frac{(a+b-c)^2}{c^2+(a+b)^2} \geq \frac{3}{5}.$$

Solution: Let us introduce x, y, z by

$$x = \frac{b+c}{a}, \quad y = \frac{c+a}{b}, \quad z = \frac{a+b}{c}.$$

Then x, y, z are positive numbers such that $xyz = x+y+z+2$. The inequality is

$$\frac{(x-1)^2}{x^2+1} + \frac{(y-1)^2}{y^2+1} + \frac{(z-1)^2}{z^2+1} \geq \frac{3}{5}.$$

Using Cauchy-Schwarz inequality, we obtain

$$\frac{(x-1)^2}{x^2+1} + \frac{(y-1)^2}{y^2+1} + \frac{(z-1)^2}{z^2+1} \geq \frac{(x+y+z-3)^2}{x^2+y^2+z^2+3}.$$

Hence it is enough to prove that

$$\frac{(x+y+z-3)^2}{x^2+y^2+z^2+3} \geq \frac{3}{5}.$$

This reduces to

$$(x+y+z)^2 - 15(x+y+z) + 3(xy+yz+zx) + 18 \geq 0.$$

Substituting $x+y+z = xyz-2$ and using $xy+yz+zx \geq 3(xyz)^{2/3}$, it is enough to prove that

$$r^2 - 19r + 9r^{2/3} + 52 \geq 0,$$

where $r = xyz$. If we substitute $u = r^{1/3}$, this reduces to

$$u^6 - 19u^3 + 9u^2 + 52 \geq 0.$$

We check that **lhs** is factorisable and

$$u^6 - 19u^3 + 9u^2 + 52 = (u-2)^2(u^4 + 4u^3 + 12u^2 + 13u + 13) \geq 0.$$

This completes the proof. ■

315. Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\sqrt{\frac{1}{a}-1}\sqrt{\frac{1}{b}-1} + \sqrt{\frac{1}{b}-1}\sqrt{\frac{1}{c}-1} + \sqrt{\frac{1}{c}-1}\sqrt{\frac{1}{a}-1} \geq 6.$$

Solution: Introducing $a = xy, b = yz$ and $c = zx$, we have $xy+yz+zx = 1$. Hence we can write

$$x = \tan\frac{\alpha}{2}, \quad y = \tan\frac{\beta}{2}, \quad x = \tan\frac{\gamma}{2},$$

where $\alpha, \beta, \gamma \in (0, \pi)$ and $\alpha+\beta+\gamma = \pi$. We observe that

$$\sqrt{\frac{1}{a}-1}\sqrt{\frac{1}{b}-1} = \sqrt{\frac{(1-a)(1-b)}{ab}} = \sqrt{\frac{(1-xy)(1-yz)}{xy^2z}}$$

$$\sqrt{\frac{(yz+zx)(zx+xy)}{xy^2z}} = \frac{\sqrt{1+y^2}}{y} = \frac{1}{\sin(\beta/2)}.$$

Similarly, we can get

$$\sqrt{\frac{1}{b}-1}\sqrt{\frac{1}{c}-1} = \frac{1}{\sin(\gamma/2)}, \quad \sqrt{\frac{1}{c}-1}\sqrt{\frac{1}{a}-1} = \frac{1}{\sin(\alpha/2)}.$$

Thus we need to prove that

$$\frac{1}{\sin(\alpha/2)} + \frac{1}{\sin(\beta/2)} + \frac{1}{\sin(\gamma/2)} \geq 6.$$

But AM-HM inequality implies that

$$\frac{1}{\sin(\alpha/2)} + \frac{1}{\sin(\beta/2)} + \frac{1}{\sin(\gamma/2)} \geq \frac{9}{\sin(\alpha/2) + \sin(\beta/2) + \sin(\gamma/2)}.$$

Hence it is enough to prove that

$$\sin(\alpha/2) + \sin(\beta/2) + \sin(\gamma/2) \leq \frac{3}{2}.$$

This follows from (3.4.10) of chapter 3.

$\blacksquare$

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Abbreviation

We have used the following abbreviations in the text.

- AMM — American Mathematical Monthly;
- CRUX — Crux Mathematicorum;
- IMO — International Mathematical Olympiad;
- CRMO — Central Regional Mathematical Olympiad(this is the first level examination for selecting students to represent India in IMO);
- INMO — Indian National Mathematical Olympiad(this is the second level examination for selecting students to represent India in IMO).

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